Divergent series and holomorphic dynamical systems

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Foreword

The first seminar I heard, a couple of years ago, about divergent series was done by Professor J.P. Ramis. He tried to show us the main difficulties which arised in proving the equivalence between the different definitions of the multisummability. At this time, the meaning of summability was not clear to me, neither were its apllications to differential equations.

Now, his theory of summability is achieved and provides fondemental results on differential equations, among others. He presents it, now, in such a way, that all its beauty and its power impose themselves to us. The self-contained text which follows is an attempt to show it.

I would like to thanks Professor J.P. Ramis for his opening mind for discussion, for his advises and for explaining the real meaning of Mathematics without drowning me under a flow of technical details.

I would also like to thanks Professor B. Malgrange for having taught me the main part of my knowledges on summation of divergent series, during his annual lecture[Mal92] at the Université Joseph Fourier in 1991-1992. He pointed out the difficulties and the means to pass over them. Moreover, I profited of his typed notes of his lectures for the present writing.

Laurent Stolovitch
The writer.

Contents

Chapter 1

Introduction

We shall talk about sommation of divergent series related to asymptotic expansions. There are close relations with the lectures of Il'yashenko, Ecalle and Moussu. The theory of summation we shall show, is strongly related to the work of Ecalle. We shall be interested, mainly, in formal power series $\hat{f}(x) = \sum_{n\geq 1} a_n x^n$ with complex coefficients. The set of these series is denoted by $C[[x]]$ and is a differential algebra. This set is "huge"; thus, we shall restrict ourselves to the series which have a practical intersest, that is that they are

1. issued from geometry (dynamical systems)
2. effectively computable from different point of view:
   (a) Computer Algebra that is formal computations,
   (b) Numerical which leads to graphical representations.

Moreover, we may assume that all the coefficients of a series are generated by an algorithm. For instance, an inital condition and an induction formula give rise to an algorithm. The key point, in the theory we have developped, is that different algorithms can generate a same formal power series (as it was already remarked by L. Euler). Let us illustrate this phenomenon:
Examples:

1. $a_0 = 1$ and, $\forall n \in N^*$, $a_n = (-1)^n$.Thus , we have $\hat{f}(x) = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$. The algorithm which gives the coefficients a_n could be replaced by the one which gives $\frac{1}{1+x}$.
2. Euler's geometrical series
   This series is defined by following datas
   (a) Initial conditions: $a_0 = 0$, $a_1 = 1$,
   (b) Induction formula: $a_n = -(n-1)a_{n-1}$

   Thus, we have defined the key example in the sequel, the Euler's series

$$\hat{f}(x) = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1}$$

Nevertheless, the previous algorithm, which builds up this series, can be replaced by the one which build up the solution of the problem

$$x^2y' + y = x$$

with the initial condition $y(0) = 1$.

There are many examples of that kind in which one has to look at the formal solutions of functional equations (linear or not):

1. Differential equations
2. Differences equations (relation between f and its different translations by $x \mapsto x+1$)
3. q-differences equations (relation between f and foh where $h : x \mapsto qx$ with $q \in C^*$)

We can mix up these different kinds of equations. Nevertheless, we shall only consider the analytical case.
Example of q-differences equation:
We give the q-differences equation similar to the Euler's equation. The way to procced is to replace the operator $x\frac{d}{dx}$ by the operator G_q, where $G_q(f(x)) = f(qx)$. We obtain

$$xf(xq) + f(x) = x$$

which solution is the q-analog of the Euler's series, that is

$$\sum_{n\geq 0} (-1)^n q^{\frac{n(n+1)}{2}} x^{n+1}$$

We shall be intersested, mainly, in divergent formal power series which look like the Euler's series in a sense that we shall define in the sequel. Our aim is to provide these series with a numerical value.

Chapter 2

k-summability

2.1 Asymptotic expansions: the classical theory

The classical theory of asymptotic expansion is due to Poincaré. He has developped it with the idea to apply it to analytical ordinary differential equations: his main motivation was to give sense to a divergent formal power series solution of such a differential equation, that is considering a formal solution as a real solution in some sense.

Definition 2.1.1 *[Was65] Let V be an open sectorial domain of the complex plane (or more generally, of the Riemann surface of the Logarithm) at 0. Let f be a holomorphic function on V and let $\hat{f} = \sum_{p=0}^{\infty} a_p x^p$ be a formal power series. The series $\hat{f}$ will be said to be the asymptotic expansion of f on V if, for all compact subsector W of $V \cup \{0\}$ and for all integer n, there exists a positive constant $M_{W,n}$ such that*

$$|x|^{-n}|f(x) - \sum_{p=0}^{n-1} a_p x^p| < M_{W,n}$$

for all $x \in W$.

The set of holomorphic functions on V having an asymptotic expansion at the origin (with the operations $+$, $.$, $x^2\frac{d}{dx}$) is a C-differential algebra denoted by $\mathcal{A}(V)$. We shall denote the Taylor jet application by $T : \mathcal{A}(V) \to C[[x]]$. We shall denote by $\mathcal{A}^{<0}(V)$ the kernel of the application T.
The main question that we are led to ask is whether there exists a holomorphic function on a sector admitting a given formal power series as asympotic expansion et the origin. The answer is affirmative and is given by the

Theorem 2.1.1 (Borel-Ritt) *[Was65] Let V be a sectorial domain at 0, which opening angle is less than 2π. Then, the Taylor jet application $T : \mathcal{A}(V) \to C[[x]]$ is onto.*

This theorem means that we can always regard a formal power series as a function. Nevertheless, the latest is not unique, it is well defined modulo the space of infinitely flat functions at the origin $\mathcal{A}^{<0}(V)$. This lack of unicity is the main problem in Poincaré's theory.

Theorem 2.1.2 (Fondamental theorem about asymptotic expansions) *Let $G(x, Y, \ldots, Y_n)$ be an analytic function of $n+2$ variables and let $\hat{f}$ be a formal series solution of the differential equation*

$$G(x, y, \ldots, y^{(n)}) = 0$$

Then, there exists a real number $k > 0$ such that, for all open sector V at the origin, in the complex plane, with an opening angle smaller than $\frac{\pi}{k}$ and of radius sufficenctly small, there exists a function f, solution of the differential equation, admitting $\hat{f}$ as an asymptotic expansion on V

We shall study this problem more closely in the sequel.

2.2 Gevrey asymptotic expansions

Let us see why we are led, in a natural way, to introduce the notion of Gevrey asymptotic expansion. First of all, we notice that almost all the formal power series $\sum_{n \geq 0} a_n z_n$ that we have to deal with, satisfy a growth's condition of the kind

$$|a_n| < C(n!)^{\frac{1}{k}} A^n$$

where $A, C, k > 0$ are suitable real numbers. This fact can be checked, for instance, on the "special functions"(cf. any good book of formulae).
On the other hand we notice that, in practice, the uncertainty about a function to be determined from a known asymptotic expansion, is not only an infinitely flat holomorphic function but more precisely, a holomorphic exponential decreasing function of a certain order $k > 0$:

$$|f(x)| < C \exp(-\frac{a}{|z|^k})$$

for some suitable constants $C, a > 0$.

Let us consider an open sectorial domain V of the complex plane C or of the Riemann surface of the logarithm. We shall also consider a covering $\{V_i\}_{i \in I}$ of V. Some assumptions are made on the coverings we use:

1. The covering is finite(i.e I is a finite discret set)
2. each V_i has the same radius as V
3. the intersection of any three elements of the covering is empty.

A covering which fulfilled the above assumptions will be called a **good covering**.
We may assume that $I = [1, m]$, and we shall order the covering clockwise. We shall also write $V_{i,i+1} = V_i \cap V_{i+1}$.
Before defining the notions of Gevrey asymptotic expansion and related topics, we shall first state the main and fondemental theorem which will make the further definitions natural.

Theorem 2.2.1 (Ramis-Sibuya) *[Sib91] Let $\{S_1, \ldots, S_p\}$ be a good covering of the punctured disk $D_r^* = \{z : \; 0 < |z| < r\}$. Let $\Phi_1, \ldots, \Phi_p$ be functions satisfying to the following conditions:*

1. Φ_l is holomorphic and bounded on S_l.

2. $|\Phi_l(z) - \Phi_{l+1}(z)| \leq a \exp(\frac{-b}{|z|^k})$ on $S_l \cap S_{l+1}$ with $a \geq 0$, $b > 0$ and $k > 0$ independant of l.

Then there exists a formal power series $\hat{\Phi} = \sum_{n=0}^{\infty} a_n z^n$ such that

1. $\forall n \geq 0$, $|a_n| \leq C(n!)^{\frac{1}{k}} A^n$

2. for each l, for each subsector W of S_l, there exists constants $C_W, A_W > 0$ such that

$$\forall z \in W, \quad |\Phi_l(z) - \sum_{n=0}^{N-1} a_n z^n| < C_W (N!)^{\frac{1}{k}} A_W^N |z|^N$$

This theorem leads us to state the following definitions:

Definition 2.2.1 *Let $\hat{f} \in C[[z]]$ be a formal power series $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$.*
*$\hat{f}$ is said to be **Gevrey of order** $s = \frac{1}{k}$ if*

$$\forall n \geq 0, \quad |a_n| < C(n!)^{\frac{1}{k}} A^n$$

for some positive constants A and C.

Definition 2.2.2 *Let V be a sectorial domain and let $f \in \mathcal{A}(V)$. Let $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$ be a Poincaré asymptotic expansion of f.*
*We shall say that $\hat{f}$ is a **Gevrey asymptotic expansion of order** $\frac{1}{k}$ if for any subsector W of V, there exist positive constants A_W, C_W such that*

$$\forall z \in W, \quad |f(z) - \sum_{n=0}^{N-1} a_n z^n| < C_W (N!)^{\frac{1}{k}} A_W^N |z|^N$$

We sall denote the set of holomorphic functions on V admitting a Gevrey asymptotic expansion of order $\frac{1}{k}$ by $\mathcal{A}_{\frac{1}{k}}(V)$.

The theorem 2.2.1 is a corollary of the following lemma.

Lemma 2.2.1 *Let $\{S_1, \ldots, S_p\}$ be a good covering of the punctured disk $D_r^* = \{z : \ 0 < |z| < r\}$. Let $\delta_1, \ldots, \delta_p$ be functions satisfying to the following conditions:*

1. δ_l is holomorphic on $S_l \cap S_{l+1}$.

2. $|\delta_l(z)| \leq a \exp(-\frac{b}{|z|^k})$ on $S_l \cap S_{l+1}$ with $a \geq 0$, $b > 0$ and $k > 0$ independant of l.

Then there exists a formal power series $\hat{\Phi} = \sum_{n=0}^{\infty} a_n z^n$ and functions $\Psi_1, \ldots, \Psi_p$ such that

1. Ψ_l is holomorphic and bounded on S_l,

2. $\delta_l(z) = \Psi_l - \Psi_{l+1}$ on $S_l \cap S_{l+1}$,

3. $\forall n \geq 0$, $|a_n| \leq C(n!)^{\frac{1}{k}} A^n$, *4. for each l, for each subsector W of S_l, there exists constants $C_W, A_W > 0$ such that*

$$\forall z \in W, \quad |\Psi_l(z) - \sum_{n=0}^{N-1} a_n z^n| < C_W (N!)^{\frac{1}{k}} A_W^N |z|^N$$

Proof of theorem 2.2.1 by means of the lemma: Since we are given the functions $\Phi_1, \ldots, \Phi_p$, let us set

$$\delta_l(z) = \Phi_l(z) - \Phi_{l+1}(z)$$

By the hypothesis of theorem 2.2.1 on the Φ's, the hypothesis of the lemma are fulfilled by the δ's. Thus, there exist functions $\Psi_1, \ldots, \Psi_p$ which satisfy, in particular,

$$\Phi_l(z) - \Phi_{l+1}(z) = \Psi_l(z) - \Psi_{l+1}(z)$$

on $S_l \cap S_{l+1}$. This means that, for each l,

$$\Phi_l(z) - \Psi_l(z) = \Phi_{l+1}(z) - \Psi_{l+1}(z)$$

We can then define the function $\Phi(z) = \Phi_l(z) - \Psi_l(z)$ on S_l. By the lemma, Φ is holomorphic and bounded on $\{z : \ 0 < |z| < r\}$. Therefore, by the Riemann's vanishing singularities theorem, Φ can be represented by a convergent power series in a neighbourhood of the origin. Since $\Phi_l(z) = \Psi_l(z) + \Phi(z)$, theorem 2.2.1 follows immediately. $\square$

Proof of lemma 2.2.1:

1. By linearity, we may assume that all but one of the δ_l's are zero. In this case, the covering of the punctured disk contains only one sectorial domain which "intersects itself". In fact, if we denote $\Sigma(\alpha, \beta, r)$ the sectorial domain on which $\delta \equiv \delta_1$ is defined, then we have to find a holomorphic function Ψ defined on a sectorial domain $\Sigma(\alpha', \beta', r')$(with $\alpha' = \alpha$, $\beta' = \beta + 2\pi$ and $0 < r' < r$) admitting an asymptotic expansion at the origin and such that the difference between its two determinations is equal to δ, i.e

$$\forall z \in \Sigma(\alpha, \beta, r), \quad \Psi(ze^{2i\pi}) - \Psi(z) = \delta(z)$$

   Let θ be a direction such that $\alpha < \theta < \beta$ and let γ be the oriented path $\{se^{i\theta},\ 0 \leq s \leq r'\}$. Then, we set

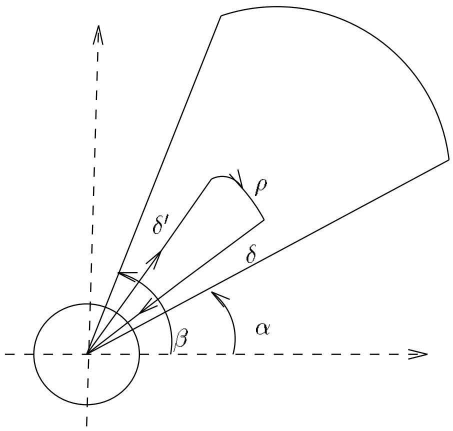
$$\Psi_\theta(x) = \frac{1}{2i\pi} \int_\gamma \frac{\delta(y)}{x - y} dy$$

   This function is defined and holomorphic on $C \setminus \gamma$. Let us consider a direction θ' close to θ. Thus, if $x \in \Sigma(\alpha', \beta', r')$ with $\arg(x) \notin [\theta', \theta]$, then the function $y \mapsto \frac{g(y)}{x-y}$ is holomorphic in the interior of the domain delimited by $\gamma + \rho - \gamma'$; where γ': $\{se^{i\gamma'},\ 0 \leq s \leq r'\}$ and ρ: $\{r'e^{i\phi},\ \theta \leq \phi \leq \theta'\}$. Therefore, we have

$$\Psi_\theta(x) - \Psi_{\theta'}(x) = \frac{-1}{2i\pi} \int_\rho \frac{\delta(y)}{x - y} dy$$

   But, the integral, in the above equality, is an analytic function on the open disk $\{x \in C |\ |x| < r'\}$. Consequently, Ψ_θ is a function Ψ on $\Sigma(\alpha', \beta', r')$. So, if $x \in S \subset \Sigma(\alpha, \beta, r)$, then we have

$$\Psi(xe^{2i\pi}) - \Psi(x) = \frac{1}{2i\pi} \int_{\partial S} \frac{\delta(y)}{x - y} dy$$



2. We have now to show that the function Ψ admits a Gevrey asymptotic expansion of order $\frac{1}{k}$ at the origin.
   Let $\Sigma"(\alpha+\epsilon, \beta-\epsilon, r')$ be the subsector of Σ', with $\epsilon > 0$. Let $\theta \in]\alpha+\epsilon, \beta+2\pi-\epsilon[$, and let γ be the oriented path $\{te^{i\theta},\ 0 \leq t \leq r'\}$.

   Since we can write
$$\frac{1}{x-y} \;=\; -\frac{1}{y}[\sum_{p=0}^{n-1}(\frac{x}{y})^p \;+\; (\frac{x}{y})^n\frac{1}{1-\frac{x}{y}}]$$
   we have
$$\Psi(x) \;=\; -\frac{1}{2i\pi}(\int_\gamma \delta(y)(\sum_{p=0}^{n-1}\frac{x^p}{y^{p+1}})dy \;+\; \int_\gamma \frac{\delta(y)x^n}{y^{n+1}-xy^n}dy)$$
   Let us set $a_p = -\frac{1}{2i\pi}\int_\gamma \frac{\delta(y)}{y^{p+1}}$ which takes sense since δ has the nule asymptotic expansion. Thus, we have
$$|\Psi(x) - \sum_{p=0}^{n-1} a_p x^p| = \frac{1}{2\pi}|\int_\gamma \frac{\delta(y)}{y^{n+1}-xy^n}dy||x|^n$$
   But, since $|argx - \theta| > \epsilon$, then we have $|x-y| > |y|\sin\epsilon$ for all $y \in \gamma$. Consequently, we have
$$|\int_\gamma \frac{\delta(y)}{y^{n+1}-y^n x}dy| \leq \frac{1}{\sin\epsilon}\int_0^{r'}\frac{\delta(te^{i\theta})}{t^{n+1}}dt$$
   Thus, by changing the angle θ, the above integral is uniformely bounded, and then the function δ admits a Poincaré asympotic expansion at the origin.

3. Assuming that δ is an exponential decreasing function of order $\geq k$, we have to show that Ψ admits a Gevrey asymptotic expansion of order $\frac{1}{k}$.
   So, we assume that, in the sector we consider, there exist two constants $A, B > 0$, such that $\delta(x) \leq Ae^{-\frac{B}{|x|^k}}$.

