Rearrangement vs Convection:

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Outline

1. A toy-model for (very fast) convection based on rearrangement theory

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2. Multidimensional rearrangement theory and generalization of the toy model

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2. Multidimensional rearrangement theory and generalization of the toy model
3. Interpretation of the model as a hydrostatic limit of the Navier-Sokes Boussinesq equations

A reminder

Given a scalar function $z(x), \; x \in D = [0,1]$**, there is a unique non decreasing function** $Z(x) = \mathrm{Rearrange}(z)(x)$ **such that,**

$$\int_D f(Z(x))dx = \int_D f(z(x))dx$$

for all test function f**.**

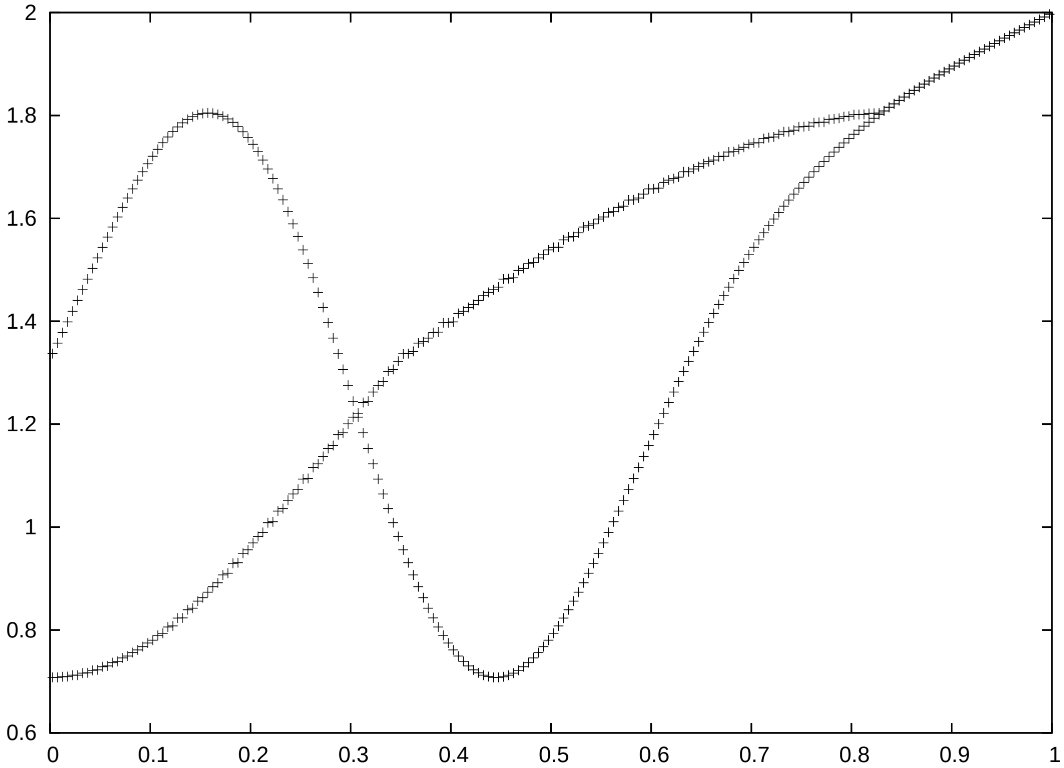
Notice that in the discrete case when

$$z(x) = z_j, \quad j/N < x < (j+1)/N, \quad j = 0, ..., N-1$$

then $Z(x) = Z_j$ **where** $(Z_1, ..., Z_N)$ **is just** $(z_1, ..., z_N)$ **sorted in increasing order.**

A function and its rearrangement

N = 200 grid points in x



A toy-model for (very fast) convection

Model:

-vertical coordinate only: $\mathbf{x = x_3 \in D = [0, 1]}$

-temperature field: $\mathbf{y(t, x)}$

-heat source term: $\mathbf{G = G(t, x, y)}$

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Time discrete scheme:

-time step $\mathrm{h} > 0$**,** $\mathbf{y}(\mathbf{t} = \mathbf{nh}, \mathbf{x}) \sim \mathbf{y_n}(\mathbf{x})$**,** $\mathbf{n} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \cdots$

-predictor (heating): $\tilde{\mathbf{y}}_{\mathbf{n+1}}(\mathbf{x}) = \mathbf{y_n}(\mathbf{x}) + \mathbf{h}\, \mathbf{G}(\mathbf{nh}, \mathbf{x}, \mathbf{y_n}(\mathbf{x}))$

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-corrector (fast convection): $\mathbf{y_{n+1}} = \mathbf{Rearrange}(\tilde{\mathbf{y}}_{\mathbf{n+1}})$

so that the temperature profile stays monotonically increasing at EACH time step. (This actually corresponds to a succession of stable equilibria modified by the source term.)

