Université Nice-Sophia Antipolis
SMEMP302 - ECUE Probabilitic computational methods, 2023-2024
Sylvain Rubenthaler (https://math.unice.fr/~rubentha/)

Final exam (2h30)

Let $T > 0$, $x \in \mathbb{R}$, $b : \mathbb{R} \mapsto \mathbb{R}$ a C_b^2 function (can be derived twice and its derivatives are all bounded) and, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $(W_t)_{t\in[0,T]}$ a standard Brownian motion in $\mathbb{R}$. We are interested in the following stochastic differential equation

$$\begin{cases} X_0 &= x\,, \\ dX_t &= dW_t + b(X_t)dt\,. \end{cases}$$

We fix $N \in \mathbb{N}^*$ and for $0 \leq k \leq N$, we set $t_k = k\Delta t$ where $\Delta t = T/N$. The Euler scheme with N steps is defined recursively by

$$\begin{cases} \overline{X}_0 = x \\ \forall 0 \leq k \leq N-1\,,\, \forall t \in [t_k, t_{k+1}]\,,\, \overline{X}_t = \overline{X}_{t_k} + (W_t - W_{t_k}) + b(\overline{X}_{t_k})(t - t_k) \end{cases}$$

(1) For $s \in [0,T]$, we set $\underline{s} = [s/\Delta t] \times \Delta t$ ($[\dots]$ =integer part), this is the last discretisation time before s). Show that

$$\forall t \in [0,T]\,,\, |X_t - \overline{X}_t| \leq \sup_{x\in\mathbb{R}} |b'(x)| \times \int_0^t |X_{\underline{s}} - \overline{X}_{\underline{s}}| ds + \left| \int_0^t b(X_s) - b(X_{\underline{s}}) ds \right|\,.$$

(Hint: $b(X_s) - b(\overline{X}_{\underline{s}}) = b(X_s) - b(X_{\underline{s}}) + b(X_{\underline{s}}) - b(\overline{X}_{\underline{s}})$.)

(2)

(a) Show that, for all $1 \leq k \leq N$,

$$\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} b'(X_u) dW_u ds = \int_{t_{k-1}}^{t_k} (t_k - u) b'(X_u) dW_u\,.$$

(b) Show that for $1 \leq k \leq N$,

$$\int_{t_{k-1}}^{t_k} b(X_s) - b(X_{t_{k-1}}) ds = \int_{t_{k-1}}^{t_k} (t_k - r)(b(X_r)b'(X_r) + \frac{1}{2} b''(X_r).0dr + \int_{t_{k-1}}^{t_k} (t_k - r) b'(X_r) dW_r\,.$$

(Hint: start by computing $b(X_s) - b(X_{t_{k-1}})$ using Itô's formula.)

(3) Check that, for all $0 \leq t \leq T$,

$$\left| \int_0^t b(X_s) - b(X_{\underline{s}}) ds - \int_0^{\underline{t}} b(X_s) - b(X_{\underline{s}}) ds \right| \leq 2 \sup_{x\in R} |b(x)| \times \Delta t\,.$$

(4)

(a) Show that, for all $0 \leq k \leq N$,

$$\int_0^{t_k} b(X_s) - b(X_{\underline{s}}) ds = \int_0^{t_k} (\underline{r} + \Delta t - r) \left(b(X_r)b'(X_r) + \frac{1}{2} b''(X_r) \right) dr + \int_0^{t_k} (\underline{r} + \Delta t - r) b'(X_r) dW_r\,,$$

(b) And that, for all $0 \leq t \leq T$,

$$\left| \int_0^t b(X_s) - b(X_{\underline{s}}) ds - \int_0^{\underline{t}} (\underline{s} + \Delta t - s) \left(b(X_s)b'(X_s) + \frac{1}{2} b''(X_s) \right) ds - \int_0^{\underline{t}} (\underline{s} + \Delta t - s) b'(X_s) dW_s \right| \leq 2 \sup_{x\in R} |b(x)| \times \Delta t\,.$$

(5) Show that

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t b(X_s)-b(X_{\underline{s}})ds\right|\right)$$
$$\leq \Delta t\left(2\sup_{x\in R}|b(x)|+\int_0^T \mathbb{E}\left(\left|b(X_s)b'(X_s)+\frac{1}{2}b''(X_s)\right|\right)+\sqrt{\int_0^T \mathbb{E}((b'(X_s)^2)ds}\right).$$

(6) We set $z(t)=\mathbb{E}\left(\sup_{u\in[0,t]}|X_u-\overline{X}_u|\right)$. Show that

$$\forall t\in[0,T]\,,\; z(t)\leq C\left(\Delta t+\int_0^t z(s)ds\right),$$

where the constant C does not depend on N.

(7) Conclude that

$$\mathbb{E}\left(\sup_{t\in[0,T]}|X_t-\overline{X}_t|\right)\leq\frac{C}{N},$$

for some constant C that does not depend on N.

(8) Suppose that f is a function defined in a `python` code. Write a `python` function that returns a simulation $f(\overline{X}_T)$ (anything vaguely looking like `python` is enough). You will define the consants you need.

(9) Write a `python` code that returns a Monte-Carlo computation of $\mathbb{E}(f(\overline{X}_T))$ (anything vaguely looking like `python` is enough).