We have

$$
\begin{aligned}
|\Psi(x) - \sum_{p=0}^{n-1} a_p x^p| &\leq \frac{A}{\sin\epsilon}\int_0^{r'} \frac{e^{-\frac{B}{t^k}}}{t^{n+1}}dt \\
&\leq \frac{A}{k\sin\epsilon}\int_0^{\infty} u^{\frac{n}{k}-1}e^{-Bu}du \\
&\leq \frac{A}{k\sin\epsilon B^{\frac{n}{k}}}.\Gamma(\frac{n}{k})
\end{aligned}
$$

since we have

$$\frac{1}{\tau^{\frac{p}{k}}} = \frac{1}{\Gamma(\frac{p}{k})}\int_0^{\infty} e^{-t\tau}t^{\frac{p}{k}-1}dt$$

□

2.3 Proprieties of the Taylor jet application

Let T be the Taylor jet application. We are interested in the proprieties of this application for Gevrey functions. First of all, we have

Proposition 2.3.1 *The Taylor jet application $T : \mathcal{A}_{\frac{1}{k}}(V) \to C[[x]]_{\frac{1}{k}}$ is well defined .*

The proprieties of T depends stongly on the opening angle of the sector V. We first state a Borel-Ritt theorem in the Gevrey case.

Theorem 2.3.1 (Borel-Ritt-Ramis) *Let V be a sectorial domain which opening angle is less than $\frac{\pi}{k}$. Then the Taylor jet application $T : \mathcal{A}_{\frac{1}{k}}(V) \to C[[x]]_{\frac{1}{k}}$ is onto.*

Proof:
Let $\hat{f} = \sum_{p\geq 0} a_p x^p$ be a Gevrey formal power series of order $\frac{1}{k}$. We may assume that the sector V is bissected by R^+, that is

$$V = \{x \in C|\ \ |argx| \leq \frac{\pi}{2k},\ 0 < |x| < R\}$$

Moreover, since the theorem is trivially true for constants, we may assume that $a_i = 0$ for $0 \leq i \leq [k]$.
Let us define the formal Borel transform of order k of $\hat{f}$ as beeing

$$\phi : t \mapsto \sum_{p\geq[k]+1} \frac{a_p t^{p-k}}{\Gamma(\frac{p}{k})}$$

Since $\hat{f}$ is a Gevrey formal power series of order $\frac{1}{k}$, ϕ has a non-zero radius of convergence R (which can be infinite). Even if it means making a change of variable, we may assume R to be greater than 1. Let us consider the incomplete "ramified" Laplace transform of ϕ

$$g(x) = k\int_0^1 \phi(t)e^{-\frac{t}{x^k}}t^{k-1}dt$$

We shall compare the function g to the series $\sum_{p\geq 0} a_p x^p$ in the domain $V_\delta = \{x \in C| \ |argx| \leq \frac{\pi}{2k} - \frac{\delta}{k},\ 0 < |x| < 1\}$ for some $\delta > 0$. We first notice that

$$x^p = \frac{1}{\Gamma(\frac{p}{k})} \int_0^\infty e^{-\frac{t}{x^k}} t^{\frac{p}{k}-1} dt$$

Thus, we have for $n - 1 \geq [k] + 1$

$$\begin{aligned} g(x) - \sum_{p=[k]+1}^{n-1} a_p x^p &= k \int_0^1 \phi(t) e^{-\frac{t^k}{x^k}} t^{k-1} dt - \sum_{p=[k]+1}^{n-1} \frac{a_p}{\Gamma(\frac{p}{k})} \int_0^\infty e^{-\frac{t}{x^k}} t^{\frac{p}{k}-1} dt \\ &= \sum_{p\geq n} \frac{a_p}{\Gamma(\frac{p}{k})} \int_0^1 e^{-\frac{u}{x^k}} u^{\frac{p}{k}-1} du - \sum_{p=[k]+1}^{n-1} \frac{a_p}{\Gamma(\frac{p}{k})} \int_1^\infty e^{-\frac{u}{x^k}} u^{\frac{p}{k}-1} du \end{aligned}$$

But, for each $x \in V_\delta$, we have $|e^{-\frac{u}{x^k}}| \leq e^{-\frac{u}{|x|^k}\sin\delta}$. Moreover, in each sum, in the above expression, t^p is bounded by t^n. Consequently, we have

$$\begin{aligned} |g(x) - \sum_{p=[k]+1}^{n-1} a_p x^p| &\leq (\sum_{p\geq 1} \frac{|a_p|}{\Gamma(\frac{p}{k})}) \int_0^\infty e^{-\frac{u}{|x|^k}\sin\delta} u^{\frac{n}{k}-1} du \\ &\leq (\sum_{p\geq 1} \frac{a_p}{\Gamma(\frac{p}{k})}) \frac{\Gamma(\frac{n}{k})}{(\sin\delta)^{\frac{n}{k}}} |x|^n \end{aligned}$$

So, the function $x \mapsto g(x)$ belongs to $\mathcal{A}_{\frac{1}{k}}(V)$ since, for every subsector $W \subset V$, there exists a suitable $\delta > 0$ such that $W \subset V_\delta$ and it admits $\sum_{p\geq 1} a_p x^p$ as Gevrey asymptotic expansion of order $\frac{1}{k}$. $\square$

Let $\mathcal{A}^{-k}(V)$ be the set of holomorphic functions on V which are exponentially decreasing of order $\geq k$ at the origin. If f is an element of that set, then, for every subsector $W \subset V$, there exist constants $M_W > 0$ and $C_W > 0$ such that

$$\forall x \in W,\ \ |f(x)| < C_W e^{-\frac{M_W}{|z|^k}}$$

We have the following caracterisation of this set by the

Proposition 2.3.2 *The kernel of the Taylor jet application* $T : \mathcal{A}_{\frac{1}{k}}(V) \to C[[x]]_{\frac{1}{k}}$ *is* $\mathcal{A}^{-k}(V)$.

Theorem 2.3.2 (Watson's lemma) *Let V be a sectorial domain which opening angle is greater than $\frac{\pi}{k}$. Then the kernel of the Taylor jet application $T : \mathcal{A}_{\frac{1}{k}}(V) \to C[[x]]_{\frac{1}{k}}$ is reduced to $\{0\}$.*

Proof:

Let f be an analytic exponential decreasing function of order $\geq k$ in a sectorial domain which opening angle is greater than $\frac{\pi}{k}$. We may assume that this function is defined on the sector S defined by $S = \{z \in C| \ 0 < |z| < R,\ |argz| \leq \frac{\pi}{2k'}\}$ with $k' < k$. Let l be a number such that $k' < l < k$, $l < 3k'$ and λ be a positive number. We have

$$\lim_{|z|\to 0, z\in S} |f(z) \exp(\frac{\lambda}{|z|^l})| = 0$$

Moreover, when $|argz| = \frac{\pi}{2k'}$, we have $|\exp(\frac{\lambda}{|z|^l})| \leq 1$. Thus, by the maximum principle, we have, for all $z \in S$, $|f(z)\exp(\frac{\lambda}{|z|^l})| \leq \sup_{z\in S}|f(z)|$. This fact implies that $f = 0$ since we can have λ as large as desired.□

2.4 Definition of the k-summability

The work we have done before leads to the following notion of summability:

Definition 2.4.1 *Let $k > 0$ and d a direction. A formal power series $\hat{f}$ will be said to be k-summable in the direction d, if there exists a holomorphic function f on a sector V, bissected by d and which opening angle is greater the $\frac{\pi}{k}$, which Geverey asymptotic expansion of order $\frac{1}{k}$ on V is $\hat{f}$.*

In those conditions, the function f is **unique** (up to the definition of the sector V), according to the Watson's lemma. This function will be said to be the k-sum of $\hat{f}$ in the direction d. It will be denoted by $f = S_{\frac{1}{k};d}\hat{f}$, and we shall denote $C\{x\}_{\frac{1}{k};d}$ the set of all formal power series which are k-sumable in the direction d. One can easely see that $C\{x\}_{\frac{1}{k};d}$ is a differential algebra of $C[[x]]_{\frac{1}{k}}$ and that the application

$$S_{k;d}: \ C\{x\}_{\frac{1}{k};d} \rightarrow \mathcal{A}_d.$$

is an injective differential algebra's homomorphism.(We have written A_d for the algebra of germs of holomorphic functions defined on open sectors, bissected by d, with arbitrary opening angle and radius.)

2.5 Laplace-Borel transforms and k-summability

This definition of k-summability is nice and useful, particulary when one has to check the k-summability of a formal solution of an analytical functional equation. Nevertheless, it does not provide any practical mean to compute the sum. For that purpose, we shall use the Borel and Laplace transforms[MR89].

2.5.1 Borel-Laplace isomorphism

Borel transform

Let f be an analytic function in a sector V at the origin, bissected by a direction d, and which opening angle Θ is greater than π.

$$V(d,\Theta) = \{x \in C \,||arg\, x - d| < \frac{\Theta}{2},\ 0 < |x| < R\}$$

We consider the following integral, for $\xi \in d$:

$$\mathcal{B}f(\xi) = \frac{1}{2i\pi}\int_{\gamma(r,\theta,\theta')} f(x)e^{\frac{\xi}{x}}\frac{dx}{x^2}$$

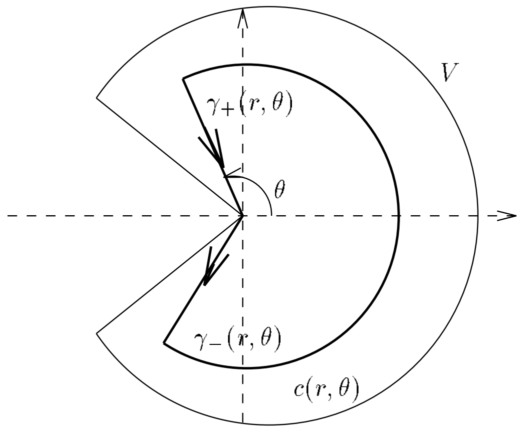


Figure 2.1: Path of integration in the Borel transform

where $\gamma(r,\theta,\theta')$ is the path of V defined by (see figure):

$$\begin{aligned}
\gamma_- &: \{te^{i(d-\theta')},\ 0 \leq t \leq r\} \\
c &: \{re^{i\phi},\ -\theta' \leq \phi - d \leq \theta\} \\
\gamma_+ &: \{te^{i(d+\theta)},\ r \geq t \geq 0\}
\end{aligned}$$

Here, θ and θ' are both chosen greater than $\frac{\pi}{2}$ and smaller than π in order to ensure that the function $\frac{e^{\frac{\xi}{x}}}{x^2}$ is rapidly decreasing when x approaches zero in the direction $d+\theta$ and $d-\theta'$ (i.e $Re(\frac{\xi}{x}) < 0$).
This integral exists, independently of γ, as soon as $|f|$ does not increase too much when $|x|$ approaches zero. It is a holomorphic function on d. Moreover, this integral exists as soon as $Re(\frac{\xi}{x}) < 0$ when x approaches zero in the direction $d+\theta$ and $d-\theta'$; thus, by moving θ and θ' from $\frac{\pi}{2}$ to Θ, it defines a holomorphic function on the "infinite" sector $\tilde{V}(d, \Theta - \pi) = \{\xi \in C || arg\,\xi - d| < \frac{\Theta-\pi}{2}\}$. This function is called the **Borel transform** of f. Let us consider the ramification application ρ_k definied by $\rho_k f(x) = f(x^{\frac{1}{k}})$. Let f be a holomorphic function in a sector, at the origin, which opening angle is greater than $\frac{\pi}{k}$. We define the **Borel transform $\mathcal{B}_k$ of order k** by

$$\mathcal{B}_k f(\xi) = \rho_{\frac{1}{k}} \circ \mathcal{B} \circ \rho_k f(\xi)$$

Example: Let $\lambda > 0$ and $f(x) = x^\lambda$. Then, in any direction d, we have

$$\mathcal{B} f(\xi) = \frac{\xi^{\lambda-1}}{\Gamma(\lambda)}$$

This is obtained by the Hankel's expression of $\Gamma(\lambda)$:

$$\frac{1}{\Gamma(\lambda)} = \frac{-1}{2i\pi} \int_{\mathcal{C}} (-t)^{-\lambda} e^{-t} dt$$

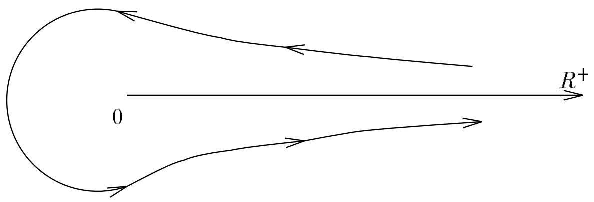


Figure 2.2: Path in the Hankel integral's representation of $\frac{1}{\Gamma(z)}$

where $\mathcal{C}$ is the path starting and ending on the real axis at the infinity, and the value of the integral is unchanged if the path is of the form $-a+iR$.□

This leads us to define the formal Borel transform of a formal power series $\hat{f}=\sum_{n\geq 0}a_nx^n$ as

$$\mathcal{B}\hat{f}=\sum_{n\geq 0}a_n\frac{x^{n-1}}{\Gamma(\lambda_n)}$$

In the same way, we define the formal Borel transform of order k as

$$\mathcal{B}_k\hat{f}=\sum_{n\geq 0}a_n\frac{x^{n-k}}{\Gamma(\frac{n}{k})}$$

More generally, if $|f(x)|<C|x|^\lambda$ on V $(\lambda>0)$, then, we have, on $\tilde{V}(d,\theta-\pi)$ (with $\pi<\theta<\Theta$),

$$|\mathcal{B}f(\xi)|\leq K(R,\Theta,\theta)C\frac{|\xi|^{\lambda-1}}{\Gamma(\lambda)}e^{\frac{|\xi|}{R}}$$

where the constant K depends only on Θ, θ and R.
Moreover, if f is defined in analytic in a sector which opneing angle is greater then $\frac{\pi}{k}$, then we have

$$|\mathcal{B}_kf(\xi)|\leq K(R,\Theta,\theta)C\frac{|\xi|^{\lambda-k}}{\Gamma(\frac{\lambda}{k})}e^{\frac{|\xi|^k}{R}}$$

Laplace transform.

Let f be a holomorphic function on an open "infinite" sector $\tilde{V}(d,\phi)$, ϕ beeing arbitrary. We consider the integral

$$\mathcal{L}f(x)=\int_d f(\xi)e^{-\frac{\xi}{x}}d\xi$$

If f is integrable in a neighbourhood of zero and at most, exponentially increasing at the infinity in the direction d, that is $|f(\xi)|<Me^{\frac{|\xi|}{B}}$, then this integral defines a holomorphic function on the open disk of axis d, of diameter B passing through zero. If this growth's condition remains true for any other direction in $\tilde{V}$, then the integral admits an analytic continuation in a sector bissected by d and which opening angle is $\phi+\pi$. This function is called the **Laplace transform** of f.

If f is exponentially increasing of order $\leq k$, then we define the **Laplace transform of order** k by $\mathcal{L}_k = \rho_{\frac{1}{k}} \mathcal{L} \rho_k$, that is

$$\mathcal{L}_k f(x) = k \int_d f(\xi) e^{-\frac{\xi^k}{x^k}} \xi^{k-1} d\xi$$

Example: If $\lambda > 0$ and $f(\xi) = \xi^{\lambda - 1}$ then, for any direction d, we have

$$\mathcal{L} f(x) = \Gamma(\lambda) x^\lambda$$

This is obtained by the usual defintion of the function Γ.□

More generally, if $|f(\xi)| \leq C|\xi|^{\lambda-1} e^{\frac{|\xi|}{R}}$ on d, then

$$|\mathcal{L} f(x)| \leq K(R, R') C \Gamma(\lambda) |x|^\lambda$$

on the disk of axis d, of diameter $R' < R$ passing through zero. The constant K depends only on R and R'.
Moreover, if $f(\xi)| \leq C|\xi|^\lambda e^{\frac{|\xi|^k}{R}}$ on d^k, then

$$|\mathcal{L}_k f(x)| \leq K(R, R') C \Gamma(\frac{\lambda}{k} + 1) |x|^{\lambda + k}$$

Main properties of Borel and Laplace transforms

1. Reciprocity

   The Borel and Laplace transforms, in a same direction d, are the inverse of each other, that is
   $$\mathcal{L}\mathcal{B} f = f \text{ and } \mathcal{B}\mathcal{L} f = f$$
   when $|f| < k|x^2| + \cdots$.

   In particular, $\mathcal{B}$ and $\mathcal{L}$ are a bijection between the following sets:

   (a) B_d: set of holomorphic functions in a small sector bissected by d and which opening angle is greater than π, satisfying to $|f(x)| < C|x|^\lambda$ $(\lambda > 0)$.

   (b) L_d: set of holomorphic functions on an infinite sector bissected by d, satsfying to

      i. $|f(\xi)| < C|\xi|^{\lambda-1} (\lambda > 0)$ for $|\xi|$ sufficiently small,

      ii. $|f(\xi)| < C' e^{\frac{|\xi|}{R}} (R > 0)$ for $|\xi|$ sufficiently large.