Heat profiles with a rough time step

$$\mathbf{G = G(x) = 1 + \exp(-25(x-0.2)^2) - \exp(-20(x-0.4)^2)}$$

$$\mathbf{t, x \in [0,1] \quad h = 0.1 \quad 500 \text{ grid points in } x}$$

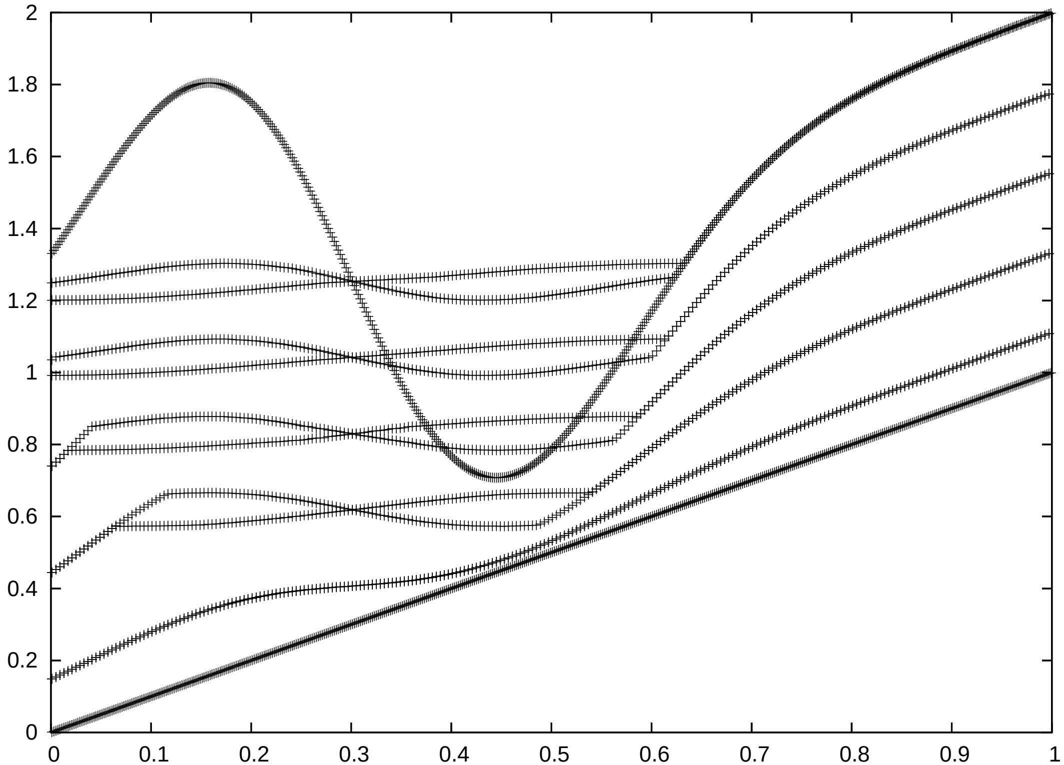

Figure 1: y=y(t,x) versus x at different t (predictor and corrector)

