Gravity and Kelvin-Helmoltz instabilities in two fluids systems

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Ecole Normale Supérieure

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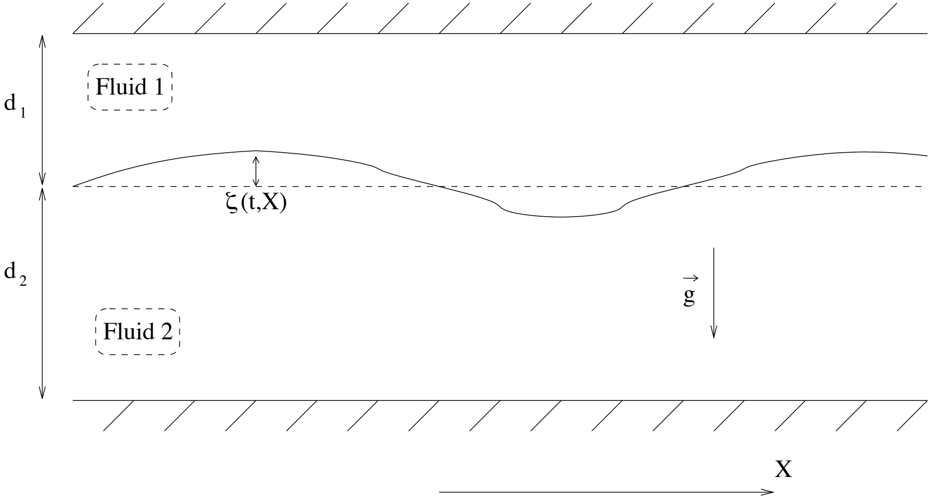
Notations



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Chandrasekhar condition: instability if $[\![V^\pm]\!]^2 > 4\frac{\langle\rho^\pm\rangle}{\underline{\rho}^+\underline{\rho}^-}(g'\sigma)^{1/2}.$

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Interface equations

Interface ($\psi^+ = \Phi^+_{|interface}$, $\psi^- = \Phi^-_{|interface}$)

- $\partial_t \zeta - \sqrt{1+|\nabla\zeta|^2}\partial_n \Phi^\pm{}_{|interface} = 0$
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Natural unknwowns

- Surface elevation ζ
- $\psi = \underline{\rho}^+\psi^+ - \underline{\rho}^-\psi^-$ (with $\underline{\rho}^\pm = \frac{\rho^\pm}{\rho^+ + \rho^-}$).

Quantities to express in terms of ζ and ψ

- Trace of the velocity potentials at the interface $\psi^\pm$
- Normal derivative of the velocity potentials $\partial_n\phi^+_{|_{interface}} = \partial_n\phi^-_{|_{interface}}$

The Dirichlet-Neumann operator

Definition (Dirichlet-Neumann operator)

Let $\dot{H}^s = \{f \in L^2_{loc}(\mathbb{R}^d), \nabla f \in H^{s-1}\}$.

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Proposition

The transmission problem (T) is well posed for $\psi \in \dot{H}^{s+1/2}$, $\zeta \in H^{s+1/2}$.

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- Chandresekhar condition: instability if $[\![V^{\pm}]\!]^2 > 4\frac{\langle\underline{\rho}^{\pm}\rangle}{\underline{\rho}^+\underline{\rho}^-}(g'\sigma)^{1/2}$.

Stabilizing factors: $\underline{\rho}^- \ll 1$ and $[\![V^{\pm}]\!] \ll 1$

Linear analysis (with $\sigma = 0$)

Linearized equation around the rest state

$$\begin{cases} \partial_t \zeta - G[0]\psi = 0 \\ \partial_t \psi + g'\zeta = 0 \end{cases}$$

with $g' = (\underline{\rho}^+ - \underline{\rho}^-)g$ and $G[0] = |D| \frac{\tanh(H^+|D|)\tanh(H^-|D|)}{\underline{\rho}^+ \tanh(H^-|D|) + \underline{\rho}^- \tanh(H^+|D|)}$.