2. Isomorphism of differential rings

   If $(f, g) \in B_d^2$, then $fg \in B_d$ and we have
   $$\mathcal{B}(fg) = \mathcal{B} f * \mathcal{B} g$$
   where $*$ denotes the radial convolution's product in L_d, defined by
   $$f * g(\xi) = \int_0^\xi f(\eta) g(\xi - \eta) d\eta$$

3. Relation with asymptotic expansions

Let $f \in B_d$ admitting $\hat{f} = \sum_{n=0} a_n x^{\lambda_n}$ as asymptotic expansion at the origin, where $\Lambda = (\lambda_n)_{n\in N}$ is a suitable familly, that is

$$|f(x) - \sum_{k=0}^{n-1} a_k x^{\lambda_k}| < A_n |x|^{\lambda_n}$$

Then, the function $\mathcal{B}f$ admits

$$\mathcal{B}\hat{f} = \sum_{n\geq 0} a_n \frac{x^{\lambda_n - 1}}{\Gamma(\lambda_n)}$$

as asymptotic expansion at the infinity related to the suitable familly $\Lambda' = (\lambda_n - 1)_{n\in N}$, that is

$$|\mathcal{B}f(\xi) - \sum_{k=0}^{n-1} a_k \frac{\xi^{\lambda_k - 1}}{\Gamma(\lambda_k)}| < K \frac{A_n}{\Gamma(\lambda_n)} |\xi|^{\lambda_n}$$

This fact is a straightforward consequence of the above results. In fact, let us set $g_n(x) = f(x) - \sum_{k=0}^{n-1} a_k x^{\lambda_k}$, then, by hypothesis, $|g_n(x)| < A_n |x|^{\lambda_n}$, thus,

$$|\mathcal{B}f(\xi) - \sum_{k=0}^{n-1} a_k \frac{x^{\lambda_k - 1}}{\Gamma(\lambda_k)}| = |\mathcal{B}g_n(\xi)| \leq K A_n \frac{|\xi|^{\lambda_n - 1}}{\Gamma(\lambda_n)} e^{\frac{|\xi|}{R}}$$

As a consequence, if $\hat{f}$ is the Gevrey asymptotic expansion of order $\frac{1}{k}$ of f then $\mathcal{B}\hat{f}$ is the Gevrey asymptotic expansion of order $\frac{1}{k'} = \frac{1}{k} - 1$ of $\mathcal{B}f$ if $\frac{1}{k} > 1$, and is convergent otherwise.
In fact, we have, in this case, $\Lambda = N$ and $A_n = C^n \Gamma(\frac{n}{k})$. Thus, the coefficient of asymptoticity of $\mathcal{B}$ is $K \frac{C^n \Gamma(\frac{n}{k})}{\Gamma(n)}$. But, it is well known that

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

the Euler's Beta function. Thus, we have, if $\frac{1}{k'} = \frac{1}{k} - 1 > 0$, $\frac{\Gamma(n)\Gamma(\frac{n}{k'})}{\Gamma(\frac{n}{k})} = B(n, \frac{n}{k'})$.

2.5.2 Caracterization of the k-summability

Using the proprieties of the Borel-Laplace transforms, we can state an equivalent definition of k-summability which is more convenient for computations.

Theorem 2.5.1 *Let $\hat{f} = \sum_{n\geq 0} a_n x^n \in C[[x]]_{\frac{1}{k}}$. Even if it means changing $\hat{f}$ into $\hat{f} - \sum_{p=0}^{[k]+1} a_p x^p$, we can assume that $a_0 = \cdots = a_{[k]+1} = 0$.*
Then, the following assertions are equivalent:

1. *$\hat{f}$ is k-summable in the direction d, and its k-sum on a sector bissected by d, and opening angle Θ is f*

2. *$\mathcal{B}_k\hat{f}$ is convergent in a neighbourhood of the origin, and admits an analytic continuation $AC\mathcal{B}_k\hat{f}$ on an infinite sector bissected by d and which opening angle is $\Theta - \frac{\pi}{k}$, into an exponential increasing function of order $\leq k$.*

Moreover, we have $f = \mathcal{L}_k AC\mathcal{B}_k\hat{f}$.

Proof:

1. (1) $\Longrightarrow$ (2)
   From the properties of $\mathcal{B}_k$, we see that $\mathcal{B}_k f$ is holomorphic on an infinite sector bissected by d, which opening angle is $\Theta - \frac{\pi}{k}$ and, is, at most, exponential increasing of order k, at the infinity. But, $\hat{f} \in C[[x]]_{\frac{1}{k}}$ and is the Gevrey asymptotic expansion of order $\frac{1}{k}$ of f. Thus, $\mathcal{B}_k\hat{f}$ converges in a neighbourhood of the origin and is the asymptotic expansion of $\mathcal{B}_k f$. Therefore, $\mathcal{B}_k f$ is the desired analytic continuation and we have $f = \mathcal{L}_k\mathcal{B}_k f$.

2. (2) $\Longrightarrow$ (1)
   If $\mathcal{B}_k\hat{f}$ is convergent in a neighbourhood of the origin, then, $\hat{f}$ is Gevrey of order $\frac{1}{k}$. Let g be its analytic continuation in the infinite sector. We have, in the

   $$|g(\xi) - \sum_{p=0}^{n-1} a_p \frac{\xi^{p-k}}{\Gamma(\frac{p}{k})}| < KA^n|\xi|^{n-k}e^{\frac{|\xi|^k}{R}}$$

   Thus,

   $$|\mathcal{L}_k g(x) - \sum_{p=0}^{n-1} a_p x^p| < KA^n\Gamma(\frac{n}{k})|\xi|^n$$

   Thus $\hat{f}$ is the Gevrey asymptotic expansion of order $\frac{1}{k}$ of f in a sector bissected by d which opening angle is greater than $\frac{\pi}{k}$. That is, $\hat{f}$ is k-summable in the direction d and its k-sum is $\mathcal{L}_k g$.

□

Fundamental exemple:**Euler series**

We consider the Euler series

$$\hat{f} = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1}$$

The formal Borel transform of $\hat{f}$ is

$$\mathcal{B}\hat{f} = \sum_{n=0}^{\infty} (-1)^n x^n$$

This series is convergent in a neighbourhood of the origin and admits an analytic continuation in every direction but R^- into the function $\frac{1}{1+x}$.
Thus, according to the previous theorem, $\hat{f}$ is 1-summable in every direction but R^-, and its sum, in such a direction d, is

$$\int_d \frac{e^{-\frac{\xi}{x}}}{1+\xi} d\xi$$

□

2.6 k-precise quasi-functions and k-summability

Now that we have stated the main theorems about the Taylor jet application and defined the notion of k-summability, we can caracterize it in a more geometrical way, using the fondemantal Ramis-Sibuya theorem and the important notion of quasi-function.
We shall consider "holomorphic 0-cochains" which consist of a family $\{f_i\}$ such that f_i is a holomorphic function on V_i and the associated "1-coboundary" which consist of a family $\{h_i\}$ such that $h_i = f_{i+1} - f_i$.

Let $k > 0$. By definition, a **k-precise quasi-function** on a sector V is a data given by a holomorphic 0-cochain $\{f_i\}$ associated to a covering $\{V_i\}$ such that the associated 1-coboundary is exponentially decreasing of order $\geq k$, that is, $h_i = f_{i+1} - f_i$, defined on $V_{i+1,i}$, is an exponentially decreasing function of order $\geq k$. Moreover, we identify two such datas $(\{f_i\}, \{V_i\})$ and $(\{g_i\}, \{W_i\})$, if, every time that the intersection $V_i \cap W_j$ is not empty, $f_i - g_j$ is an exponential decreasing function of order k on that intersection.
In other words, the notion of k-precise quasi-function formalize the notion of holomorphic function known up an exponential decreasing function of order $\geq k$. A k-precise quasi-function will be said bounded if each f_i are bounded. A k-precise quasi-function $(\{f_i\}, \{V_i\})$ will be said to admit a Gevrey (resp. Poincaré) asymptotic expansion of order $\frac{1}{k}$, if, for all i, $f_i \in \mathcal{A}_{\frac{1}{k}}(V_i)$ (resp. $f_i \in \mathcal{A}(V_i)$); that is, each f_i admits a Gevrey asymptotic expansion of order $\frac{1}{k}$ (resp. Poincaré asymptotic expansion). In the sequel, every 0-cochain used will be assumed to admit a Poincaré asymptotic expansion. A corrolary of the theorem 2.2.1 can be stated as follow

Corollary 2.6.1 *Let $\{V_i\}$ be a good covering of $D_r^* = \{z : \ 0 < |z| < r\}$ and $(\{f_i\}, \{V_i\})$ be a k-precise quasi-function admitting a Poincaré asymptotic expansion. Then it admits a Gevrey asymptotic expansion of order $\frac{1}{k}$.*

Using a good covering of the punctured disk D_r^*, which sectorial domains have an opening angle smaller than $\frac{\pi}{k}$, and the Borel-Ritt-Ramis theorem, we conclude that we can associate a unique k-precise quasi-function to every Gevrey formal power series of order $\frac{1}{k}$: its quasi-sum; if $\{f_i\}$ represents the quasi-sum of $\hat{f}$, then f_i admits $\hat{f}$ as Gevrey asymptotic expansion of order $\frac{1}{k}$. As every bounded holomorphic function on the punctured disk is the sum of a convergent series, a bounded k-precise quasi-function on the punctured disk is the quasi-sum of a Gevrey formal power series of order $\frac{1}{k}$. Thus, we can establish a caracterisation of the k-summability in the following way

Corollary 2.6.2 (Caracterisation of the k-summability) *Let $\hat{f} \in C[[x]]_{\frac{1}{k}}$. Let F be the k-precise quasi-function which is the quasi-sum of $\hat{f}$ on a punctured disk. Let V be an open sectorial domain which opening angle is greater than $\frac{\pi}{k}$.*
Let f be a holomorphic function on V. Then, the following proprieties are equivalent:

1. *The function f admits $\hat{f}$ as a Gevrey asymptotic expansion of order $\frac{1}{k}$, that is $\hat{f}$ is k-summable on V*

2. *f mod $\mathcal{A}^{\leq -k} = F_{|V}$*

Chapter 3

Multisummability

After the discovery, by Borel, of the Borel-summability[Bor28], quickly generalized to "k-summability", by Leroy and Nevanlinna, many mathematicians had tried to prove that the formal power series solutions of algebraic differential equations were "k-summable". Unfortunetely, as announced in 1979 in [Ram79], this was shown to be false. Roughly speaking, if $k \neq k'$, the way to proceed k-summability and k'-summability are quite different: if $k < k'$, then the asymptotic estimations required for the k'-summability are more strict, whereas the opening required for the sector is smaller. This intuition is confirmed by the following "Tauberian" theorem.

Theorem 3.0.1 *Let k and k' be real numbers such that $k' > k > 0$. Then,*

$$C\{x\}_{\frac{1}{k}} \cap C\{x\}_{\frac{1}{k'}} = C\{x\}.$$

We shall prove this theorem in the sequel.

In order to find a counter-example to our problem, we are led to consider the sum $\hat{f}$ of a k_1-summable formal solution $\hat{f}_1$ of a linear algebraic differential operator D_1 and of a k_2-summable formal solution $\hat{f}_2$ of a linear algebraic differential operator D_2. This sum is a solution of a linear algebraic differential operator D.
The simplest example is made by considering as $\hat{f}_1$ as the Euler's series and as $\hat{f}_2$ the Euler's series where x has been changed to x^2. Thus $\hat{f}_1$ is 1-summable and $\hat{f}_2$ is 2-summable. Nevertheless,

$$\begin{aligned} \hat{f} &= \hat{f}_1 + \hat{f}_2 \\ &= \sum_{n=0}^{\infty} (-1)^n n! x^{n+1} + \sum_{n=0}^{\infty} (-1)^n n! x^{2n+2} \end{aligned}$$

is not k-summable for any real $k > 0$. Moreover, $\hat{f}$ is a solution of the following linear algebraic differential equations

$$Dy = 0$$

with

$$D = (\frac{d}{dx})^5 (x^5(2-x)\frac{d^2}{dx^2} - x^2(2x^3 - 5x^2 - 4)\frac{d}{dx} + 2(x^2 - x + 2))$$

and

$$x^5(2-x)y" - x^2(2x^3-5x^2-4)y' + 2(x^2-x+2)y \;=\; -3x^4+10x^3+2x^2+4x$$

If we want to find a process of summation able to sum formal solutions of algebraic differential equations (even only linear), it seems reasonnable to look for a process able to sum a finite sum of k-summable series (in a same given direction) for different values of k. One could think that, since $\hat{f} \;=\; \hat{f}_1+\hat{f}_2$, $\hat{f}$ could be defined as the sum of $\hat{f}_1$ and $\hat{f}_2$. Nevertheless, this leads to a difficulty: one has to prove that the sum $\hat{f}$ doesn't depend on the decomposition(i.e of the choice of $\hat{f}_1$ and $\hat{f}_2$). Moreover we want to work with differential algebra. Thus we shall have to study the decomposition into a sum of products. At last, if we want to apply all this to differential equations (whether they are linear or not), we shall have to study the decomposition into sum of k-summable series (in the same direction) of a formal series solution of such an equation. After many years of intense research, all these questions have been answered.

3.1 Definitions of the multisummability

The simplest way to define the notion of "multisummability" is to use the following theorem

Theorem 3.1.1 *Let d be a direction from the origin. The sum*

$$\sum_{k>\frac{1}{2}} C\{x\}_{\frac{1}{k};d}$$

of complex sub-spaces of $C[[x]]$ is a differential sub-algebra of $(C[[x]],x^2\frac{d}{dx})$. Moreover, there exists an unique injective homomorphism of differential algebra

$$S_d:\;\sum_{k>\frac{1}{2}} C\{x\}_{\frac{1}{k};d}\to\mathcal{A}_d$$

such that the restriction of S_d to each fo the spaces $C\{x\}_{\frac{1}{k};d}$ (with $k>\frac{1}{2}$) is the k-summation operator in the direction d:

$$S_{k;d}:\;C\{x\}_{\frac{1}{k};d}\to\mathcal{A}_d.$$

By the mean of a ramification, we obtain an injective homeomorphism of differential algebra

$$S_d:\;\bigcup_{m\in N^*}\sum_{k>\frac{1}{2}} C\{x^{\frac{1}{m}}\}_{\frac{1}{k};d}\to\mathcal{A}_d$$

We shall denote

$$C\{x\}_{.;d} \;=\; \bigcup_{m\in N^*}\sum_{k>\frac{1}{2}} C\{x^{\frac{1}{m}}\}_{\frac{1}{k};d}\bigcap C[[x]]$$

This set contains, for all $k>0$, the algebra $C\{x\}_{\frac{1}{k};d}$ since $C\{x^{\frac{1}{m}}\}_{\frac{1}{k};d}\cap C[[x]]=C\{x\}_{\frac{1}{mk};d}$; and we obtain an injective homomorphism of differential algebras $S_{.;d}\;:\;C\{x\}_{.;d}\to A_d$ which restriction to each algebra $C\{x\}_{\frac{1}{k};d}$ ($k>0$) is equal to k-summation operator in the direction d. This leads us to the following definition

Definition 3.1.1 *A formal power series belonging to $C\{x\}_{.;d}$ will be said* ***multisummable in the direction*** *d.*
Let $k_1, \dots, k_r$ be real numbers such that $0 < k_1 < \dots < k_r$, and d a direction from the origin. A formal power series $\hat{f}$ will be said $(k_1, \dots, k_r)$-summable in the direction d if it belongs to the set $\sum_{i=1}^{r} C\{x\}_{\frac{1}{k_i};d}$. In this case, we shall write $\hat{f} \in C\{x\}_{\frac{1}{k_1},\dots,\frac{1}{k_r};d}$.

There exist two proofs of the previous theorem which are quite different. The first uses the acceleration's theory of J. Ecalle and generalizes the approach, by the mean of integral formulae and analytic continuation, of the k-summability. The last, due to B. Malgrange and J.P. Ramis, is more "geometrical". It uses the notion of exponentially small corrections. This is this approach that we shall follow here.
First, we shall define the notion of multi-summability in the following way:

Definition 3.1.2 *Let $k_1, \dots, k_r$ be real numbers such that $0 < k_1 < \dots < k_r$. Let $V_r \subset \dots \subset V_1$ be opened sectors with the same radius and bissected by the same direction d. Moreover, we assume that the openings are such that $|V_i| > \frac{\pi}{k_i}$.*
Let $\hat{f}$ be a Gevrey formal power series of order $\frac{1}{k_1}$ and f_0 be the quasi-function k_1-precise associated to $\hat{f}$. A $(k_1, \dots, k_r)$-sequence associated to $\hat{f}$, is a sequence $(f_1, \dots, f_r)$ where f_r is an analytic function on V_r and f_i $(1 \le i < r)$ a quasi-function k_{i+1}-precise defined on V_i. The formal power series $\hat{f}$ will be said $(k_1, \dots, k_r)$-summable on $(V_1, \dots, V_r)$, with the sum $(f_1, \dots, f_r)$, if, for all $1 \le i < r$, the restriction of f_{i-1} to V_i is equal to the quasi-function k_i-precise associated to the quasi-function k_{i+1}-precise f_i, i.e $f_{i-1}|V_i = f_i \bmod \mathcal{A}^{\le -k_i}$.

3.2 Uniqueness of the multisum

In the study of the k-summability, we have shown that if a holomorphic function f, defined on a sectorial domain which opening angle is greater than $\frac{\pi}{k}$, which is exponential decreasing of order $\ge k$, then it is the null function. This theorem, known as the Watson's lemma, was fondamental in order to define the "k-sum" of formal power series, since it provides unicity. In the case of the "multisummability", we consider formal series which are the sum of formal series which are k-summable, in a given direction, for different value of k. In order to get unicity for the "multisum" we need an extension of the Watson's lemma. This theorem, known as the "Relative Watson's lemma", is not a consequence of the Watson's lemma and is due to B. Malgrange[RM92][Mal92].