Heat profiles with a fine time step

$$\mathbf{t, x \in [0,1] \quad h = 0.005 \quad 500\ grid\ points\ in\ x}$$

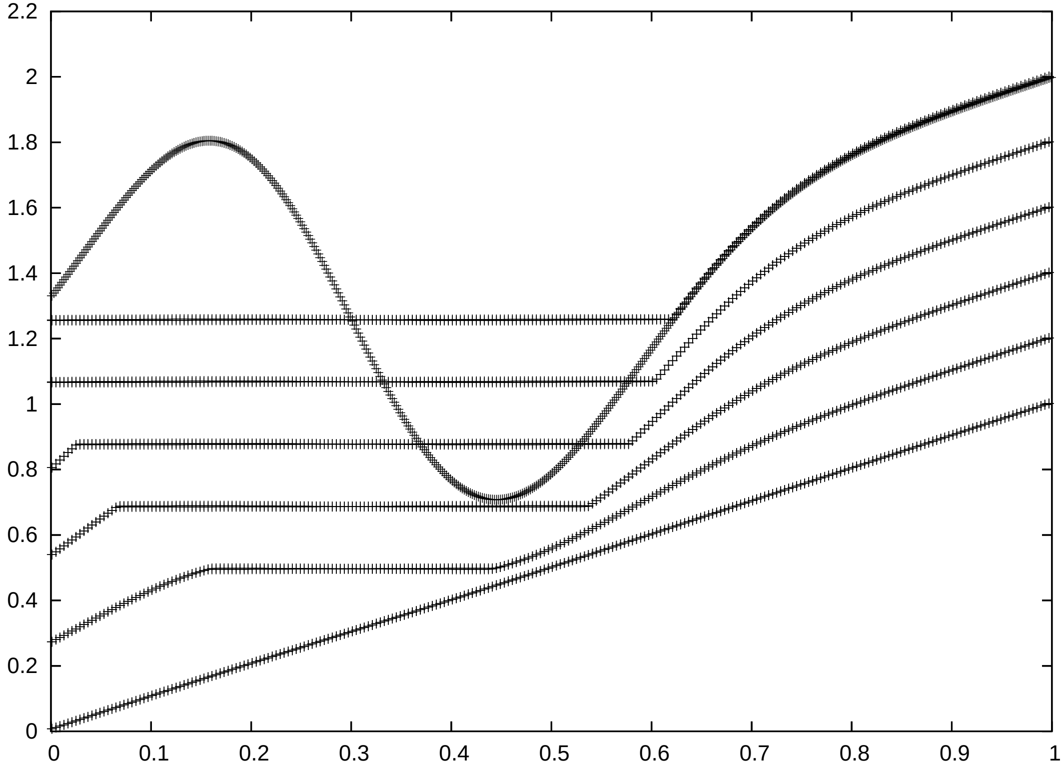

Figure 2: y=y(t,x) versus x at different t

mixing of the fluid parcels

$$\mathbf{t}, \mathbf{x} \in [0,1] \quad \mathbf{h} = 0.005 \quad \mathbf{500\ grid\ points\ in\ x}$$