Shallow water limit

The depths $H^\pm$ are small compared to the "typical wavelength" λ. Then

$$G[0] \sim -\underline{H}\Delta, \qquad \text{with} \qquad \underline{H} = \frac{H^+H^-}{\underline{\rho}^+ H^- + \underline{\rho}^- H^+}.$$

Wave equation: $\partial_t^2 \zeta - c^2 \Delta \zeta = 0, \qquad c^2 = g'\underline{H}.$

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- Shallowness parameter $\mu = \frac{\underline{H}^2}{\lambda^2}$
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III. Nondimensionalization (using linear analysis)

- $\widetilde{\zeta} = \frac{\zeta}{a}$, $\psi = \frac{\psi}{\psi_0}$
- $\widetilde{X} = \frac{X}{\lambda}$, $\widetilde{t} = \frac{t}{\lambda/c}$.

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The Bond number $Bo = \frac{\langle\underline{\rho}^\pm\rangle g'\lambda^2}{\sigma}$

1. Coastal flows (Water-Air interface) $Bo \sim \frac{10^0.(10.10^0)10^2}{10^{-2}} = 10^5$.
2. Internal waves ($\sim$ Water-Brine interface) $Bo \sim \frac{10^0(10.10^{-3})10^4}{10^{-2}} = 10^4$.

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2. Internal waves ($\sim$ Water-Brine interface) $Bo \sim \frac{10^0(10.10^{-3})10^4}{10^{-2}} = 10^4$.

No role of surface tension for propagation of (internal) waves

Benjamin67,....,BonaLannesSaut08,Duchene09

Condition for the existence of "stable" solutions

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G_\mu[\varepsilon\zeta]\psi = 0, \\ \partial_t \psi + \zeta + \varepsilon \frac{1}{2} [\![\underline{\rho}^\pm |\nabla\psi^\pm|^2]\!] \\ \qquad -\varepsilon\mu\frac{1}{2}(1+\varepsilon^2\mu|\nabla\zeta|^2)) [\![\underline{\rho}^\pm (w_\mu^\pm[\varepsilon\zeta]\psi^\pm)^2]\!] = -\frac{2}{Bo}\frac{1}{\varepsilon\sqrt{\mu}} \kappa(\varepsilon\sqrt{\mu}\zeta). \end{cases}$$

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- Do this uniformly with respect to the relevant asymptotics (ε, μ...)

The uniformity constraint and the operator $G_\mu[\varepsilon\zeta]$

Construction

$G_\mu[\varepsilon\zeta]$ is constructed in such a way that

$$\begin{aligned} G_\mu[\varepsilon\zeta]\psi &= G_\mu^-[\varepsilon\zeta, H^-]\psi^- = G_\mu^+[\varepsilon\zeta, H^+]\psi^+ \qquad (\psi = \underline{\rho}^+\psi^+ - \underline{\rho}^-\psi^-) \\ &= G_\mu^+[\varepsilon\zeta, H^+]J_\mu[\varepsilon\zeta]^{-1}\psi \end{aligned}$$

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$$\begin{cases} \partial_z^2\Phi^+ + \mu\Delta\Phi^+ = 0, & -H^+/\underline{H} < z < \varepsilon\zeta, \\ \Phi^+_{|_{z=\varepsilon\zeta}} = \psi^+, & \partial_z\Phi^+_{|_{z=-H^+/\underline{H}}} = 0 \end{cases}$$

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Proposition

Let $|f|_{\dot{H}_*^{s+1/2}} = |\frac{|D|}{(1+\sqrt{\mu}|D|)^{1/2}} f|_{H^s}$.

The operator $J_\mu[\varepsilon\zeta] : \dot{H}_*^{s+1/2} \to \dot{H}_*^{s+1/2}$ *is uniformly bijective.*

Linear analysis (with $\sigma > 0$)

Linearized equation around the rest state

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G_\mu[0]\psi = 0 \\ \partial_t \psi + \zeta - \frac{2}{Bo}\Delta\zeta = 0 \end{cases}$$

with $G_\mu[0] = \sqrt{\mu}|D| \frac{\tanh(H^+\sqrt{\mu}|D|)\tanh(H^-\sqrt{\mu}|D|)}{\underline{\rho}^+ H^+ \tanh(H^-\sqrt{\mu}|D|) + \underline{\rho}^- H^- \tanh(H^+\sqrt{\mu}|D|)}$.