Theorem 3.2.1 (Relative Watson's theorem) *Let $k > k' > 0$. Let V be an open sectorial domain of the complex plane (or of the Riemann's surface of the Logarithm) at the origin, with an opening angle greater then $\frac{\pi}{k'}$. Let f be a quasi-function on V, k-precise and which is exponential decreasing of order k' on V (i.e f is represented by a 0-cochain $\{f_i\}$, with f_i an exponential decreasing function of order k' on V_i and $f_{i+1,i}$ an exponential decreasing function of order k on $V_{i+1,i}$).*
Then, f is a null quasi-function k-precise on V (i.e. the f_i's are exponential deacreasing functions of order k on V_i).

This theorem means that, if we have functions f_i defined on V_i which are exponentially decreasing of order k and if $\{V_i\}$ is a covering of V, if we assume that these functions glue

together up to $\mathcal{A}^{\leq -k'}$ in $V_{i+1} \cap V_i$, then the f_i's are exponentially decreasing functions of order k' on V_i as soon as the opening angle of V is large enough. We shall prove it in the next section.
This theorem is equivalent to a "Tauberian" theorem due to Martinet and Ramis:

Theorem 3.2.2 *[MR89] Let $0 < k' < k$, $\kappa = \frac{kk'}{k-k'}$ and d a direction from the origin. If the formal power series $\hat{f} \in C[[x]]_{\frac{1}{k}}$ is k'-summable in the direction d, then it is k-summable in all directions of the sector bissected by d and which opening angle is $\frac{\pi}{\kappa}$.*

Proof: The proof we shall give is based on the proprites of the Laplace-Borel transforms.

Let f be the k'-sum of $\hat{f}$ in the direction d. Then, $\mathcal{B}_{\frac{1}{k}} f$ admits $\mathcal{B}_{\frac{1}{k}} \hat{f}$ as Gevrey asymptotic expansion of order $\frac{1}{k'} - \frac{1}{k} = \frac{1}{\kappa}$. But, by assumption, this series converges. Thus, $\mathcal{B}_{\frac{1}{k}} f$ is its sum and is an analytic in an infinite sector bissected by d and which opening angle is $\frac{\pi}{\kappa}$ Thus, according to theorem 2.5.1, $\hat{f}$ is k-summable in all directions of the sector bissected by d and which opening angle is $\frac{\pi}{\kappa}$.□

Proof of the equivalence between theorem 3.2.1 and theorem 3.2.2:

1. "Relative Watson's theorem" $\Longrightarrow$ "Tauberian theorem"
   Since $\hat{f} \in C[[x]]_{\frac{1}{k}}$, there exist $\{(f_i, V_i)\}$ where $\{V_i\}$ is a good covering of a punctured disk, $f_i \in \mathcal{A}_{\frac{1}{k}}(V_i)$ and $Tf_i = \hat{f}$. We denote J the set of index such that $\{V_i\}_{i \in J}$ covers the sector V bissected by d and which opening angle is greater than $\frac{\pi}{k}$ on which a function f admits $\hat{f}$ as asymptotic expansion. On V_i, $i \in J$, we have $f_i - f \in \mathcal{A}^{\leq -k'}(V_i)$ because $\hat{f}$ is Gevrey of order $\frac{1}{k'}$, and is the asymptotic expansion of both fi and $\hat{f}$. But, if $V_i \cap V_j \neq \emptyset$, with $(i,j) \in J^2$, $(f_i - f) - (f_j - f) = f_i - f_j \in \mathcal{A}^{\leq -k}(V_i \cap V_j)$. According to the Relative Watson's theorem, $f_i - f \in \mathcal{A}^{\leq -k}(V_i)$ and thus, $\hat{f}$ is k-summable in all direction θ such that $|\theta - d| < \frac{\pi}{k'} - \frac{\pi}{k}$.

2. "Tauberian theorem" $\Longrightarrow$ "Relative Watson's theorem"
   Let $\{V_i\}_{i \in I}$ be a good covering of V and let $\{(f_i, V_i)\}_{i \in I}$, be an exponential decreasing of order k' 0-cochain, such that its associated 1-cocycle is exponential decreasing of order k.
   We complete this covering into a good covering $\{V_i\}_{i \in J} (I \subset J)$ of a punctured disk. Then we define the 1-cocycle $\{(h_{i,j}\}_{(i,j) \in J^2}$ with (i,j) such that $V_i \cap V_j \neq \emptyset$, by

   (a) $h_{j,k} = -h_{k,j}$
   (b) if $(j,k) \in I^2$, then, $h_{j,k} = f_j - f_k$
   (c) $h_{j,k} = 0$ otherwise.

   By Ramis-Sibuya theorem, there exists a 0-cochain $\{h_i\}_{i \in J}$ such that $h_i - h_j = h_{i,j}$ on $V_i \cap V_j$. Moreover, each h_i admit an asymptotic expansion of order $\frac{1}{k}$. We have, for $(i,j) \in I^2$, $h_i - f_i = h_j - f_j$ on $V_i \cap V_j$, thus, this define a function f on V. The power series Tf is Gevrey of order $\frac{1}{k}$, thus Gevrey of order $\frac{1}{k'}$ and k'-summable on V. Thus, by theorem 3.2.2, it is k-summable on V with the same sum. Thus, f_i's are exponentially decreasing of order $\geq k$.

□

Proof of theorem 3.0.1:

Let $k > k' > 0$, and $\hat{C}[[x]]_{\frac{1}{k}} \cap C\{x\}_{\frac{1}{k'}}$. Thus, according to theorem 3.2.2, $\hat{f}$ is k-summable in the same direction in a sector which opening angle is greater then $\frac{\pi}{k'}$. Let $\{\theta_1, \ldots, \theta_n\}$ be the singular directions. Let $\epsilon > 0$ such that $\frac{\pi}{k'} - 2\epsilon > \frac{\pi}{k}$. Let V^+(resp. V^-) be a sector bissected by $\theta_i + \epsilon$(resp. $\theta_i - \epsilon$), and which opening angle is greater than $\frac{\pi}{k'}$. Let f^+(resp. f^-) be the k'-sum of $\hat{f}$ on V^+(resp. V^-). But f^+(resp f^-) is the k-sum of $\hat{f}$ on V^+(resp. V^-). Thus, on $V^+ \cap V^-$, $f^+ - f^-$ is an exponentially decreasing function of order k. But, the opening angle of $V^+ \cap V^-$ is greater than $\frac{\pi}{k'} - 2\epsilon > \frac{\pi}{k}$, thus, according to the Watson lemma, $f^+ - f^- = 0$ on $V^+ \cap V^-$.

By doing this operation for all singular directions, we see that the sums glue together, thus they define a bounded holomorphic function in a neighbourhood of the origin. Consequently, $\hat{f}$ is convergent. $\square$

Proposition 3.2.1 *If a formal power series $\hat{f}$ is $(k_1, \ldots, k_r)$-summable on $(V_1, \ldots, V_r)$, then the multi-sum $(f_1, \ldots, f_r)$, defined above, is unique.*

Proof: Let $(f_1, \ldots, f_r)$ and $(\tilde{f}_1, \ldots, \tilde{f}_r)$ be two multisums of $\hat{f}$ on $(V_1, \ldots, V_r)$. Then, we have

$$f_1 - \tilde{f}_1 (mod\, \mathcal{A}^{\leq -k_1}) \;=\; 0$$

But, both f_1 and $\tilde{f}_1$ are quasi-functions k_2-precise on V_1 wich opening angle is greater than $\frac{\pi}{k_1}$. Thus, by the relative Watson's lemma, $f_1 - \tilde{f}_1$ is the null quasi-function k_2-precise on V_1. Then, we have

$$f_2 - \tilde{f}_2 (mod\, \mathcal{A}^{\leq -k_2}) \;=\; (f_1 - \tilde{f}_1)_{|V_2} \;=\; 0$$

By the same way, we obtain that $f_2 - \tilde{f}_2$ is the null quasi-function k_3-precise on V_2. By induction, we obtain that, for all $1 \leq j < r$, $f_j - \tilde{f}_j$ is the null quasi-function k_{j+1}-precise on V_j. For the last rank r, we have

$$f_r - \tilde{f}_r (mod\, \mathcal{A}^{\leq -k_r}) \;=\; (f_{r-1} - \tilde{f}_{r-1})_{|V_r} \;=\; 0$$

This means that $f_r - \tilde{f}_r$ is an exponentially decreasing function of order $\geq k_r$ on V_r, which opening angle is greater than $\frac{\pi}{k}$. Thus, according to the "ordinary" Watson's lemma, $f_r - \tilde{f}_r$ is the null function on V_r. $\square$

3.3 Multisum as sum of k-summable functions for different values of k

We give the theorem which states that the definitions 3.1.1 and 3.1.2 are equivalent.

Theorem 3.3.1 *Let $k_1, \ldots, k_r$ be real numbers such that $0 < k_1 < \cdots < k_r$. Let $V_r \subset \cdots \subset V_1$ be opened sectors with the same radius and bissected by the same direction d. Moreover, we assume that the openings are such that $|V_i| > \frac{\pi}{k_i}$.*
Let $\hat{f}$ be a Gevrey formal series of order $\frac{1}{k_1}$. The following statements are equivalent:

1. *$\hat{f}$ is $(k_1, \ldots, k_r)$-summable on $(V_1, \ldots, V_r)$ in the sense of definition 3.1.2*

2. $\hat{f}$ *can be writen* $\hat{f} = f_1 + \cdots + f_r$ *where, for all* $1 \leq j \leq r$, $f_j \in C[[x]]_{\frac{1}{k_j}}$ *is* k_j*-summable on* V_j*.*

Proof:

1. Sufficient condition

   We state two lemmas in which we keep on using the above notations.

   Lemma 3.3.1 *If* $\hat{g} \in C[[x]]_{\frac{1}{k_j}}$ *is* k_j*-summable on* V_j $(1 \leq j < r)$ *then* $\hat{g}$ *is* $(k_1, \ldots, k_r)$*-summable on* $(V_1, \ldots, V_r)$*.*

   Proof: Let g be the quasi-sum of $\hat{g}$ on a punctured disk. According to the corrolary 2.6.2, there exists a 0-cochain g_1 admitting a Poincaré asymptotic expansion such that $g_{|V_i} = g_1 \ mod \ \mathcal{A}^{\leq -k_j}$. Then $\hat{g}$ is $(k_1, \ldots, k_r)$-multisummable on $(V_1, \ldots, V_r)$, its multisum beeing $(g_1 \ mod \ \mathcal{A}^{\leq -k_2}, \ldots, g_1 \ mod \ \mathcal{A}^{\leq -k_r}, g_{1|V_r})$.□

   Lemma 3.3.2 *If* $\hat{g} \in C[[x]]_{\frac{1}{k_r}}$ *is* k_r*-summable on* V_r *then* $\hat{g}$ *is* $(k_1, \ldots, k_r)$*-summable on* $(V_1, \ldots, V_r)$*.*

   Proof: Let g be the quasi-sum of $\hat{g}$ on a punctured disk. According to the corrolary 2.6.2, there exists a 0-cochain g_1 admitting a Poincaré asymptotic expansion such that $g_{|V_r} = g_1 \ mod \ \mathcal{A}^{\leq -k_r}$. Then $\hat{g}$ is $(k_1, \ldots, k_r)$-multisommable on $(V_1, \ldots, V_r)$, its multisum beeing $(g \ mod \ \mathcal{A}_{|V_1}^{\leq -k_2}, \ldots, g \ mod \ \mathcal{A}_{|V_{r-1}}^{\leq -k_{r-1}}, g_{|V_{r-1}}, g_1)$.□

2. Necessary condition

   By induction, it is sufficient to state that $\hat{f} = \hat{f}' + \hat{f}$", where $\hat{f}'$ is $(k_1, \ldots, k_{r-1})$-summable on $(V_1, \ldots, V_{r-1})$, and $\hat{f}$" k_r-summable on V_r. We need the following lemma

   Lemma 3.3.3 *Let* g *be a* k_r*-precise quasi-function on* V_{r-1}*. Then we can write* $g = g'_{|V_{r-1}} + g" \ mod \ \mathcal{A}^{\leq -k_r}$ *where* g' *is a* k_r*-precise quasi-function on a punctured disk, and* g" *a function on* V_{r-1}*.*

   Proof: Let $\{U_i\}_{i \in I}$ be a good covering of V_r. We may assume that g is represented by functions g_i defined on U_i such that, if $U_i \cap U_j \neq \emptyset$ then, $g_i - g_j$ is an exponential decreasing function of order k_r on $U_i \cap U_j$. We complete the covering $\{U_i\}_{i \in I}$ into a good covering $\{U_i\}_{i \in J}$ $(I \subset J)$ of a punctured disk. Then we construct functions $h_{j,k}$ on $U_i \cap U_j$ if this set is not empty, which are exponentially decreasing functions of order k_r on $U_j \cap U_k$, satisfying to the following conditions:

   (a) $h_{j,k} = -h_{k,j}$

   (b) if $(j, k) \in I^2$, then, $h_{j,k} = g_j - g_k$

   (c) $h_{j,k} = 0$ otherwise.

According to the Ramis-Sibuya theorem (theorem 2.2.1), there exist functions h_j defined on U_j which admit an Gevrey asymptotic expansion of order $\frac{1}{k_r}$ such that $h_{j,k} = h_j - h_k$. Since the functions $g_j - h_j$ glue together on V_r, they define a 0-cochain g' on V_r. Furthermore, the h_j *mod* $\mathcal{A}^{\leq -k_r}$'s glue together on a punctured disk, thus they define a k_r-precise quasi-function g" on the same set. This provide the desired equality.□

We can now conclude the proof of the theorem. In fact, let us assume that $\hat{f}$ is multisummable in the sense of the definition 3.1.2; There exists a sequence $(f_1, \dots, f_r)$, where f_r is an analytic function on V_r and f_i $(1 \leq i < r)$ a quasi-function k_{i+1}-precise defined on V_i, such that, for all $1 \leq i < r$, the restriction of f_{i-1} to V_i is equal to the quasi-function k_i-precise associated to the quasi-function k_{i+1}-precise. Thus, according to the previous lemma, we can write $f_{r-1} = f'_{|V_{r-1}} + f$" *mod* $\mathcal{A}^{\leq -k_r}$ where f' is a k_r-precise quasi-function on a punctured disk, and f" a function on V_{r-1}. But, we have $f_{r-1}|V_r = f_r$ *mod* $\mathcal{A}^{\leq -k_r}$; thus, we obtain $f'_{|V_r} = (f_r - f"_{|V_r})$ *mod* $\mathcal{A}^{\leq -k_r}$. Moreover, $f_r - f"_{|V_r}$ is a function on V_r. Consequently, according to corrolary 2.6.2, f' is k_r-summable on V_r. Hence, f" is the $(k_1, \dots, k_{r-1})$-sum of $f_{r-1} - f'$ on $(V_1, \dots, V_{r-1})$.

□

3.4 Multisummability and Borel-Laplace transforms

The main drawback of the "geometrical" appraoch of the multisummability that we have sketched is that it doesn't provide any useful explicit process in order to compute the multisums. There are two ways to obtain such a process. First of all, we can use the acceleration method. We can also use the iteration of Laplace's transforms (of different levels), a method due to Balser. We proceed as follow: Let $k_r > \dots > k_1 > 0$. We define the real positive numbers κ_i, $1 \leq i \leq r$ such that $\frac{1}{\kappa_r} = \frac{1}{k_r}$ and $\frac{1}{\kappa_i} = \frac{1}{k_{i+1}} - \frac{1}{k_i}$ $(1 \leq i < r)$. Thus, we have

$$\frac{1}{k_i} = \frac{1}{\kappa_i} + \dots + \frac{1}{\kappa_r}$$

Let us denote the operator of analytic continuation in the direction d by AC_d. Then, we have

Theorem 3.4.1 *Let d be a direction from the origin. Let $k_r > \dots > k_1 > 0$ and $\kappa_1, \dots, \kappa_r$ as above. For all formal power series $\hat{f}$, the following propositions are equivalent:*

1. *$\hat{f}$ is $(k_1, \dots, k_r)$-summable in the direction d;*
2. *$\hat{\mathcal{B}}_{\kappa_1} \dots \hat{\mathcal{B}}_{\kappa_r} \hat{f}$ is a convergent series and, for all $1 \leq i < r$,*

$$AC_d \mathcal{L}_{\kappa_i} AC_d \dots \mathcal{L}_{\kappa_1} AC_d S \hat{\mathcal{B}}_{\kappa_1} \dots \hat{\mathcal{B}}_{\kappa_r} \hat{f}$$

is a holomorphic exponential increasing function of order κ_{i+1} in a suitable sectorial domain bissected by d.

If those conditions are satisfied then

$$\mathcal{L}_{\kappa_r} AC_d \dots \mathcal{L}_{\kappa_1} AC_d S \hat{\mathcal{B}}_{\kappa_1} \dots \hat{\mathcal{B}}_{\kappa_r} \hat{f}$$

exists and is the sum in the sens of the multisummability of $\hat{f}$ in the direction d.

3.5 Proof of the Relative Watson's theorem

We shall follow the proof of Malgrange[Mal92][RM92].

Before proving the theorem, we shall make some inessential assumptions in order to simplify the proof. Thus, we shall assume that

1. The sector V considered will have an opening angle greater then $\frac{\pi}{k}$.
2. By using the ramification $y = x^k$, we may suppose $k = 1$ and thus, π will be the opening angle of V, which can be assumed to be bissected by R^+.
3. We shall only consider objects "at infinity", that is, by the mean of the change of variable $z = \frac{1}{y}$, we shall consider exponential decreasing functions at infinity and so on.