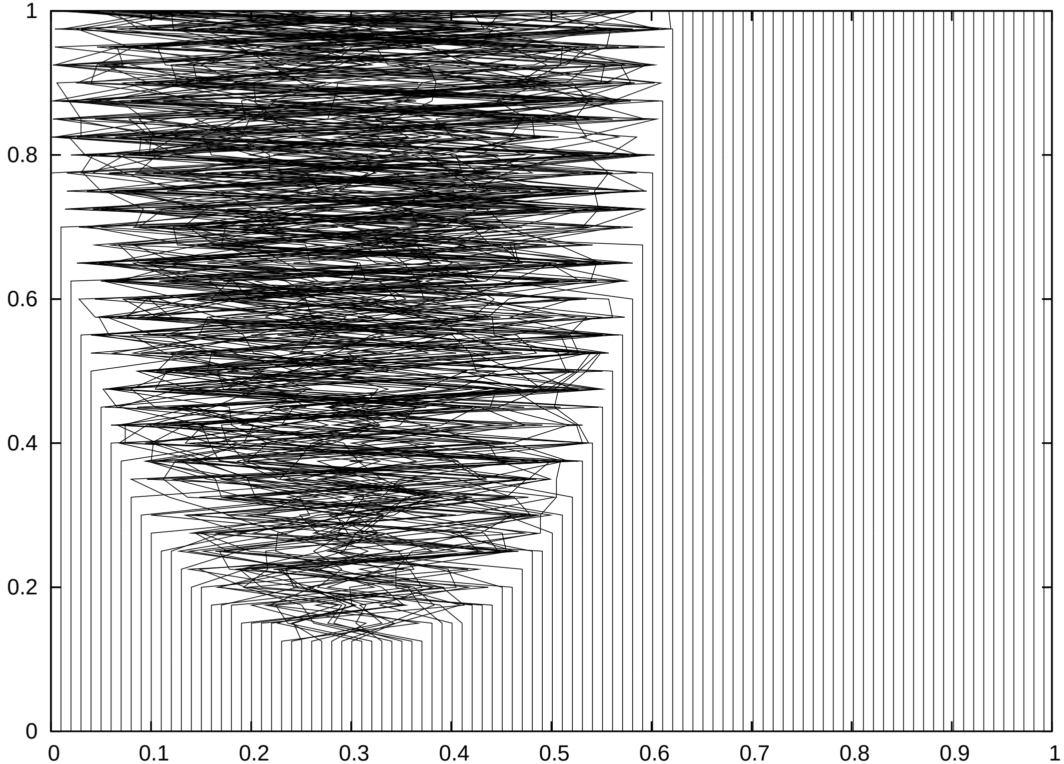

Figure 3: t versus x up to rearrangement

Convergence analysis

Theorem

As $h \to 0$, the time-discrete scheme has a unique limit y in space $C^{0}(R_+,\ L^{2}(D, R^{d}))$ that satisfies the subdifferential inclusion:

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$$\boxed{\mathbf{G}(\mathbf{t}, \mathbf{x}, \mathbf{y}) \in \partial_{\mathbf{t}}\mathbf{y} + \partial\boldsymbol{\Psi}[\mathbf{y}]}$$

where $\Psi[y] = 0$ if y is non decreasing as a function of x and $\Psi[y] = +\infty$ otherwise.

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where $\Psi[y] = 0$ if y is non decreasing as a function of x and $\Psi[y] = +\infty$ otherwise.
In addition, choosing g so that $G(t, x, y) = \partial_x g(t, x, y)$, we check that the pseudo-inverse $x = u(t, y)$ is an entropy solution to the scalar conservation law

$$\partial_{\mathbf{t}} \mathbf{u} + \partial_{\mathbf{y}}(\mathbf{g}(\mathbf{t}, \mathbf{u}(\mathbf{t}, \mathbf{y}), \mathbf{y})) - (\partial_{\mathbf{y}} \mathbf{g})(\mathbf{t}, \mathbf{u}(\mathbf{t}, \mathbf{y}), \mathbf{y}) = \mathbf{0},$$

This is an example of the more general L^2 formulation of multidimensional scalar conservation laws, YB 2006 to appear in ARMA 2009

Multidimensional rearrangement

Theorem

Given a bounded domain $\mathrm{D} \subset \mathrm{R}^{\mathrm{d}}$
and an $\mathrm{L}^{\mathbf{2}}$ **map** $\mathbf{x} \in \mathrm{D} \rightarrow \mathbf{z}(\mathbf{x}) \in \mathrm{R}^{\mathrm{d}}$,

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$$\mathrm{Rearrange}(\mathbf{z})(\mathbf{x}) = \nabla \mathrm{p}(\mathbf{x}),$$

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$\mathbf{Rearrange}(\mathbf{z})(\mathbf{x}) = \nabla \mathbf{p}(\mathbf{x})$,
$\mathbf{p}(\mathbf{x})$ **lsc convex in** $x \in \mathbf{R}^{\mathbf{d}}$**, a.e. differentiable on** $\mathbf{D}$**, such that**

$$\int_{\mathbf{D}} \mathbf{f}(\nabla \mathbf{p}(\mathbf{x}))\mathbf{dx} = \int_{\mathbf{D}} \mathbf{f}(\mathbf{z}(\mathbf{x}))\mathbf{dx}$$

for all continuous function $\mathbf{f}$ **such that** $|\mathbf{f}(\mathbf{x})| \leq 1 + |\mathbf{x}|^{\mathbf{2}}$

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This is a typical result in optimal transport theory, see YB, CRAS Paris 1987 and CPAM 1991, Smith and Knott, JOTA 1987, cf. Villani, Topics in optimal transportation, AMS, 2003, see also papers, lecture notes and books by Rachev-Rüschendorf, Evans, Caffarelli, Urbas, Gangbo-McCann, Otto, Ambrosio-Savaré, Villani, Trudinger-Wang and many others contributions...