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Symmetrizer

$$S[0] = \begin{pmatrix} 1-\frac{2}{Bo}\Delta & 0 \\ 0 & G_\mu[0] \end{pmatrix}$$

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Energy

$$\begin{array}{rcl} E_{lin}(U) & = & (U, S[0]U) \\ & \sim & |\zeta|^2_{H^1_\sigma} + |\psi|^2_{\dot{H}^{1/2}_*}. \end{array}$$

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$$|\zeta|^2_{H^1_\sigma} = |\zeta|_2^2 + \frac{2}{Bo}|\nabla\zeta|_2^2.$$

The first equation and the “good unknowns”

$$\partial^\alpha(\partial_t\zeta - \frac{1}{\mu}G_\mu[\varepsilon\zeta]\psi) = 0 \rightsquigarrow ???$$

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The "..." must be controled in H^1_σ norm by the energy

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2. There is an explicit formula for the shape derivative

The first equation and the “good unknowns”

$$dG_\mu[\varepsilon\zeta](h)\psi = \underbrace{-\varepsilon G_\mu[\varepsilon\zeta](h[\![\underline{\rho}^{\pm} w_\mu^{\pm}]\!]) - \nabla\cdot(hV^+) + \underline{\rho}^- G_\mu[\varepsilon\zeta]\circ(G^-)^{-1}(\nabla\cdot(h[\![V^{\pm}]\!]))}_{:=\mathcal{T}h}$$

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2. There is an explicit formula for the shape derivative
3. The energy

$$E^N(U) = |\nabla\psi|_{H^{t_0+2}} + \sum_{|\alpha|\leq N} |\partial^\alpha\zeta|^2_{H^1_\sigma} + |\psi_{(\alpha)}|^2_{\dot{H}^{1/2}_*}.$$

The second equation and the instabilities

$$\partial^\alpha(\partial_t\psi + \zeta + \varepsilon\frac{1}{2}[\![\underline{\rho}^\pm|\nabla\psi^\pm|^2]\!] + \dots) = ???$$

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$$\partial_t\psi_{(\alpha)} + \mathfrak{a}\partial^\alpha\zeta + [\![\underline{\rho}^\pm V^\pm \cdot \nabla\psi^\pm_{(\alpha)}]\!] \sim -\frac{2}{Bo}\frac{1}{\varepsilon\sqrt{\mu}}\partial^\alpha \kappa(\varepsilon\sqrt{\mu}\zeta).$$

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3. $[\![\underline{\rho}^\pm V^\pm \cdot \nabla\psi^\pm_{(\alpha)}]\!]$ as a function of ζ and $\psi_{(\alpha)}$?

$$\begin{aligned}[\![\underline{\rho}^\pm V^\pm \cdot \nabla\psi^\pm_{(\alpha)}]\!] &= \langle V^\pm\rangle\cdot\nabla[\![\underline{\rho}^\pm\psi^\pm_{(\alpha)}]\!] + [\![V^\pm]\!]\cdot\nabla\langle\underline{\rho}^\pm\psi^\pm_{(\alpha)}\rangle \\ &= \langle V^\pm\rangle\cdot\psi_{(\alpha)} + \text{☠☠☠}\end{aligned}$$

The Rayleigh-Taylor instability operator

The second equation

$$\partial_t \psi_{(\alpha)} + \mathcal{T}^* \psi_{(\alpha)} + \mathcal{RT}\partial^\alpha \zeta + \frac{2}{Bo} K_{N+1}(\zeta) \ \sim 0 \ ,$$

with $\quad \mathcal{RT} f = \mathfrak{a} f + \underline{\rho}^+ \underline{\rho}^- \varepsilon^2 \mu [\![V^\pm]\!] \cdot E_\mu[\zeta](f [\![V^\pm]\!]) - \dfrac{2}{Bo} \nabla \cdot K \nabla f.$

The Rayleigh-Taylor instability operator

Controled by the energy, uniformly in $\dot{H}_*^{1/2}$

The second equation

$$\partial_t \psi_{(\alpha)} + \mathcal{T}^* \psi_{(\alpha)} + \mathcal{R}\mathcal{T} \partial^\alpha \zeta + \frac{2}{Bo} K_{N+1}(\zeta) \sim 0 ,$$

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The operator $E_\mu[\zeta]$

$$E_\mu[\zeta] = \nabla \circ (G^-)^{-1} G (G^+)^{-1} \circ \nabla^T.$$

- Symbolic analysis yields $E_\mu[\zeta] \sim -|D| \quad (d = 1)$.
- $E_\mu[\zeta]$ costs $\sqrt{\mu}$ at high frequencies, μ at low frequencies.