Let $V = \{z \in C |\ argz \in]\alpha, \beta[,\ |z| > r\}$. Let $\{V_i\}$ be a good covering of V, that is
$V_i = \{z \in C |\ argz \in]\alpha_i, \beta_i[,\ |z| > r\}$, with
$\alpha = \alpha_1 < \alpha_2 < \beta_1 < \alpha_3 < \ldots < \alpha_n < \beta_{n-1} < \beta_n = \beta$ and $\alpha_1 < -\frac{\pi}{2}$, $\beta_n > \frac{\pi}{2}$.
So, we are given functions f_i's defined on V_i such that

1. $\forall i, \exists A_i, B_i > 0, \forall z \in V_i\ \ |f_i(z)| < A_i e^{-B_i|y|}$
2. if $V_i \cap V_j \neq \emptyset$ then

$$\exists C_{i,j}, B_{i,j} > 0, \forall z \in V_i \cap V_j\ \ |f_i(z) - f_j(z)| < C_{i,j} e^{-B_{i,j}|z|^{k'}}$$

We want to prove that the f_i's are exponential decreasing functions of order k'.

3.5.1 Fondamental lemma

The goal of this section, is to see if a given function is exponentially decreasing of order $\geq k'$ using its Laplace transform.
We shall consider a holomorphic function f in a neighbourhood of 0, admitting an analytic continuation in the sector $\{y \in C |\ argy \in]\alpha, \beta[\}$ with $\beta < \alpha + \pi$. Moreover, we assume that the analytic continuation of f is bounded in the sector. Let $\theta \in]\alpha, \beta[$ be a direction. Let δ_θ be the oriented half-line from the origin which angle from the positive x-coordinate half-axis is θ. We define the function

$$g_\theta(\eta) = \int_{\delta_\theta} f(y) e^{-y\eta} dy$$

This integral is convergent for all η such that $|arg\eta + \theta| < \frac{\pi}{2}$. By changing the angle θ, we get a bounded holomorphic function g in the sector $\arg \eta \in]\alpha - \frac{\pi}{2}, \beta + \frac{\pi}{2}[$. We can state the main result

Proposition 3.5.1 *Using the above notation, f is an exponentially decreasing function of order $\geq k$ if and only if the function g admits an analytic continuation into an entire function of order k' (with $\frac{1}{k}+\frac{1}{k'}=1$), that is*

$$\exists A, B>0, \forall \eta \in C \quad |g(\eta)|<Ae^{B|\eta|^{k'}}$$

3.5.2 End of the proof of the relative Watson's theorem

According to the previous hypothesis, we consider an open good covering $\{V_j\}$ of the "sector" V for which, if $y \in V$ then $-\frac{\pi}{2} \leq \arg y \leq \frac{\pi}{2}$. We set $V_j = \{y \in C \,|\, \alpha_j < \arg y < \beta_j,\ |y|>r\}$ where r is a given positive number independant of j. Moreover, we assume that $\alpha_1 < -\frac{\pi}{2} < \beta_1$, $\alpha_n < \frac{\pi}{2} < \beta_n$ and that

$$\alpha=\alpha_1<\alpha_2<\beta_1<\alpha_3<\ldots<\alpha_n<\beta_{n-1}<\beta_n=\beta.$$

We are given functions f_j's defined respectively on V_j which satisfy the growth hypothesis given above.
Let a_j be a point in V_j. Let δ_{θ_j} be a half-line issued from a_j and which angle with the positive x-coordinates axis is θ_j. We chose θ_j such that $\alpha_{j+1} \leq \theta_j < \beta_j$; this mean that all but a finite segment of the half line δ_{θ_j} belongs to $V_{j+1} \cap V_j$.
Then, we define

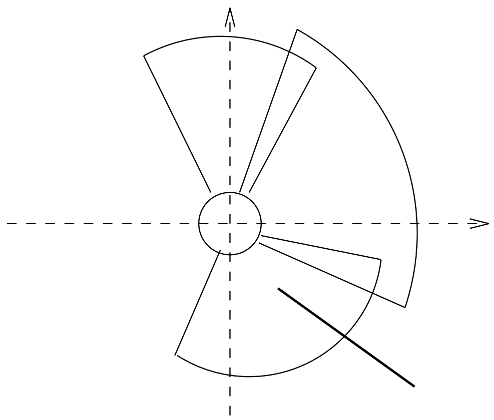


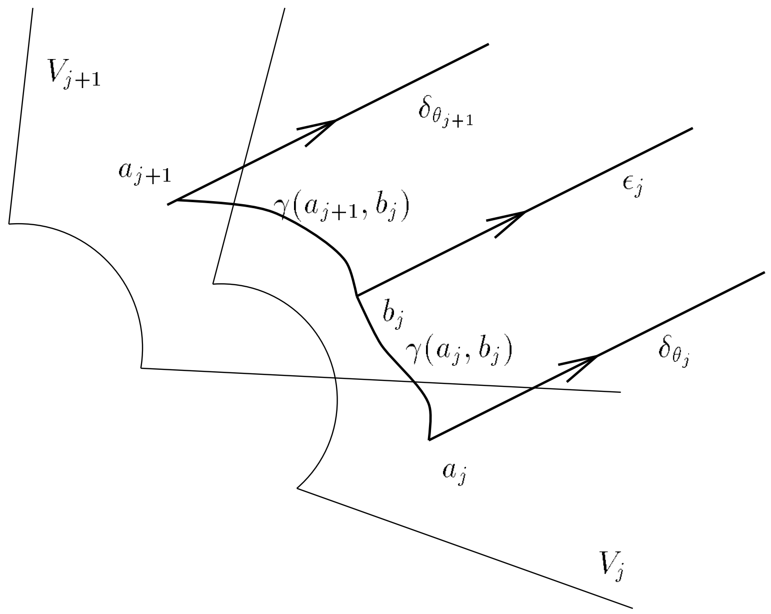
Figure 3.1: Covering

$$g_j(\eta)=\int_{\delta_{\theta_j}} f(y)e^{-y\eta}dy$$

that is,

$$g_j(\eta) \;=\; e^{-a_j\eta}\int_0^{\infty} f_j(\sigma e^{i\theta_j}+a_j)e^{-\eta\sigma e^{i\theta_j}}d\sigma$$

We have to show that the f_j's are exponentially decreasing functions of order k. We shall use the proposition above in order to prove that.
Each function g_j is analytic in the open set $D_{\theta_j}=\{\eta \in C \,|\, -Re(\eta)\cos\theta_j + Im(\eta)\sin\theta_j < B\}$ where B is the lowest of the B_j(the rate of growth of f_j) defined above. Thus, all the g_j's are defined and analytic on a same neighbourhood of the origin. Nevertheless, the function



g_j doesn't depend on the angle θ_j. In fact, let us consider two angles θ_j and $\tilde{\theta}_j$ both belonging to the interval $[\alpha_{j+1}, \beta_j[$. The functions g_j and $\tilde{g}_j$, the latest beeing the Laplace transform of f on the path $\delta_{\tilde{\theta}_j}$, are both defined and analytic on a same angle-independant neighbourhood W of the origin. It is then easy to see, using a third variable path from δ_{θ_j} to $\delta_{\tilde{\theta}_j}$, that, since f_j is an exponential decreasing function on V_j, the two functions g_j and $\tilde{g}_j$ are equal on W. Therefore they are an analytic continuation of each other. Consequently, changing θ_j from α_{j+1} to β_j, g_j admit an analytic continuation on the set

$$D_j = \bigcup_{\theta \in [\alpha_{j+1}, \beta_j[} \{Re(\eta e^{i\theta}) > -B\}$$

We shall first prove that these functions admit an anlalytic continuation into the whole complex plane, that is they are entire functions, which are of order $\leq k' = \frac{k}{k-1}$. Then we shall apply the previous proposition in order to conclude.

Lemma 3.5.1 *With the above notation, $g_{j+1} - g_j$ admits an analytic continuation into an entire function of order at most k'.*

Proof: Let b_j be a point in $V_{j+1} \cap V_j$. Let us chose $\theta_j \in [\alpha_{j+1}, \beta_j[$ and set $\theta_{j+1} = \theta_j$. Let ϵ_j be the half-line starting at b_j and parallel to δ_{θ_j}.
According the the previous proposition, $F : \eta \mapsto \int_{\epsilon_j} (f_{j+1}(y) - f_j(y)) e^{-y\eta} dy$ is an entire function of order $k' = \frac{k}{k-1}$ since, by hypothesis, $f_{j+1} - f_j$ is an exponential decreasinf function of order k in $V_{j+1} \cap V_j$. Moreover, as before, the Laplace transform of f over $\delta_{\theta_{j+1}}$ and ϵ_j are both analytic, at least, on a same neighbourhood W of the origin. Using a

variable path from ϵ_j to $\delta_{\theta_{j+1}}$, we see that

$$\forall \eta \in W, \ \int_{\epsilon_j} f_{j+1}(y)e^{-y\eta}dy \ = \ \int_{\delta_{\theta_{j+1}}} f_{j+1}(y)e^{-y\eta}dy + \int_{\gamma(b_j,a_{j+1})} f_{j+1}(y)e^{-y\eta}dy$$

where $\gamma(b_j, a_{j+1})$ is a path included in V_{j+1}, starting at b_j and ending at a_{j+1}.
By the same way, we obtain that

$$\forall \eta \in W, \ \int_{\epsilon_j} f_j(y)e^{-y\eta}dy \ = \ \int_{\delta_{\theta_j}} f_j(y)e^{-y\eta}dy + \int_{\gamma(b_j,a_j)} f_j(y)e^{-y\eta}dy$$

where $\gamma(b_j, a_j)$ is a path included in V_j, starting at b_j and ending at a_j.
Both integrals over the finite lenght paths γ's are entire functions of order 1. Thus, the function

$$g_{j+1} - g_j \ = \ \int_{\epsilon_j} (f_{j+1}(y) - f_j(y))e^{-y\eta}dy - \int_{\gamma(b_j,a_{j+1})} f_{j+1}(y)e^{-y\eta}dy + \int_{\gamma(b_j,a_j)} f_j(y)e^{-y\eta}dy$$

admit an analytic continuation into an entire function of order at most equal to k'.□

Corollary 3.5.1 *For all j, g_j admit an analytic continuation into an entire function of order at most equal to k'.*

Proof: Applying (n-1) times the previous lemma, we conclude that the function $g_n - g_1$ admit an analytic continuation into an entire function h of order at most equal to k'. From the hypothesis made on the angles α_1 and β_n, the domains D_1 and D_n, defined above as the definition's domains of f_1 and f_n respectively, make up a covering of the complex plane. In the domain where g_1 is not defined, we set $g_1 = g_n - h$. We do the same for g_n. Consequently, both g_1 and g_n admit an analytic continuation into an entire function of order at most equal to k'. By induction, we conclude in the same way, that each g_j admits an analytic continuation into an entire function of order at most equal to k'. □
We can end the proof of the theorem, using the sufficient condition of the proposition, applyed to each g_j.□

Chapter 4

Differential equations

4.1 Linear differential equations

For the theory of linear ordinary differential equations and its link with Laplace transform, we refer to the book of Malgrange[Mal91].

4.1.1 Newton polygon of a differential operator

Let us denote by K the field $C((x)) = C[[x]][x^{-1}]$ of formal meromorphic series with coefficients in the complex field C. We denote, by $\mathcal{D}$, the ring $K[\frac{d}{dx}]$ of differential operator with coefficients in K, that is the set of finite sum $D = \sum a_i(x)\frac{d^k}{dx^k}$ where $a_k \in K$. We shall define the degree or the order of D as $deg\, D = \sup\{i|\ a_i \neq 0\}$. At last, if $a(x) = \sum a_j x^j \in K$, then we define the valuation of a by $v(a) = \inf\{j|\ a_j \neq 0\}$.

Let (u, v) be a point in R^2. We define the set $Q(u, v) = \{(x, y) \in R^2 \mid x \leq u,\ y \geq v\}$. Let D be a differential operator $D = \sum a_i(x)\frac{d^k}{dx^k} \in \mathcal{D}$ with $a_i(x) = \sum a_{i,j}x^j \in K$. Let us consider the set in R^2:

$$M(D) = \bigcup_{a_{i,j}\neq 0} Q(i, j - i)$$

Then, we define the Newton polygon, denoted by $N(D)$, as the convex hull of $M(D)$(positive slopes).

Of course, we define, in the same way, the Newton polygon of a differential operator with coefficient in K^n.

Example: Let us consider the Euler operator

$$D = x^2\frac{d}{dx} + 1.$$

Hence, the Newton polygon of D is the convex hull of $Q(0, 0) \cup Q(1, 2-1)$. The only positive slope of $N(D)$ is 1.

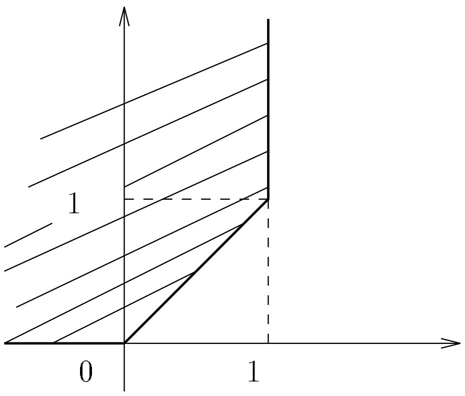


Figure 4.1: Newton polygon of the Euler operator

4.1.2 First order singular linear differential equation

Throughout the sequel of this chapter, by a first order singular linear differential equation, we mean a differential equation of the form

$$\frac{dY}{dx} = M(x)Y$$

where x is a complex independant variable and M is an $m \times m$-matrix which coefficients are meromorphic functions at $x = 0$, that is, $M(x) \in \mathcal{M}_m(C[[x]][x^{-1}])$. This equation is associated to the differential operator $D = \frac{d}{dx} - M$.

It is well known that such an equation admit a formal fundamental solution $H(x)$, this is a matrix which columns are linearly independant and solutions of the equation, which can be written under the following form:

$$H(x) = F(x)x^L exp(Q(x))$$

where

1. $F(x) \in GL_m(C[[t]])$ with $t^p = x$, p beeing a positive integer,
2. $L \in \mathcal{M}_m(C)$ and $x^L = exp(L \log x)$,
3. $Q(x) = diag(q_1(x), \ldots, q_m(x))$ with $q_i(x) \in \frac{1}{t}C[\frac{1}{t}]$,

For each l such that q_l is not identically equal to zero, consider the rational degree d_l of q_l with respect to x^{-1}. Then, the set of all these rational numbers is identical with the set of all positive slopes of the Newton polygon $N(D)$ of D.

Let us denote by a_l the coefficient of x^{d_l} in q_l. We shall call Stokes direction of D, a direction θ such that $Re(a_l e^{i d_l \theta}) = 0$ for some l such that q_l is not identically equal to zero. The Stokes directions of level h will denote the set of Stokes directions concerning the q_l's of equal degree equal to h.

4.1.3 Gevrey character of formal solutions

Let $D = \frac{d}{dx} - M$ be a differential system in a neighbourhood of the origin of the complex plane, where M a $m \times m$ meromorphic matrix at 0. The Newton polygon $N(D)$ has, at most, a finite number of positive slopes:

$$0 < h_q < h_{q-1} < \cdots < h_1 < +\infty$$

There is no positive slope if and only if, $z = 0$ is, at worst, a regular singular point of the differential equation $Dy = 0$. If $z = 0$ is a regular singular point and $Df \in (C\{z\}[z^{-1}])^m$, then f is convergent. Thus, we are interested in the case where the singular point is not regular. Hence, we may assume that the Newton polygon $N(D)$ has, at least, one positive slope.

We shall recall some fondamental results for our purpose concerning properties of linear differential equations. First of all, we have a result which is a particuliar case of Ramis-Sibuya theorems

Theorem 4.1.1 *[RS89][Mal91][Was65] The operators $D : \mathcal{A}^{<0} \to \mathcal{A}^{<0}$ and $D : \mathcal{A}^{\leq -k} \to \mathcal{A}^{\leq -k}$ are both onto.*

As we shall see, this theorem is fondamental.

Corollary 4.1.1 *Let h be one of the Newton polygon's slopes of D. Then the application*

$$\mathcal{D} \ : \ (C[[x]]_{\frac{1}{h}}[x^{-1}])^m \to (C[[x]]_{\frac{1}{h}}[x^{-1}])^m/(C\{x\}[x^{-1}])^m$$

is surjective.

<u>Proof:</u> Let $\hat{g} \in (C[[x]]_{\frac{1}{h}}[x^{-1}])^m$; we shall prove that there exists $\hat{h} \in (C[[x]]_{\frac{1}{h}}[x^{-1}])^m$ such that $D\hat{h} - \hat{g} \in (C\{x\}[x^{-1}])^m$.

According to the Borel-Ritt-Ramis theorem, there exist a good covering $\{V_i\}$ of a punctured disk by sectors, and functions $\{g_i\}$ defined on V_i, such that $Tg_i = \hat{g}$ and, if $V_i \cap V_j \neq \emptyset$, then $g_{i,j} = g_i - g_j \in (\mathcal{A}^{\leq -h}(V_i \cap V_j)[x^{-1}])^m$. Thus, on $V_i \cap V_j \neq \emptyset$, we have $Dg_{i,j} = 0$. But, according to theorem 4.1.1, there exist $h_{i,j} \in (\mathcal{A}^{\leq -h}(V_i \cap V_j)[x^{-1}])^m$ such that $Dh_{i,j} = g_{i,j}$. Moreover, according to Ramis-Sibuya, there exist functions $h_i \in (\mathcal{A}_{\frac{1}{h}}(V_i)[x^{-1}])^m$ such that, on $V_i \cap V_j \neq \emptyset$, $h_i - h_j = h_{i,j}$.

