

## Answers for final exam (2h30)

(1) We compute

$$\begin{aligned} X_t - \bar{X}_t &= \int_0^t b(X_s) - b(\bar{X}_s) ds \\ &= \int_0^t b(X_s) - b(\bar{X}_s) ds + \int_0^t b(\bar{X}_s) ds - \int_0^t b(\bar{X}_s) ds, \end{aligned}$$

then, by basic triangular inequality and mean value's inequality

$$|X_t - \bar{X}_t| \leq \left| \int_0^t b(X_s) - b(\bar{X}_s) ds \right| + \sup_{x \in \mathbb{R}} |b'(x)| \int_0^t |\bar{X}_s - \bar{X}_s| ds.$$

(2)

(a) We use the integration by parts equation (for the processes  $s \mapsto t_k - s$ ,  $s \mapsto \int_{t_{k-1}}^s b'(X_u) dW_u$ )

$$\begin{aligned} (t_k - t_k) \times \int_{t_{k-1}}^{t_k} b'(X_u) dW_u - (t_k - t_{k-1}) \times \int_{t_{k-1}}^{t_{k-1}} b'(X_u) dW_u = \\ - \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s b'(X_u) dW_u ds + \int_{t_{k-1}}^{t_k} (t_k - u) b'(X_u) dW_u + 0 \end{aligned}$$

(quadratic variation is zero).

(b) We compute, using Itô's formula, for  $t_{k-1} \leq s \leq t_k$ ,

$$b(X_s) - b(X_{t_{k-1}}) = \int_{t_{k-1}}^s b'(X_u) (b(X_u) du + dW_u) + \frac{1}{2} \int_{t_{k-1}}^s b''(X_u) du.$$

So

$$\begin{aligned} \int_{t_{k-1}}^{t_k} b(X_s) - b(X_{t_{k-1}}) ds = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s b'(X_u) b(X_u) + \frac{1}{2} b''(X_u) du ds \\ + \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s b'(X_u) dW_u ds. \end{aligned}$$

We can use Fubini's Theorem on the first part (as  $b'$  and  $b''$  are bounded):

$$\begin{aligned} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s b'(X_u) b(X_u) + \frac{1}{2} b''(X_u) du ds &= \int_{t_{k-1}}^{t_k} \left( b'(X_u) b(X_u) + \frac{1}{2} b''(X_u) \right) \times \int_u^{t_k} ds du \\ &= \int_{t_{k-1}}^{t_k} \left( b'(X_u) b(X_u) + \frac{1}{2} b''(X_u) \right) (t_k - u) du. \end{aligned}$$

For the second part, we use the previous question. So

$$\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s b'(X_u) dW_u ds = \int_{t_{k-1}}^{t_k} (t_k - u) b'(X_u) dW_u.$$

So we get

$$\begin{aligned} \int_{t_{k-1}}^{t_k} b(X_s) - b(X_{t_{k-1}}) ds = \int_{t_{k-1}}^{t_k} \left( b'(X_u) b(X_u) + \frac{1}{2} b''(X_u) \right) (t_k - u) du \\ + \int_{t_{k-1}}^{t_k} (t_k - u) b'(X_u) dW_u. \end{aligned}$$

(3) We have

$$\begin{aligned} \left| \int_0^t b(X_s) - b(X_{\underline{s}}) ds - \int_0^{\underline{t}} b(X_s) - b(X_{\underline{s}}) ds \right| &= \left| \int_{\underline{t}}^t b(X_s) - b(X_{\underline{s}}) ds \right| \\ \text{(triangular inequality)} &\leq \int_{\underline{t}}^t |b(X_s)| + |b(X_{\underline{s}})| ds \\ &\leq 2 \sup_{x \in \mathbb{R}} |b(x)| \times \Delta t. \end{aligned}$$

(4)

(a) Let  $0 \leq k \leq N$ . We deduce from the previous question that

$$\begin{aligned} \int_0^{t_k} b(X_s) - b(X_{\underline{s}}) ds &= \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} b(X_s) - b(X_{t_j}) ds \\ &= \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (\underline{u} + \Delta t - u) \left( b'(X_u) b(X_u) + \frac{1}{2} b''(X_u) \right) du \\ &\quad + \int_{t_j}^{t_{j+1}} (\underline{u} + \Delta t - u) b'(X_u) dW_u \\ &= \int_0^{t_k} (\underline{u} + \Delta t - u) \left( b'(X_u) b(X_u) + \frac{1}{2} b''(X_u) \right) du \\ &\quad + \int_0^{t_k} (\underline{u} + \Delta t - u) b'(X_u) dW_u. \end{aligned}$$

(b) This is a consequence of question 3.