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The coefficient $\mathfrak{a}$

One can check that "$\mathfrak{a} = [\![\partial_z P]\!]$".

The quasilinear system

The system in $\partial^\alpha\zeta$ and $\psi_{(\alpha)} = \partial^\alpha\psi - \varepsilon w \partial^\alpha\zeta$

$$\begin{cases} \partial_t\partial^\alpha\zeta + \varepsilon\mathcal{T}\partial^\alpha\zeta - \frac{1}{\mu}G_\mu[\varepsilon\zeta]\psi_{(\alpha)} - G_N[\zeta]\psi \sim 0, \\ \partial_t\psi_{(\alpha)} + \mathcal{T}^*\psi_{(\alpha)} + \mathcal{R}\mathcal{T}\partial^\alpha\zeta + \frac{2}{Bo}K_{N+1}(\zeta) \sim 0 \end{cases}$$

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Symmetrization

- $\mathcal{R}\mathcal{T}$ is a second order operator $\rightsquigarrow$ problem with subprincipal terms in the commutator with $G_\mu[\varepsilon\zeta]$.

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Symmetrization

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  1. Clever commutator estimate (symbolic analysis): MeiZhang08
  2. Use the DN and curvature operators to differentiate: ShatahZeng08
  3. Use paradifferential calculus: AlazardBurqZuily09
  4. Put the time derivatives in the energy: RoussetTzvetkov09

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- Symmetrizer

$$S[U] = \begin{pmatrix} \mathcal{R}\mathcal{T} & 0 \\ 0 & \frac{1}{\mu} G_\mu[\varepsilon\zeta] \end{pmatrix}$$

A key result

Proposition

1. *One has* $(\frac{1}{\mu}G_\mu[\varepsilon\zeta]\psi,\psi) \sim |\psi|^2_{\dot H^{1/2}_*}$.
2. *If the following condition is satisfied*

$$\text{(Stab)}\qquad \varepsilon^2\mu|[\![V^\pm]\!]|^2_\infty < \frac{1}{\underline{\rho}^+\underline{\rho}^-}\frac{1}{\|E_\mu[\zeta]\|_{\dot H^1\to L^2}}\Big(\frac{8}{Bo}\frac{[\![\partial_z P]\!]}{(1+\varepsilon^2|\nabla\zeta|^2)^{3/2}}\Big)^{1/2}$$

then $(\mathcal{RT}\zeta,\zeta)\sim|\zeta|^2_{H^1_\sigma}$.

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then $(\mathcal{RT}\zeta,\zeta) \sim |\zeta|^2_{H^1_\sigma}$.

Proof.

Use gravity to control low frequencies and surface tension to control high frequencies... □

Main result

$$\text{(Stab)} \quad \varepsilon^2\mu|[\![V^\pm]\!]|_\infty^2 < \frac{1}{\underline{\rho}^+\underline{\rho}^-}\frac{1}{\|E_\mu[\zeta]\|_{\dot{H}^1\to L^2}}\Big(\frac{8}{Bo}\frac{[\![\partial_z P]\!]}{(1+\varepsilon^2|\nabla\zeta|^2)^{3/2}}\Big)^{1/2}$$

Theorem

Under (Stab), the interfacial waves equations are well posed in $(\zeta,\psi)\in H^N\times\dot{H}^{N+1/2}$ ($N>d+5$) on a time that depend on σ through (Stab) only (and uniformly with respect to ε, μ)

Applications

- Coastal flows: $\varepsilon^2\sqrt{\mu}\lesssim 10^{-2}$.
- Internal wave: $\varepsilon^2\sqrt{\mu}\lesssim 10^{-2}$ or 10^{-3}.