Consequently, on $V_i \cap V_j \neq \emptyset$, we have $Dh_i - g_i = Dh_j - g_j$. By the Riemann vanishing singularity theorem, there exists a function $G \in (C\{x\}[x^{-1}])^m$ such that, on V_i, $G = Dh_i - g_i$. Then, if we set $\hat{h} = Th_i$, then, we have $D\hat{h} - \hat{g} \in (C\{x\}[x^{-1}])^m$, and $\hat{h} \in (C[[x]]_{\frac{1}{h}}[x^{-1}])^m$.$\square$

Proposition 4.1.1 (Poincaré asymptotic expansion theorem) *[Ram78][RM92]*

Let $\hat{f}$ be a formal solution of the homogenous equation $Dy = 0$, i.e $\hat{f} \in (C[[z]])^m$ and $D\hat{f} = 0$. Then, for any direction d, there exists a sector V_d, bissected by d, and a holomorphic function f on V_d such that f is solution of the differential equation $Dy = 0$ and admits $\hat{f}$ as asymptotic expansion in V_d.

<u>Proof:</u> According to Borel-Ritt theorem, for every direction d, there exists a sector V_d, bissected by d, and a holomorphic function $\tilde{f}$ on V_d such that $T\tilde{f} = \hat{f}$(T denotes, as usual, the Taylor jet application). Hence, $D\tilde{f} \in (\mathcal{A}^{<0}(V_d))^m$. Thus, according to theorem 4.1.1, there exists $g \in (\mathcal{A}^{<0}(V_d))^m$ such that $Dg = D\tilde{f}$.

The function $f = \tilde{f} - g$ answers to the question since $Df = 0$, $f \in (\mathcal{A}(V_d))^m$ and $Tf = \hat{f}$.$\square$

Proposition 4.1.2 (Gevrey asymptotic expansion theorem) *Let $k > 0$ and let $\hat{f}$ be a formal Gevrey solution of order $\frac{1}{k}$, of the homogenous equation $Dy = 0$, i.e $\hat{f} \in (C[[z]]_{\frac{1}{k}})^m$ and $D\hat{f} = 0$. Then, for any direction d, there exists a sector V_d, bissected by d and which opening angle is smaller than $\frac{\pi}{k}$, and a holomorphic function f on V_d such that f is solution of the differential equation $Dy = 0$ and admits $\hat{f}$ as Gevrey asymptotic expansion of order $\frac{1}{k}$ in V_d.*

Proof: According to Borel-Ritt-Ramis theorem, for every direction d, there exists a sector V_d, bissected by d and which opening angle is smaller than $\frac{\pi}{k}$, and a holomorphic function $\tilde{f} \in (\mathcal{A}_{\frac{1}{k}}(V_d))^m$ such that $T\tilde{f} = \hat{f}$(T denotes, as usual, the Taylor jet application). Hence, $D\tilde{f} \in (\mathcal{A}^{<-k}(V_d))^m$. Thus, according to theorem 4.1.1, there exists $g \in (\mathcal{A}^{<-k}(V_d))^m$ such that $Dg = D\tilde{f}$.
The function $f = \tilde{f} - g$ answers to the question since $Df = 0$, $f \in (\mathcal{A}_{\frac{1}{k}}(V_d))^m$ and $Tf = \hat{f}$.□

Proposition 4.1.3 *[Ram84]*
Let $\hat{f} \in (C[[x]])^m$ such that $D\hat{f}$ converges, i.e $D\hat{f} \in (C\{z\}[z^{-1}])^n$. Then, $\hat{f} \in C[[x]]_{\frac{1}{h}}$, where h is one of the Newton polygon's slopes.

Proof: Let $G = D\hat{f}$. According to the Borel-Ritt theorem, there exist a good covering $\{V_i\}$ of a punctured disk by sectors and functions $\tilde{f}_i \in (\mathcal{A}(V_i))^m$ admiting $\hat{f}$ as asymptotic expansion on V_i. Moreover, we have, for all i, $D\tilde{f}_i - G \in (\mathcal{A}^{<0}(V_i))^m$. Thus there exist functions $f_i' \in (\mathcal{A}^{<0}(V_i))^m$ such that $Df_i' = D\tilde{f}_i - G$. Let us set $f_i = \tilde{f}_i - f_i'$ on V_i; then, $f_i \in (\mathcal{A}(V_i))^m$, $Tf_i = \hat{f}$ and $Df_i = D\hat{f}$. But, if $V_i \cap V_j \neq \emptyset$, then, $(f_i - f_j)_{V_i \cap V_j} \in (\mathcal{A}^{<0}(V_i \cap V_j))^m$ and $D(f_i - f_j)_{V_i \cap V_j} = 0$. Consequently, there exists $1 \leq l \leq q$ such that, for all (i,j) for which $V_i \cap V_j \neq \emptyset$, $(f_i - f_j)_{V_i \cap V_j} \in (\mathcal{A}^{\leq -h_l}(V_i \cap V_j))^m$. According to the Ramis-Sibuya theorem, for all i, $f_i \in (\mathcal{A}_{\frac{1}{h_l}}(V_i))^m$ and thus, $\hat{f}$ is a Gevrey formal power series of order $\frac{1}{h_l}$.

Moreover, this result is precise. In fact, let us assume that $\hat{f} \in (C[[x]]_s)^m$ with $s < \frac{1}{h_l}$. We have to face two cases:

1. $\frac{1}{h_{l-1}} < s < \frac{1}{h_l}$

   We shall proceed as before. Using the Borel-Ritt-Ramis theorem (theorem 2.3.1) instead of the Borel-Ritt thorem, and the second part of theorem 4.1.1 concerning the flateness of solutions non-homogenous linear differential equations, we find a good covering $\{V_i\}$ of a punctured disk, and functions $f_i \in (\mathcal{A}(V_i))^m$ such that $Tf_i = \hat{f}$ and, if $V_i \cap V_j \neq \emptyset$, then $D(f_i - f_j)_{V_i \cap V_j} = 0$ and $(f_i - f_j)_{V_i \cap V_j} \in (\mathcal{A}^{\leq -\frac{1}{s}}(V_i \cap V_j))^m$. But, according the flateness proprieties of solutions of the differential equation $Dy = 0$, we must have $(f_i - f_j)_{V_i \cap V_j} \in (\mathcal{A}^{\leq -h_{s-1}}(V_i \cap V_j))^m$. Then, according to the Ramis-Sibuya thorem, we have $\hat{f} \in (C[[x]]_{\frac{1}{h_{l-1}}})^m$.

2. $s < \frac{1}{h_1}$

   We proceed as in the previous case. Nevertheless, this time, we use the fact that, the only solution of $Dy = 0$ which is exponential decreasing of order greater than q_1, is the null solution. Thus, the functions f_i stick back together in the intersections of the covering. According to the Riemann vanishing singularities theorem, we conclude that the f_i's are restrictions of an analytic function f, defined in a neighbourhood of the origin, and satisfying $Df = G$ and, having $\hat{f}$ as asymtoptic expansion. Thus, $\hat{f}$ converges.

□

4.1.4 Summability of formal solutions

Theorem 4.1.2 *[Ram79][RS89] Let h be the greatest slope of the Newton polygon $N(D)$ of D. Let $\hat{f}$ be a formal solution of the equation $Dy = 0$. Let d be a direction from the origin.*

Then, if $\hat{f} \in C[[x]]_{\frac{1}{h}}$ and d is not a Stokes direction of level h, then $\hat{f}$ is h-summable in the direction d and its sum satisfies the same equation.

<u>Remark:</u> We have shown that a formal solution of $Dy = 0$, is a Gevrey power series of order $\frac{1}{k}$, where k is one of the Newton polygon's slopes. The number k may mot be the greatest one, so, we cannot conclude about the k-summability of the formal solution.□

Before proving this theorem, we shall need a lemma which deals with exponential decreasing solution of order the greatest slope of thte Newton polygon of the differential operator it is solution of.

Proposition 4.1.4 *Let D be a differential operator and h the greatest slope of its Newton polygon. Let d be a direction which is not a Stokes direction a level h.*

Then, there exist a positive number $\epsilon(d)$ such that if f is an exponential decreasing solution of order h of the equation $Dy = 0$, in a sector bissected by d and which opening angle is $\frac{\pi}{h} - \epsilon(d)$, then f is identically null.

<u>Idea of the proof:</u>

According to propostion 4.1.3, any formal solution of $Dy = 0$ is Gevrey of order $\frac{1}{k}$, where k is one of the Newton polygon slopes of D. Further more, according to proposition 4.1.2, if $Dy = 0$ admits a formal Gevrey solution $\hat{g}$ of order $\frac{1}{k}$, then the differential equation admits a solution g in an sector, which opening angle is lower than $\frac{\pi}{k}$, which admits $\hat{g}$ as Gevrey asymptotic expansion of order $\frac{1}{k}$. Consequently, if h is the greatest slope of the Newton polygon of D, then there exists a basis $\{f_i\}_{1 \le i \le m}$ of solutions in a sector $V_{d,h}$ bissected by d and which opening angle is $\frac{\pi}{h}$. Moreover, these solutions are of the form

$$f_i(x) = x^{\alpha_i} \psi_i(t) e^{q_i(\frac{1}{t})}$$

where $t = x^{\frac{1}{\mu}}$, $\psi_i(t) = \sum_j \phi_{i,j} (\ln x)^j$, $\phi_{i,j} \in \mathcal{A}(V_{d,k})$ and q_i beeing a polynomial without constant term.

Let d' be a generic direction in the sector $V_{d,h}$ (there is only a finite number of forbidden directions, thatt is the Stokes directions); we can order the f_i's in such a way that each of them is negligible compared to its successor. This order depends on d' and is locally constant. Thus, on a sufficiently small sector bissected by d', any function of the form $f = \sum_{i=1}^{m} \lambda_i f_i$, where λ_i's are complex numbers, is equivalent to one of the $\lambda_i f_i$'s, denoted by $\lambda_{i(d')} f_{i(d')}$.

If d is not a Stokes direction of level h, each function $e^{q_i(\frac{1}{t})}$, which degree in the variable x is equal to h, is exponentially increasing of order h in some subsector of $V_{d,h}$. Thus,

if $Dy = 0$ has a solution which is exponentially decreasing of order h in $V_{d,h}$, then it is identically null.
Proof of the theorem 4.1.2: Let d be a direction which is not a Stoke direction of level h. Let δ be a positive number. We define the following two open sectors bissected, respecttively, by $d - \delta$ and $d + \delta$, which opening angles are both equal to $\frac{\pi}{h}$

$$V^{\pm} = \{x \in C | \ |\arg x - (d \pm \delta)| < \frac{\pi}{h},\ 0 < |x| < R\}$$

According to proposition 4.1.2, there exist functions f^+ and f^- such that

$$f^{\pm} \in (\mathcal{A}_{\frac{1}{h}}(V^{\pm}))^m, \quad Df^{\pm} = 0$$

and $f^{\pm}$ admit $\hat{f}$ as Gevery asymptotic expansion of order $\frac{1}{h}$ on $V^{\pm}$.

Moreover, the opening angle of the intersection $V^- \cap V^+$ is equal to $\frac{\pi}{h} - 2\delta$; the function $f^+ - f^-$ is solution of the equation $Dy = 0$ on $V^- \cap V^+$ and exponential decreasing of order k on this sector. Thus, according to the previous proposition, if we choose $\delta < \epsilon(d)$, then $f^+ - f^+ = 0$ on $V^- \cap V^+$. Thus, $f^{\pm}$ define an analytic function f on $V^- \cup V^+$ which is solution of $Dy = 0$ and which admits $\hat{f}$ as Gevrey asymptotic expansion of order $\frac{1}{h}$ on $V^- \cup V^+$; this means that $\hat{f}$ is h-summable in the direction d, its h-sum f beeing a solution of the equation.□

4.1.5 On the non k-summability of formal solutions

In this section, we shall give an example of non-k-summability of formal power series solutions. In fact, this example has already been given as an introduction to the Multisummability chapter. At this place, we have state some facts. Here, we shall prove them. We refer to [RS89].

Lemma 4.1.1 *Let $\hat{f}_1$ be the Euler power series*

$$\hat{f}_1 = \sum_{n=0}^{+\infty} (-1)^n n! x^{n+1}$$

Then,

1. *$\hat{f}_1$ is not convergent,*
2. *$\hat{f}_1 \in C[[x]]_1$ and $\hat{f}_1$ does not belong to $C[[x]]_s$ for any $s < 1$,*
3. *$\hat{f}_1$ is 1-summable but not k-summable for any $k \neq 1$,*
4. *$\hat{f}_1$ satistfies the linear differential equation $D_1\hat{f}_1 = x$ where $D_1 = x^2\frac{d}{dx} + 1$.*

Proof: The two first points are clear. Thus, $\hat{f}_1$ cannot be k-summable for any $k > 1$. Moreover, as previously shown, $\hat{f}_1$ is 1-summable.

Let us assume that $\hat{f}_1$ is k-summable for some $0 < k < 1$. Then, since it is not a convergent series, its set of singular direction $S(\hat{f}_1)$ is not empty. Let $S(\hat{f}_1) = \{\theta_1, \ldots, \theta_p\}$, with $0 \leq \theta_1 < \cdots < \theta_p < 2\pi$. Then, we define the sectors, for $1 \leq l \leq p$,

$$V_l = \{x \in C | \ \theta_{l-1} - \frac{\pi}{2k} < arg\, x < \theta_l + \frac{\pi}{2k},\ 0 < |x| < r\}$$

with $\theta_0 = \theta_p - 2\pi$.
Then, since $\hat{f}_1$ is k-summable on each direction $\frac{(\theta_l+\theta_{l-1})}{2}$, then there exist functions $f_l \in \mathcal{A}_{\frac{1}{k}}(V_l)$ such that $D_1 f_l = x$ on V_l and $Tf_l = \hat{f}_1$. Consequently, on $V_l \cap V_{l+1}$, we have

1. $D_1(f_{l+1} - f_l) = 0$
2. $f_{l+1} - f_l \in \mathcal{A}^{\leq -k}(V_{l+1} \cap V_l)$

Thus, we conclude that, on $V_{l+1} \cap V_l$, $f_{l+1} - f_l$ is an exponential decreasing function of order 1. But, the opening angle of $V_{l+1} \cap V_l$ is $\frac{\pi}{k} > \pi$, thus, according to the Watson's lemma, on $V_{l+1} \cap V_l$, $f_{l+1} - f_l = 0$. Consequently, we have found a function $f \in \mathcal{A}_{\frac{1}{k}}(V_{l+1} \cup V_l)$, such that $D_1 f = x$, and $Tf = \hat{f}_1$ on $V_{l+1} \cup V_l$ which contain a sector bissected by θ_l and which opening angle is greater than $\frac{\pi}{k}$. Hence, according to the definition, θ_l is not a singular direction and this is a contradiction.□

The following lemma is a consequence of the previous one:

Lemma 4.1.2 *Let $\hat{f}_2$ be the Euler power series where x has been changed by x^2:*

$$\hat{f}_2 \;=\; \sum_{n=0}^{+\infty} (-1)^n n! x^{2n+2}$$

Then,

1. *$\hat{f}_2$ is not convergent,*
2. *$\hat{f}_2 \in C[[x]]_{\frac{1}{2}}$ and $\hat{f}_2$ does not belong to $C[[x]]_s$ for any $s < \frac{1}{2}$,*
3. *$\hat{f}_2$ is 2-summable but not k-summable for any $k \neq 2$,*
4. *$\hat{f}_2$ satistfies the linear differential equation $D_2\hat{f}_2 = 2x^2$ where $D_2 = x^3\frac{d}{dx} + 2$.*

Using straightforward computation, we obtain the following result

Lemma 4.1.3 *Set*

$$D_0 \;=\; x^5(2-x)\frac{d^2}{dx^2} - x^2(2x^3 - 5x^2 - 4)\frac{d}{dx} + 2(x^2 - x + 2)$$

Then,

$$D_0 \;=\; \{x^3(2-x)\frac{d}{dx} + 2x^2 - 2x + 4\}D_1$$

and

$$D_0 \;=\; \{x^2(2-x)\frac{d}{dx} + x^2 - x + 2\}D_2$$

We shall give in the following theorem an example of non-k-summable solution

Theorem 4.1.3 *Set*

$$\hat{f} = \hat{f}_1 + \hat{f}_2 = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1} + \sum_{n=0}^{\infty} (-1)^n n! x^{2n+2}$$

and

$$D = (\frac{d}{dx})^5 D_0 = (\frac{d}{dx})^5\{x^5(2-x)\frac{d^2}{dx^2} - x^2(2x^3 - 5x^2 - 4)\frac{d}{dx} + 2(x^2 - x + 2)\}$$

Then, $D\hat{f} = 0$ and $\hat{f}$ is not k-summable for any $k > 0$.

Proof: It is clear that $\hat{f}$ is a Gevrey power series of order 1 and not of order lower than 1, and hence $\hat{f}$ is not k-summable for any $k > 1$. Moreover, $\hat{f}$ is not 1-summable, since $\hat{f}_1$ is 1-summable but $\hat{f}_2$ is not.