Multi-d generalization of the toy-model

Model:

-a smooth bounded domain $x \in D \subset R^d$

-a vector-valued field: $y(t,x) \in R^d$ **(generalized temperature)**

-a source term: $G = G(t,x,y) \in R^d$ **with bounded derivatives**

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-time step $\mathbf{h} > \mathbf{0}$**,** $\;\mathbf{y}(\mathbf{t} = \mathbf{n}\mathbf{h}, \mathbf{x}) \sim \mathbf{y}_{\mathbf{n}}(\mathbf{x})$**,** $\quad \mathbf{n} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \cdots$

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as the unique rearrangement with convex potential $\mathbf{y}_{\mathbf{n+1}} = \nabla \mathbf{p}_{\mathbf{n+1}}$

Main property of the scheme

Take a smooth function $\mathbf{f}$. Then

$$\int_{\mathbf{D}} \mathbf{f}(\mathbf{y}_{\mathbf{n+1}}(\mathbf{x}))\mathbf{dx} = \int_{\mathbf{D}} \mathbf{f}(\tilde{\mathbf{y}}_{\mathbf{n+1}}(\mathbf{x}))\mathbf{dx}$$

(because $\mathbf{y}_{\mathbf{n+1}}$ is a rearrangement of $\tilde{\mathbf{y}}_{\mathbf{n+1}}$)

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$$= \int_D f(y_n(x))dx + h\int_D (\nabla f)(y_n(x)) \cdot G(nh, x, y_n(x))dx \quad + o(h)$$

Convergence of the scheme

Theorem

As $h \to 0$, the time-discrete scheme has converging subsequences.
Each limit y belongs to the space $C^0(R_+,\ L^2(D, R^d))$ and has a convex potential $p(t, \cdot)$ for each $t \geq 0$.

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In addition,

$$\frac{d}{dt}\int_D f(y(t,x))dx = \int_D (\nabla f)(y(t,x)) \cdot G(t,x,y(t,x))dx$$

for all smooth function f such that $|f(x)| \leq 1 + |x|^2$

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See YB, 2008, to appear in JNLS. Notice that the system is self-consistent, thanks to the rearrangement theorem. However, our global existence result does not imply uniqueness

The special case $G(t, x, y) = -x$

In this special case, which in 1d would correspond to the invisicid Burgers equation, the discrete model turns out to be the discrete version of the subdifferential equation in $L^2(D, R^d)$

$$\mathbf{0} \in \partial_{\mathbf{t}}\mathbf{z} + \partial\mathbf{K}[\mathbf{z}]$$

$$\mathbf{K}[\mathbf{z}] = \sup_{\mathbf{s} \in \mathbf{S}(\mathbf{D})} \int_{\mathbf{D}} \mathbf{s}(\mathbf{x}) \cdot \mathbf{z}(\mathbf{x})\, \mathbf{dx}$$

where $S(D)$ denotes the set of all Lebesgue measure preserving maps of D.

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$$0 \in \partial_t z + \partial K[z]$$

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Interpretation of the multi-d toy model

The formulation we have obtained for the multidimensional toy model

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbf{D}} \mathbf{f}(\mathbf{y}(\mathbf{t},\mathbf{x}))\mathbf{dx} = \int_{\mathbf{D}} (\nabla \mathbf{f})(\mathbf{y}(\mathbf{t},\mathbf{x}))\cdot \mathbf{G}(\mathbf{t},\mathbf{x},\mathbf{y}(\mathbf{t},\mathbf{x}))\mathbf{dx}$$

for all smooth function f, with $\mathrm{y} = \nabla \mathrm{p}$,

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$$\frac{d}{dt}\int_D f(y(t,x))dx = \int_D (\nabla f)(y(t,x)) \cdot G(t,x,y(t,x))dx$$

for all smooth function f, with $y = \nabla p$,
in some sense means that there exists a GAUGE field $v(t,x)$ such that

$$\partial_t y + (v \cdot \nabla)y = G(t,x,y), \qquad \nabla \cdot v = 0, \quad v // \partial D$$

which, continuously in time, rearranges $y(t,x)$ so that y stays a map with a convex potential at any time.