(5) Using the triangular inequality and the above question, we get

$$\begin{aligned} \sup_{u \in [0, T]} \left| \int_0^u b(X_s) - b(X_{\underline{s}}) ds \right| &\leq 2 \sup_{x \in \mathbb{R}} |b(x)| \times \Delta t + \sup_{0 \leq k \leq N} \left| \int_0^{t_k} (\underline{u} + \Delta t - u) \left( b'(X_u) b(X_u) + \frac{1}{2} b''(X_u) \right) du \right| \\ &\quad + \sup_{0 \leq k \leq N} \left| \int_0^{t_k} (\underline{u} + \Delta t - u) b'(X_u) dW_u \right| \end{aligned}$$

We compute the following bounds

$$\begin{aligned} \mathbb{E} \left( \sup_{0 \leq k \leq N} \left| \int_0^{t_k} (\underline{u} + \Delta t - u) b'(X_u) dW_u \right|^2 \right) &\leq \mathbb{E} \left( \sup_{u \in [0, T]} \left| \int_0^u (\underline{r} + \Delta t - r) b'(X_r) dW_r \right|^2 \right) \\ \text{(Cauchy-Schwarz)} &\leq \mathbb{E} \left( \sup_{u \in [0, T]} \left( \int_0^u (\underline{r} + \Delta t - r) b'(X_r) dW_r \right)^2 \right) \\ \text{(Burckholder-Davis-Gundy)} &\leq \mathbb{E} \left( \int_0^T \Delta t^2 \times |b'(X_r)|^2 dr \right). \end{aligned}$$

So we get

$$\begin{aligned} \mathbb{E} \left( \sup_{u \in [0, T]} \left| \int_0^u b(X_s) - b(X_{\underline{s}}) ds \right|^2 \right) &\leq 2 \sup_{x \in \mathbb{R}} |b(x)| \times \Delta t + \Delta t \int_0^T \mathbb{E} \left( \left| b(X_s) b'(X_s) + \frac{1}{2} b''(X_s) \right|^2 \right) ds \\ &\quad + \Delta t \sqrt{\int_0^T \mathbb{E}((b'(X_s))^2) ds}. \end{aligned}$$

(6) By question 1, we have for all  $0 \leq t \leq T$ ,

$$\begin{aligned} |X_t - \bar{X}_t| &\leq \left| \int_0^t b(X_s) - b(\bar{X}_s) ds \right| + \sup_{x \in \mathbb{R}} |b'(x)| \int_0^t |\bar{X}_s - \bar{X}_{\underline{s}}| ds \\ &\leq \left| \int_0^t b(X_s) - b(\bar{X}_s) ds \right| + \sup_{x \in \mathbb{R}} |b'(x)| \int_0^t \sup_{u \in [0, s]} |\bar{X}_u - \bar{X}_{\underline{u}}| ds. \end{aligned}$$

So there exists a constant  $C$  such that

$$z(t) \leq C \left( \Delta t + \int_0^t z(s) ds \right).$$

(7) From Gronwall's Lemma, we get, for all  $0 \leq t \leq T$ ,

$$z(t) \leq C \Delta t e^{CT}.$$

This proves the desired result.

```
(8) import numpy as np
import scipy.stats as sps
T=1; x0=1
def b(x):
    return(np.sin(x))
def f(x):
    return(x)
def euler(N):
    N=int(N)
    z=sps.norm.rvs(size=(N))
    x=x0
    for k in range(N):
        x=x+b(x)*T/N+np.sqrt(T/N)*z[k]
    return(f(x))
```

```
(9)
def mc(N,M):
    tab=np.ones(M)*N
    return(np.mean(list(map(euler, tab))))
```