Let us assume that $\hat{f}$ is k-summable for some $0 < k < 1$. We shall proceed as in the lemma 4.1.1. Then, we have functions f_l defined on the sector V_l such that $Df_l = 0$ on V_l and $Tf_l = \hat{f}$. Consequently, on $V_l \cap V_{l+1}$, we have

1. $D(f_{l+1} - f_l) = 0$
2. $f_{l+1} - f_l \in \mathcal{A}^{\leq -k}(V_{l+1} \cap V_l)$

Thus, we conclude that, on $V_{l+1} \cap V_l$, $f_{l+1} - f_l$ is an exponentially decreasing function of order one of the positive slopes of the Newton polygon $N(D)$, that is 1 or 2. But, the opening angle of $V_{l+1} \cap V_l$ is $\frac{\pi}{k} > \pi > \frac{\pi}{2}$, thus, according to the Watson's lemma, on $V_{l+1} \cap V_l$, $f_{l+1} - f_l = 0$. Consequently, we have got a function $f \in \mathcal{A}_{\frac{1}{k}}(V_{l+1} \cup V_l)$, such that $Df = 0$, and $Tf = \hat{f}$ on $V_{l+1} \cup V_l$ which contain a sector bissected by θ_l and which opening angle is greater than $\frac{\pi}{k}$. Hence, according to the definition, θ_l is not a singular direction and this is a contradiction.□

4.1.6 Multisummability of formal solutions

Considering a differential equation of the form $Dy = G$ with $G \in (C\{x\}[x^{-1}])^m$, we have shown that, a formal solution $\hat{f}$ was Geverey of order $\frac{1}{h_l}$ where h_l was one of Newton polygon's $N(D)$ slopes; moreover, this value is precise.

As shown previously, $\hat{f}$ may not be k-summable in a direction for some $k > 0$. Nevertheless, we shall give a fondemental result concerning the multisummability of this formal solution.

Theorem 4.1.4 (Multisummability of formal solutions theorem) *[BBRS91]*

With the above notation, let us consider "closed" sectors at the origin, $V_1 \subset V_2 \subset \cdots \subset V_l$ which opening angle are, respectively, $\frac{\pi}{h_1}, \ldots, \frac{\pi}{h_l}$. We assume that the lowest and the greatest angle of each V_i $(1 \leq i \leq l)$ are not a Stokes direction of level h_i.

Then, $\hat{f}$ is $(h_l, h_{l-1}, \ldots, h_1)$-summable on $(V_l, \ldots, V_1)$.
More precisely, we have the decomposition

$$\hat{f} \;=\; \hat{f}_l + \cdots + \hat{f}_1$$

where, for all $1 \leq i \leq l$, $\hat{f}_i$ is h_i-summable on V_i and $D\hat{f}_i$ converges.

By induction, it is sufficient to prove the following proposition:

Proposition 4.1.5 *If a formal power series $\hat{f} \in (C[[x]]_{\frac{1}{h_l}})^m$ is such that $D\hat{f} \in (C\{x\}[x^{-1}])^m$, if d is not a Stokes direction of level h_l $h_l \geq \frac{1}{2}$, then there exist $\hat{g} \in (C[[x]]_{\frac{1}{h_{l-1}}})^m$ and $\hat{f}_l \in (C\{x\}_{h_l,d})^m$ such that $\hat{f} \;=\; \hat{g} + \hat{f}_l$ and $D\hat{g} \in (C\{x\}[x^{-1}])^m$.*

It is clear that the theorem follows immediately from that proposition. We shall prove the proposition by using the following lemma:

Lemma 4.1.4 *Let $\hat{f} \in (C[[x]]_{\frac{1}{h_l}})^m$ such that $D\hat{f} \in (C\{x\}[x^{-1}])^m$, d a direction is not a Stokes direction of level h_l. Let V be a "closed" sector which opening angle is $\frac{\pi}{h_l}$.*
Then, there exist

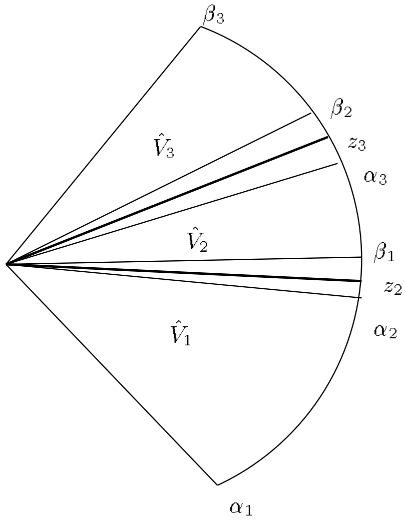
1. *a good covering $\{V_i\}_{1\leq i\leq n}$ of V by open sectors such that*
$$V_i = \{x \in C|\ \alpha_i < arg\ x < \beta_i,\ 0 < |x| < r\}$$
with $\alpha_i < \beta_{i-1} < \alpha_{i+1} < \beta_i$, if $1 \leq i \leq n-1$, and $\alpha_1 < d - \frac{\pi}{2h_l}$ and $d + \frac{\pi}{2h_l} < \beta_n$;
2. *functions $\{\phi_i\}$ such that*
   (a) *$\phi_i \in (\mathcal{A}_{\frac{1}{h_l}}(V_i))^m$,*
   (b) *$(\phi_i - \phi_j)_{V_i\cap V_j} \in (\mathcal{A}^{\leq -h_{l-1}}(V_i \cap V_j))^m$ if $V_i \cap V_j \neq \emptyset$,*
   (c) *$T\phi_i = \hat{f}$ and $D\phi_i = D\hat{f}$.*

Proof of proposition 4.1.5 using lemma 4.1.4

We shall proceed in the same way as we did in the proof of the Ramis-Sibuya theorem. For each $2 \leq i \leq n$, let us fix a point z_i in the interior of $V_i \cap V_{i-1}$. We assume that all the z_i are at equal distance from the origin, i.e $|z_j| = |z_i| = r$. We denote by $\hat{L}_i$ the segment joining 0 to z_i. These segments divide the intersection of the punctured disk of radius r and $\cup V_i$ into n open sectors $\hat{V}_1, \ldots, \hat{V}_n$ with
$$\hat{V}_i \ = \ \{x \in C|\ arg\ z_i < arg\ x < arg\ z_{i+1},\ 0 < |x| < r\}$$
if $2 \leq i \leq n-1$; and
$$\begin{aligned}\hat{V}_n \ &= \ \{x \in C|\ arg\ z_n < arg\ x < \beta_n,\ 0 < |x| < r\}\\ \hat{V}_1 \ &= \ \{x \in C|\ \alpha_1 < arg\ x < arg\ z_2,\ 0 < |x| < r\}\end{aligned}$$

Then, we define n functions ψ_i, defined respectively on each $\hat{V}_i$ by
$$\psi_i(x) \ = \ \frac{-1}{2i\pi}\sum_{k=2}^{n}\int_{\hat{L}_k}\frac{\phi_k(\xi)-\phi_{k-1}(\xi)}{\xi - x}d\xi$$
which is holomorphic on $\hat{V}_i$. By deforming $\hat{L}_i$ and $\hat{L}_{i-1}$(resp. $\hat{L}_2$),(resp. $\hat{L}_n$) without moving both endpoints, we can continue analytically ψ_i(resp. ψ_1), (resp. ψ_n) onto V_i(resp. V_1),(resp. V_n).
Concerning the asymptotic proprieties, we proceed exactly as in the proof of the Ramis-Sibuya theorem. We leave the details to the reader. Thus, we have obtained the desired result: for all i, $\psi_i \in (\mathcal{A}_{\frac{1}{h_{l-1}}}(V_i))^m$.

Let us compute the difference $\psi_i - \psi_{i-1}$ on $V_i \cap V_{i-1}$. In order to do this, we write these functions in the following way
$$\begin{aligned}\psi_i(x) \ &= \ \frac{-1}{2i\pi}\int_{L_i}\frac{\phi_i(\xi)-\phi_{i-1}(\xi)}{\xi - x}d\xi \ + \ \frac{-1}{2i\pi}\sum_{k\neq i}\int_{\hat{L}_k}\frac{\phi_k(\xi)-\phi_{k-1}(\xi)}{\xi - x}d\xi\\ \psi_{i-1}(x) \ &= \ \frac{-1}{2i\pi}\int_{L_{i-1}}\frac{\phi_i(\xi)-\phi_{i-1}(\xi)}{\xi - x}d\xi \ + \ \frac{-1}{2i\pi}\sum_{k\neq i}\int_{\hat{L}_k}\frac{\phi_k(\xi)-\phi_{k-1}(\xi)}{\xi - x}d\xi\end{aligned}$$



where L_i and L_{i-1} are both paths obtained by deforming $\hat{L}_i$ without moving its endpoints. We construct these paths such that they are both contained in the intersection $V_i \cap V_{i-1}$, and the path $L_{i-1} - L_i$ is a simple closed curve which interior contain the point z.
By substracting the two expressions above, we obtain

$$\psi_i(x) - \psi_{i-1}(x) \;=\; \frac{1}{2i\pi}\int_{-\hat{L}_i+\hat{L}_{i-1}} \frac{\phi_i(\xi) - \phi_{i-1}(\xi)}{\xi - x} d\xi \;=\; \phi_i(x) - \phi_{i-1}(x)$$

Thus, on $V_i \cap V_{i-1}$, we have $\psi_i - \psi_{i-1} = \phi_i - \phi_{i-1}$. Moreover, we have $D\psi_i \in (C\{x\}[x^{-1}])^m$.

According to the equality found on the intersection $V_i \cap V_{i-1}$, we have

$$\phi_i - \psi_i = \phi_{i-1} - \psi_{i-1}$$

Then, we define the function ϕ by $\phi = \phi_i - \psi_i$ on V_i. Thus, we have $\phi \in (\mathcal{A}_{\frac{1}{h_l}}(\cup V_i))^m$. Let us set $\hat{f}_l = T\phi$, where T is the Taylor jet application. Since, the opening angle of the sector $\cup V_i$ is greater than $\frac{\pi}{h_l}$, then $\hat{f}_l$ is h_l-summable in the direction d, i.e $\hat{f}_l \in (C\{x\}_{h_l,d})^m$. Since $T\phi_i = \hat{f}$, $T\psi_i \in (C[[x]]_{\frac{1}{h_{l-1}}})^m$ and $\phi_i = \phi + \psi_i$, we obtain expected result.□

We shall prove the lemma we have used.

<u>Proof of the lemma 4.1.4</u>

We can assume, without any loss of generality, that the Newton polygon's slopes of D are integers. It is known that there exists an $m \times m$-matrix $P(x)$ such that

1. $P(x) \in GL_m(C[[x]]_{\frac{1}{h_{l-1}}}[x^{-1}])$

2. $N(x) = M^P(x) = P(x)^{-1}M(x)P(x) - P(x)^{-1}\frac{dP(x)}{dx} = diag(N_1(x), N_2(x))$ where N_1 and N_2 satisfy the following conditions:

   (a) $N_1 \in GL_{m_1}(C\{x\}[x^{-1}])$, $N_2 \in GL_{m_2}(C\{x\}[x^{-1}])$, with $m_1 + m_2 = m$,

(b) The positive slopes of the Newton polygone of the differential operator $\mathcal{D}_1 = \frac{d}{dx} - N_1(x)$ are $h_1, h_2, \ldots, h_{l-1}$,

(c) The positive slopes of the Newton polygone of the differential operator $\mathcal{D}_2 = \frac{d}{dx} - N_2(x)$ are $h_l, h_{l+1}, \ldots, h_q$.

Let us consider the differential operator $\mathcal{D} = \frac{d}{dx} - N(x)$ and the formal power series $\hat{g}$ defined by $\hat{f}(x) = P(x)\hat{g}(x)$. Then, according to our assumptions, we have $\hat{g} \in (C[[x]]_{\frac{1}{h_l}}[x^{-1}])^m$. But, we have $D\hat{f} = P(x)\mathcal{D}\hat{g}$, thus, $\mathcal{D}\hat{g} \in (C[[x]]_{\frac{1}{h_{l-1}}}[x^{-1}])^m$. Since the application

$$\mathcal{D} \ : \ (C[[x]]_{\frac{1}{h_{l-1}}}[x^{-1}])^m \to (C[[x]]_{\frac{1}{h_{l-1}}}[x^{-1}])^m/(C\{x\}[x^{-1}])^m$$

is surjective, according to corrolary 4.1.1, then there exist a formal power series $\hat{u} \in (C[[x]]_{\frac{1}{h_{l-1}}}[x^{-1}])^m$ such that
$\mathcal{D}(\hat{g} + \hat{u}) \in (C\{x\}[x^{-1}])^m$. We set $\hat{v} = \hat{g} + \hat{u}$. We write it under the form $\hat{v} = [\hat{v}_1, \hat{v}_2]^t$, with $\hat{v}_1 \in (C[[x]]_{\frac{1}{h_l}}[x^{-1}])^{m_1}$ and $\hat{v}_2 \in (C[[x]]_{\frac{1}{h_l}}[x^{-1}])^{m_2}$. We notice that $\hat{f} = P(x)\hat{v} - P(x)\hat{u}$.
Since we have $\mathcal{D}u = -P(x)^{-1}D\hat{f} + \mathcal{D}v$, we have $\mathcal{D}_1(\hat{v}_1) \in (C\{x\}[x^{-1}])^{m_1}$ and $\mathcal{D}_2(\hat{v}_2) \in (C\{x\}[x^{-1}])^{m_2}$. Thus, according to the proposition 4.1.3, we have $\hat{v}_1 \in (C[[x]]_{\frac{1}{h_{l-1}}}[x^{-1}])^{m_1}$. Moreover, according to the theorem on summability of solutions 4.1.2, we have, since d is not a singular direction of level h_l, $\hat{v}_2 \in (C\{x\}_{h_l,d}[x^{-1}])^{m_2}$.
Consequently, there exist a good covering $\{V_i\}_{1\leq i\leq n}$ as defined in the statement of the lemma, and $m \times m$-matrix $\{P_i\}_{1\leq i\leq n}$, functions $\{u_i\}_{1\leq i\leq n}$, $\{V_{1,i}\}_{1\leq i\leq n}$ and v_2 such that

1.
$$\begin{aligned} &P_i \in M_m(\mathcal{A}_{\frac{1}{h_{l-1}}}(V_i)), \qquad u_i \in (\mathcal{A}_{\frac{1}{h_{l-1}}}(V_i))^m, \\ &v_{1,i} \in (\mathcal{A}_{\frac{1}{h_{l-1}}}(V_i))^{m_1}, \qquad v_2 \in (\mathcal{A}_{\frac{1}{h_{l-1}}}(\cup V_i))^{m_2}, \end{aligned}$$

2. $TP_i = \hat{P}$, $Tu_i = \hat{u}$, $Tv_{1,i} = \hat{v}_1$, $Tv_2 = \hat{v}_2$,

3.
$$\begin{aligned} \frac{dP_i}{dz} &= MP_i - P_iN \text{ on } V_i, \\ \mathcal{D}u_i &= -P_iD\hat{f} + \mathcal{D}\hat{v} \text{ on } V_i, \\ \mathcal{D}_1 v_{1,i} &= \mathcal{D}_1\hat{v}_1 \text{ on } V_i, \\ \mathcal{D}_2 v_2 &= \mathcal{D}_2\hat{v}_2 \text{ on } \cup V_i. \end{aligned}$$

We set $v_i = [v_{1,i}, v_2]^t$ and $f_i = P_iv_i - P_iu_i$ on V_i. Then, we have $f_i \in (\mathcal{A}_{\frac{1}{h_l}}(V_i))^m$ and $Tf_i = \hat{P}\hat{v} - \hat{P}\hat{u} = \hat{f}$. Moreover, we have $f_i - f_{i-1} = (P_i - P_{i-1})v_i + P_{i-1}(v_i - v_{i-1}) - (P_iu_i - P_{i-1}u_{i-1})$; thus, $f_i - f_{i-1} \in (\mathcal{A}^{\leq -h_{l-1}}(V_i \cap V_{i-1}))^m$. At last, we have $Df_i = P_i\mathcal{D}(v_i - u_i) = P_i\mathcal{D}\hat{v} + D\hat{f} - P_i\mathcal{D}v_i = D\hat{f}$. This end the proof a the lemma.□

4.2 Non-linear differential equations

In this section, we shall be concerned with the properties of formal solutions of a differential equation of the form $G(x, y, y', \ldots, y^{(n)}) = 0$ where y is the unknown, and G, an analytic function. We shall set the assumptions more precisely in the sequel.

First, we assume that there is a formal power series $\hat{f}$ which satisfy such an equation. Is there a "real" function which satisfy the equation and which admit $\hat{f}$ as asymptotic expansion? Furthermore, could such a formal solution be anything, or, for instance, is it a Gevrey formal power series of a certain order?

We shall try to answer these questions which arosed one century ago. The first one concerns the existence of "real" solution of differential equation. The answer is provided by the fondemental theorem

Theorem 4.2.1 (Asymptotic expansion's fundamental theorem) *[RS89]*

Let $G(x, Y_0, \ldots, Y_n)$ be a holomorphic function of $n+2$ variables and let $\hat{f}$ be a formal power series solution of the differential equation $G(x, y, y', \ldots, y^{(n)}) = 0$.
Then, there exists a real number $k > 0$, such that, for all open sector V at the origin of the complex plane, which opening angle is less than $\frac{\pi}{k}$, and of radius sufficently small, there exists an analytic function f on V solution of the equation and, which admit $\hat{f}$ as asymtpotic expansion in V.

This result, under that presentation, that is, without any restrictive assumptions, is relatively recent and is due to Ramis and Sibuya. The first particular cases where established by Poincaré.
This theorem is the key point for further investigations as we shall see. The next question is whether a formal power series solution has its coefficients compelled to follow a growth condition. An answer was first given by Maillet:

Theorem 4.2.2 (Maillet's theorem) *Let $G(x, Y_0, \ldots, Y_n)$ be a holomorphic function of $n+2$ variables and let $\hat{f}$ be a formal power series solution of the differential equation $G(x, y, y', \ldots, y^{(n)}) = 0$.*
Then, there exists a real number $k > 0$, such that, $\hat{f}$ is a Gevrey formal power series of order $\frac{1}{k}$.