Interpretation of the multi-d toy model

It turns out that the model can be interpreted as a singular limit of the Navier-Stokes Boussinesq equations with vector-valued buoyancy forces. This is what we are now going to explain in the last part of the talk

The NS-Boussinesq model

Let D be a smooth bounded domain $\mathrm{D} \subset \mathrm{R}^{\mathbf{3}}$ in which moves an incompressible fluid of velocity $\mathrm{v}(\mathrm{t}, \mathrm{x})$ at $\mathrm{x} \in \mathrm{D}, \quad \mathrm{t} \geq 0$, subject to the Navier-Stokes equations

$$\text{NSB} \quad \epsilon(\partial_{\mathbf{t}}\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) - \nu\boldsymbol{\Delta}\mathbf{v} + \nabla\mathbf{p} = \mathbf{y} \qquad \nabla \cdot \mathbf{v} = \mathbf{0}$$

with $\epsilon, \nu > 0$ and $\mathbf{v} = 0$ along $\partial\mathrm{D}$.

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The force field $\mathbf{y}$ is subject to the advection equation

$$\partial_{\mathbf{t}}\mathbf{y} + (\mathbf{v}\cdot\nabla)\mathbf{y} = \mathbf{G}(\mathbf{t}, \mathbf{x}, \mathbf{y})$$

where G is a given smooth function with bounded derivatives.

Classical Convection Theory

Classical Convection Theory corresponds to the special case:

$$\mathbf{G} = \mathbf{0}, \quad \mathbf{y} // \mathbf{e_3}, \quad \mathbf{y} = \eta \mathbf{e_3}, \quad \eta = \eta(\mathbf{t}, \mathbf{x}) \in \mathbf{R}$$ **namely:**

$$\epsilon(\partial_{\mathbf{t}} \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) - \boldsymbol{\Delta} \mathbf{v} + \nabla \mathbf{p} = \eta \mathbf{e_3}, \quad \nabla \cdot \mathbf{v} = \mathbf{0}$$

$$\partial_{\mathbf{t}} \eta + (\mathbf{v} \cdot \nabla)\eta = \mu \boldsymbol{\Delta} \eta$$

with $\mu \geq \mathbf{0}$**.**

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with $\mu \geq \mathbf{0}$**.**

For $\mu = \mathbf{0}$, **global existence of weak solutions in 3D follows from Leray/Diperna-Lions theory, while global existence of smooth solutions in 2D follows from Hou-Li 2005 and Chae 2006. (See also recent work by Danchin-Paicu.)**

Singular limits of the NSB model

While keeping unchanged

$$\boxed{\partial_{\mathbf{t}}\mathbf{y} + (\mathbf{v}\cdot\nabla)\mathbf{y} = \mathbf{G}(\mathbf{t},\mathbf{x},\mathbf{y}) \qquad \nabla\cdot\mathbf{v} = \mathbf{0}}$$

and dropping inertia terms, we consider two possible limits:

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$$\text{HYDROSTATIC HB}: \quad \epsilon = \nu = \mathbf{0} \quad \Rightarrow \quad \nabla\mathbf{p} = \mathbf{y}$$

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The 2nd one HB CORRESPONDS TO OUR MULTI-d TOY MODEL!

A convexity condition for the HB model

The Hydrostatic Boussinesq HB system

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looks strange since there is no direct equation for $\mathbf{v}$.

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Let us consider, for simplicity, the case of 2 space variables $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2})$ and write $\mathbf{v} = (-\partial_{\mathbf{1}}\theta, \partial_{\mathbf{2}}\theta)$, where $\theta(\mathbf{t}, \mathbf{x_1}, \mathbf{x_2}) \in \mathbf{R}$.

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Take the 2D curl of the evolution equation in $\mathbf{y} = (\partial_1\mathbf{p}, \partial_2\mathbf{p})$:

$$\partial_{11}\mathbf{p}\,\partial_{22}\theta + \partial_{22}\mathbf{p}\,\partial_{11}\theta - 2\partial_{12}\mathbf{p}\,\partial_{12}\theta = \partial_1(\mathbf{G_2}) - \partial_2(\mathbf{G_1})$$

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a well posed linear elliptic equation in θ whenever $\mathbf{D}^2_{\mathbf{x}}\mathbf{p}(\mathbf{t},\mathbf{x}) > \mathbf{0}$

Observables in Boussinesq systems

For each suitable test function f, consider the 'observable'

$$\mathbf{t} \longrightarrow \rho_{\mathbf{f}}(\mathbf{t}) = \int_{\mathbf{D}} \mathbf{f}(\mathbf{y}(\mathbf{t}, \mathbf{x}))\mathbf{dx}$$

where y is solution to one of the Boussinesq systems NSB,SB,HB.

