Université Nice-Sophia Antipolis
SMEMP302 - ECUE Probabilitic computational methods, 2023-2024
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Home Project

We are interested in the following SDE (dimension 1)

$$X_t = W_t - \int_0^t X_s^3 ds \tag{0.1}$$

((W_t) is a standard Brownian motion). We admit that this equation has a unique solution $(X_t)_{t\in[0,T]}$ on the interval $[0,T]$ ($T>0$) and that this solution satisfies $\mathbb{E}(\sup_{t\in[0,T]}|X_t|^4) < +\infty$. We write $(\overline{X}_t^N)_{0\le t\le T}$ for the continuous Euler scheme associated to Equation (0.1), with stepsize T/N. The discretization steps are $0, t_1 = T/N, t_2 = 2T/N, \dots$. We define the event :

$$A_N = \{|W_{t_1}| \ge \frac{3N}{T}\,,\ \sup_{1\le k\le N-1} |W_{t_{k+1}} - W_{t_k}| \le 1\}\,.$$

(1) Show that $\mathbb{P}(A_N) \ge \mathbb{P}(|W_{t_1}| \ge 3N/T)\mathbb{P}(\sup_{t\in[0,T]}|W_t| \le 1/2)$.

(2) By observing that for all $x>0$,

$$\frac{e^{-x^2/2}}{x} = \int_x^{+\infty} \left(1+\frac{1}{y^2}\right) e^{-y^2/2} dy\,,$$

show that

$$\int_x^{+\infty} e^{-y^2/2} dy \ge \frac{x e^{-x^2/2}}{1+x^2}\,.$$

(3) Show that

$$\mathbb{P}(A_N) \ge \mathbb{P}(\sup_{t\in[0,T]}|W_t| \le 1/2) \times \frac{6(NT)^{3/2}}{T^3+9N^3} \times \frac{e^{-9N^3/(2T^3)}}{\sqrt{2\pi}}\,.$$

(4) Show that, for $\omega \in A_N$, if $k \ge 1$,

$$|\overline{X}_{t_k}^N(\omega)| \ge 1 \Rightarrow |\overline{X}_{t_{k+1}}^N(\omega)| \ge |\overline{X}_{t_k}^N(\omega)|^2 \left(\frac{T}{N}|\overline{X}_{t_k}^N(\omega)| - 2\right)\,.$$

(5) Show that, if $N \ge T/3$, for $\omega \in A_N$, $\forall k \in \{1,\dots,N\}$,

$$|\overline{X}_{t_k}^N(\omega)| \ge \left(\frac{3N}{T}\right)^{2^{k-1}}\,.$$

(6) Show that $\lim_{N\to+\infty} \mathbb{E}(|\overline{X}_T^N|) = +\infty$.

(7) Find $\lim_{N\to+\infty} \mathbb{E}(|\overline{X}_T^N - X_T|)$ and $\lim_{N\to+\infty} \mathbb{E}(\sup_{t\in[0,T]}|\overline{X}_t^N - X_t|)$.

(8) Observing that the law of $\overline{X}_T^N$ is symmetric, show that, for $K>0$,

$$\mathbb{E}((\overline{X}_T^N - K)^+) \ge \frac{1}{2}\mathbb{E}(|\overline{X}_T^N|) - K\,.$$

(9) Show that $\mathbb{E}((\overline{X}_T^N - K)^+)$ does not converge towards $\mathbb{E}((X_T-K)^+)$ when $N \to +\infty$.

(10) Write a `python` function that compute $\overline{X}_T^N$ for a given N (take $T=10$).

(11) Write a `python` function that compute a Monte-Carlo approximation of $\mathbb{E}(|\overline{X}_T^N|)$ with a sum of M terms, for given N, M.

(12) BONUS QUESTION (more difficult) : Write a program in `python` that will illustrate the result of question 6. Do not forget to include confidence interval in a potential graphic.