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Then, we get

$$\frac{\mathbf{d}}{\mathbf{dt}} \int_{\mathbf{D}} \mathbf{f}(\mathbf{y}(\mathbf{t},\mathbf{x}))\mathbf{dx} = \int_{\mathbf{D}} (\nabla \mathbf{f})(\mathbf{y}(\mathbf{t},\mathbf{x})) \cdot \mathbf{G}(\mathbf{t},\mathbf{x},\mathbf{y}(\mathbf{t},\mathbf{x}))\mathbf{dx}$$

since $\partial_{\mathbf{t}}\mathbf{y} + (\mathbf{v}\cdot\nabla)\mathbf{y} = \mathbf{G}(\mathbf{t},\mathbf{x},\mathbf{y})$ where $\nabla\cdot\mathbf{v} = 0, \quad \mathbf{v}//\partial\mathbf{D}.$

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This directly follows from the MULTI-D REARRANGEMENT THEOREM

Global solutions to the HB system

With this formulation the HB system coincides with our multi-d toy model! Thus we conclude:

THEOREM

Assume G to be a smooth function with bounded first derivatives. Let C be the convex cone of all maps $y \in L^2(D, R^3)$ such that $y(x) = \nabla p(x)$ a.e. in D for some CONVEX function p. We say that $(t \to y(t, \cdot)) \in C^0(R_+, L^2(D, R^3))$ valued in the cone C is a solution to the HB system if

$$\frac{d}{dt}\int_D f(y(t,x))dx = \int_D (\nabla f)(y(t,x)) \cdot G(t,x,y(t,x))dx, \quad \forall f$$

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Then, for each $y_0 \in C$, there is always a GLOBAL solution such that $y(t = 0, \cdot) = y_0$

Rigorous derivation of the HB model

Theorem

Assume $(\mathrm{y}, \mathrm{p}, \mathrm{v})$ to be a smooth solution of the HB hydrostatic Boussinesq model, with $\mathrm{p}(\mathrm{t}, \mathrm{x})$ strongly convex in $\mathrm{x} \in \mathrm{D}$.
Then, as $\nu = \epsilon \rightarrow 0$, any Leray solution $(\mathrm{y}^\epsilon, \mathrm{p}^\epsilon, \mathrm{v}^\epsilon)$ to the full NSB Navier-Stokes Boussinesq equations, with same initial condition, converges to $(\mathrm{y}, \mathrm{p}, \mathrm{v})$.

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Idea of the proof: **Estimate:**

$$\frac{\mathbf{d}}{\mathbf{dt}} \int_{\mathbf{D}} \{\mathbf{K}(\mathbf{t}, \mathbf{y}^\epsilon(\mathbf{t}, \mathbf{x}), \mathbf{y}(\mathbf{t}, \mathbf{x})) + \frac{\epsilon}{\mathbf{2}} |\mathbf{v}^\epsilon - \mathbf{v}|^{\mathbf{2}}\} \mathbf{dx}$$

$$\mathbf{K}(\mathbf{t}, \mathbf{y}', \mathbf{y}) = \mathbf{p}^*(\mathbf{t}, \mathbf{y}') - \mathbf{p}^*(\mathbf{t}, \mathbf{y}) - \nabla \mathbf{p}^*(\mathbf{t}, \mathbf{y}) \cdot (\mathbf{y}' - \mathbf{y}) \sim |\mathbf{y} - \mathbf{y}'|^{\mathbf{2}}$$

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Idea of the proof: **Estimate:**

$$\frac{d}{dt}\int_D \{K(t, y^\epsilon(t,x), y(t,x)) + \frac{\epsilon}{2}|v^\epsilon - v|^2\}dx$$

$$K(t, y', y) = p^*(t, y') - p^*(t, y) - \nabla p^*(t, y) \cdot (y' - y) \sim |y - y'|^2$$

where $p^*(t, z) = \sup_{x \in D} x \cdot z - p(t, x)$ is the Legendre-Fenchel transform of p.

LONG LIVE GDR MOAD!