Maillet as proved this theorem by using direct estimations. The result is far from beeing optimal, that is, the number $\frac{1}{k}$ found is not the smallest possible. In order to inprove the estimation of the Gevrey order of the formal solution $\hat{f}$, we introduce the linear differential equation L which is the linearized of $G(x, Y_0, \ldots, Y_n)$ along $\hat{f}$. Let us consider the Newton polygon $N(L)$ of L. Malgrange has first shown, in his paper [Mal89], that the formal solution $\hat{f}$ was Gevrey of order $\frac{1}{k'}$, k' beeing the smallest slope of $N(L)$. This result was an improvement comparing to what was found by Maillet. A more precise theorem had been given by Ramis and Sibuya:

Theorem 4.2.3 *[Sib91]*

Let $\hat{f}$ be a formal power series solution of the differential equation $G(x, y, y', \ldots, y^{(n)}) = 0$, where $G(x, Y_0, \ldots, Y_n)$ is a analytic function of $n+2$ variables. The above equation can be rewritten in the following form

$$\hat{G}(x, y, \delta y, \ldots, \delta^n y) = L(y) + x^m g(x, y, \delta y, \ldots, \delta^n y) = 0 \tag{4.1}$$

where $\delta = x\frac{d}{dx}$ *and* $L = \sum_{k=0}^{n} a_k(x)\delta^k$.
Furthermore, we can assume, without any lost of generality that:

1. *the functions* a_k*'s are analytic and* $a_n \neq 0$
2. *the integer* m *is such that, for any linear differential operator* R *of order smaller than* n*, the Newton polygon* $N(L + x^m R)$ *is the same as the Newton polygon* $N(L)$.

Then, if $\hat{f}$ *is a formal solution of the equation 4.1, we have to cases:*

1. *if the only slope of* $N(L)$ *is* 0*:*
   The series $\hat{f}$ *is a convergent power series.*
2. *if* $N(L)$ *has positive slopes:*
   $\hat{f}$ *is a Gevrey power series of order* $\frac{1}{h}$ *where* h *is one of the Newton polygon's slopes of* L.

Proof:
According to the Asymptotic expansion's fundemental theorem 4.2.1, there exist a good covering $\{V_i\}_{1\leq i\leq n}$ of a punctured disk and functions $\{f_i\}_{1\leq i\leq n}$ such that, for all $1 \leq i \leq n$,

1. $f_i \in \mathcal{A}(V_i)$
2. $Tf_i = \hat{f}$
3. f_i is solution of the equation 4.1 on V_i.

We shall estimate, on each non-empty intersection $V_i \cap V_j$, the order of exponential decrease of the function $f_{i,j} = f_i - f_j$.
On $V_i \cap V_j$, we apply the Taylor's formula with integral rest to $\hat{G}$; we obtain:

$$\begin{aligned} 0 &= \hat{G}(x, f_i, \delta f_i, \ldots, \delta^n f_i) + L(x, f_{j,i}, \delta f_{j,i}, \ldots, \delta^n f_{j,i}) - \hat{G}(x, f_j, \delta f_j, \ldots, \delta^n f_j) \\ &\quad + x^m \sum_{k=0}^{n} (\int_0^1 \frac{\partial g}{\partial Y_k}(x, f_i, \ldots, \delta^{(k-1)} f_i, (1-t)\delta^k f_i + t\delta^k f_j, \delta^{(k+1)} f_j, \ldots, \delta^n f_j)dt)\delta^k f_{j,i} \end{aligned}$$

Since, f_i and f_j are solutions of the differential equation, $f_{j,i}$ is solution of the linear differential equation

$$(L + x^m \mathcal{L})y = 0$$

where $L = L(x, \delta^0, \delta, \ldots, \delta^n)$ and

$$\mathcal{L} = \sum_{k=0}^{n} (\int_0^1 \frac{\partial g}{\partial Y_k}(x, f_i, \ldots, \delta^{(k-1)} f_i, (1-t)\delta^k f_i + t\delta^k f_j, \delta^{(k+1)} f_j, \ldots, \delta^n f_j)dt)\delta^k$$

1. if 0 is the only slope of L
   Then, $f_{i,j} = 0$ since the only slope of $L + x^m\mathcal{L}$ is 0, that is $f_i = f_j$ on $V_i \cap V_j$. According to the Riemann vanishing singularities theorem, the f_i's define a function f wich is analytic in a neighbourhood of the origin solution of the equation and such that $f = f_i$ on V_i. Thus, $\hat{f} = Tf$ is a convergent power series.

2. if the Newton polygon of L has positive slopes
   Since we have $f_{i,j} \in \mathcal{A}^{<0}(V_i \cap V_j)$, according to the proprieties vefified by the f_i's, then $f_{i,j} \in \mathcal{A}^{\leq -h}(V_i \cap V_j)$ where h is one of the slopes of the Newton polygon $N(L + x^m \mathcal{L})$. But, according to our assuptions, we have $N(L) = N(L + x^m \mathcal{L})$. Thus, $f_{i,j}$ is an exponential decreasing function of order h one of the slopes of the Newton polygon of L. Consequently, according to the Ramis-Sibuya theorem, $f_i \in \mathcal{A}_{\frac{1}{h}}(V_i)$ and $\hat{f} \in C[[x]]_{\frac{1}{h}}$.

This end the proof of the theorem.$\square$

Chapter 5

Normal forms of diffeomorphisms

Let us consider, for the moment, the case of holomorphic local diffeomorphisms at the origin of the complex plane. Such a diffeomorphism $f\ :(U,0)\to(V,0)$, where both U and V are neighbourhood of the origin in the complex plane, is defined by a convergent series

$$f(x)\ =\ \lambda x+\sum_{n=1}^{+\infty}a_nx^n\quad\in xC\{x\}$$

where $\lambda\in C^*$.

For $|\lambda|\neq 1$, Poincaré has shown that such a diffeomorphism was analytically conjugated, at the origin, to its linear tangent application $T_0f\ :x\mapsto\lambda x$. In other words, there exists an analytic change of coordinates $x=\psi(t)$ ($\psi'(0)\neq 0$) such that $f=\psi\circ T_0f\circ\psi^{-1}$.

We shall be concerned with diffeomorphisms which "linear part" is a root of the unity. We shall, first, single out the formal conjugacy classes, then the analytic classification.

Following [RM83], we shall denote by F(resp. $\hat F$) the group of holomorphic local(resp. formal) diffeomorphisms at the origin; an element $\hat f\in\hat F$ will called **resonant** if $\hat f(x)=\nu x+\cdots$ where ν is a root of the unity.

We shall denote by $G\subset F$(resp. $\hat G\subset\hat F$) the sub-group of diffeomorphism for which $\nu=1$, that is, which are tangent to the identity.

5.1 Formal classification of resonant diffeomorphisms

In this section, we shall investigate the formal classification of diffeomorphisms. This mean that we shall find out a class of diffeormorphisms, called normal forms, such that each diffeomorphism is conjuguated, by a formal diffeomorphism, to one of these normal forms.

5.1.1 Conjugacy classes of $\hat G$ by $\hat G$

Let $\hat{\mathcal{X}}$ be the Lie algebra of complex formal vector fields which are 1-flat at the origin, that is, which are of the form

$$\hat X=\hat f(x)\frac{d}{dx}\text{ where }\hat f\in C[[x]]\text{ with }\hat f(0)=\hat f'(0)=0.$$

We recall, without proof, the following proposition, which enables us to say that conjugacy classes of $\hat G$ by $\hat G$ correspond to equivalenc classes of $\hat{\mathcal{X}}$ by $\hat G$.

Proposition 5.1.1 *The exponential map exp :* $\hat{\mathcal{X}} \to \hat{G}$ *is one-to-one.*

We shall work with vector fields and find out the equivalent classes by the adjoint action of $\hat{g}$.

Proposition 5.1.2 *Each non-null vector field* $\hat{X} \in \hat{\mathcal{X}}$ *is equivalent, by* $\hat{G}$*, to one and only one of the following vector fields, called normal forms:*

$$X_{\beta,k,\lambda} = \beta \frac{x^k}{1+\lambda x^k} x \frac{d}{dx}$$

where $\beta \in C^*$*,* k *is a positive integer, and* $\lambda \in C$*.*

Proof: Let $\hat{X} = \hat{f}(x)\frac{d}{dx}$ be an element of $\hat{\mathcal{X}}$. It is equivalent to $X_{\beta,k,\lambda}$ by $\hat{G}$, if and only if there exists $\hat{g} \in \hat{G}$, such that

$$\hat{f}(x)\frac{d\hat{g}}{dx} = \beta \frac{\hat{g}^k}{1+\lambda \hat{g}^k}\hat{g}$$

that is

$$(\frac{1}{\hat{g}^{k+1}} + \frac{\lambda}{\hat{g}})d\hat{g} = \beta \frac{dx}{\hat{f}}$$

After integrating and setting $\hat{g}(x) = x\hat{\phi}$ with $\hat{\phi} \in C[[x]]$ and $\hat{\phi}(0) = 1$, we set $\hat{u} = \hat{\phi}^k$; then, we obtain

$$\lambda x^k \hat{u} \ln \hat{u} - 1 + kx^k \hat{u}(\lambda \ln x - \beta \int \frac{dx}{\hat{f}(x)}) = 0$$

Moreover, we can write

$$\frac{1}{\hat{f}(x)} = \frac{1}{\beta'}(\frac{1}{x^{k'+1}} + \frac{a_1}{x^{k'}} + \cdots + \frac{\lambda'}{x} + \cdots)$$

Thus,we can write

$$G(\hat{u}, x) = \lambda x^k \hat{u} \ln \hat{u} - 1 + kx^k \hat{u}((\lambda - \frac{\beta}{\beta'}\lambda') \ln x - \beta \hat{H}) = 0$$

where

$$\hat{H}(x) = \frac{1}{\beta'}(-\frac{1}{k' x^{k'}} - \frac{a_1}{(k'-1)x^{k'-1}} - \cdots - \frac{a_{k'-1}}{x} + a_{k+1}x + \cdots).$$

But, we want a solution which satisfies $\hat{u}(0) = 1$. This could be possible if and only if $k = k'$, $\beta = \beta'$, and $\lambda - \frac{\beta}{\beta'}\lambda' = 0$, that is, $\lambda = \lambda'$. The differential of G with respect to $\hat{u}$ at the point $(1, 0)$ is the identity. Thus, it is invertible and, by the formal implicit function theorem, the equation $G(\hat{u}, x) = 0$ admits a unique solution $\hat{u}(x)$ such that $\hat{u}(0) = 1$. This complete the proof of the proposition.□

The proposition 5.1.1 and the previous one allow us to find out the conjugacy classes of $\hat{G}$:

Corollary 5.1.1 *Each element of* $\hat{G}$*, different from identity, is conjugated, by an element of* $\hat{G}$*, to one and only one of the diffeomorphisms* $\exp X_{\beta,k,\lambda}$*, called normal forms.*

5.1.2 Conjugacy classes of $\hat{G}$ by $\hat{F}$

The group $\hat{F}$ is obtained, from $\hat{G}$, by adding the applications $x \mapsto ax (a \neq 0)$. Such an application transforms the vector field $X_{\beta,k,\lambda}$ into the vector field $X_{\beta',k,\lambda'}$ with $\beta' = \frac{\beta}{a^k}$ and $\lambda' = \frac{\lambda}{a^k}$. Consequently, every $\hat{X} \in \hat{\mathcal{X}}$, non-null, is equivalent, by the adjoint action of $\hat{F}$ to a unique vector field $X_{2i\pi,k,\lambda}$. The reason of the normalization $\beta = 2i\pi$ will appear in the next section.

In the same way, every diffeomorphism $\hat{g} \in \hat{G}$, different from Identity, is conjugated, by an element of $\hat{F}$, to a unique diffeomorphism $g_{2i\pi,k,\lambda} = \exp X_{2i\pi,k,\lambda} = \exp$.

In the sequel, we shall set

$$X_{k,\lambda} = X_{2i\pi,k,\lambda} \text{ and } g_{k,\lambda} = g_{2i\pi,k,\lambda}$$

5.1.3 Conjugacy classes of formal resonant diffeomorphisms

Let $\hat{f} \in \hat{F}$ be a formal resonant diffeomorphism, that is, $\hat{f}(x) = \nu x + \cdots$, where $\nu = e^{\frac{2i\pi p}{q}}$, p and q beeing relatively prime, $q \geq 2$.

Proposition 5.1.3 *The diffeomorphism admits the following unique decomposition* $\hat{f} = \hat{\sigma} \circ \hat{n}$ *where*

1. $\hat{\sigma} \in \hat{F}$ *and* $\hat{\sigma}^q = \underbrace{\hat{\sigma} \circ \cdots \circ \hat{\sigma}}_{q \text{ times}} = Identity,$

2. $\hat{n} \in \hat{G}$,

3. $\hat{\sigma}$ *and* $\hat{n}$ *commute.*

Proof: Let $\hat{g} = \hat{f}^q = \underbrace{\hat{f} \circ \cdots \circ \hat{f}}_{\text{q times}}$. Hence, $\hat{g} \in G$, and by proposition 5.1.1, there exists $\hat{X} \in \hat{\mathcal{X}}$ such that $\hat{g} = \exp \hat{X}$. Since, $\hat{f}$ and $\hat{g}$ commute, then, $\hat{f} * \hat{X} = \hat{X}$. Thus, $\hat{n} = \exp \frac{\hat{X}}{q}$ commutes with $\hat{f}$. If we set $\hat{\sigma} = \hat{f} \circ \hat{n}$, thern we have the desired proprieties. Moreover, the u nicity comes from the fact that $\hat{n} = \exp \frac{\hat{X}}{q}$ is the unique element of $\hat{G}$ such that $\hat{n}^q = \hat{g}$. $\square$

Corollary 5.1.2 *Let* $\hat{f} \in \hat{F}$ *be a formal resonant diffeomorphism,* $\hat{f}(x) = e^{\frac{2ip\pi}{q}} x + \cdots$. *Then,* $\hat{f}$ *is conjugated, in* $\hat{F}$, *to one and only one of the following elements:*

1. $\sigma(x) = e^{\frac{2ip\pi}{q}} x$ *if* $\hat{f}^q = Identity,$

2. $\sigma \circ g_{k,\lambda}$ *where* $g_{k,\lambda}$ *is one of the normal forms defined previously,* k *beeing a multiple of* q.

5.2 Analytic classification of resonant diffeomorphisms

In this section, we shall intend to find out the invariants which caracterize the conjugacy classes, modulo F, of analytic resonant diffeomorphisms. The definition of these invariants depends on the conjugacy classe, modulo $\hat{F}$, caracterized by one of the normal forms obtained in the previous section. After setting some notations and stating a fondemental result, we shall study in detail the case of the normal form $g_{0,1}$.

5.2.1 A non-commatative Ramis-Sibuya theorem

Let V be an open sector at the origin. Let $G^k(V) \subset \mathcal{A}(V)$ be the set of analytic function g on V which admit an asymptotic expansion of the form

$$\hat{g} = Tg = x + \sum_{n \geq k+1} a_n x^n$$

By the implicit function theorem, $G^k(V)$ is a group under the (non-commutative) composition law of application. We shall denote by $G^{<0}(V)$, the intesection of all $G^k(V)$, that is the set of analytic function on V which are infinitely tangent to Identity. We shall denote the set $G^1(V)$ by $G(V)$. Moreover, we shall denote by $G^{<-k}(V)$ the set of function g such that $g - Id \in \mathcal{A}^{<-k}(V)$; this means that g is equal to Identity, up to an exponential decreasing function of order $\geq k$.

We can state a theorem, due to Malgrange, and which is fundemental in the sequel. This result can be regarded, somehow, as a non-commutative version of the fundemental Ramis-Sibuya theorem.

Theorem 5.2.1 *[Mal82][RM82] Let $\{V_i\}_{1 \leq i \leq n}$ be a good covering of a punctured disk by open sectors. Let $\{f_{i,j}\}$ be functions such that $f_{i,j} \in G^{<0}(V_i \cap V_j)$, if $V_i \cap V_j \neq \emptyset$. Moreover, we assume that $f_{i,j} = f_{j,i}^{-1}$.*

Then, there exist functions $f_i \in G(V_i)$, $i = 1 \dots n$, such that

1. *the f_i's have the same asymptotic expansion,*
2. *if $V_i \cap V_j \neq \emptyset$, then $f_i \circ f_j^{-1} = f_{i,j}$ on $V_i \cap V_j$.*

Moreover, let $\{V_i\}$ be a good covering of a punctured disk, $\{f_i\}$ and $\{g_i\}$ be functions such that $(f_i, g_i) \in G(V_i)^2$, the f_i's admitting a same asymtpotic expansion $\hat{f}$ and the g_i's admitting a same asymtpotic expansion $\hat{g}$. Then, for all (i,j) such that $V_i \cap V_j \neq \emptyset$, there exists $h_{i,j}$ such that $h_{i,j} \circ (g_i \circ g_j^{-1}) \circ h_{i,j}^{-1} = f_i \circ f_j^{-1}$ if and only if, there exists $h \in G$ such that $\hat{f} = f \circ \hat{g}$.

5.2.2 Analytic classification of diffeomorphism formally conjugated to $g_{1,0}$

The solution $\phi(t,x)$ passing through x at $t = 0$ of the differential equation $\frac{du}{dt} = X_{1,0} = 2i\pi u^2$ is

$$\phi(t,x) = \frac{x}{1 - 2i\pi t x}.$$

The "phase portrait" of the orbits of $X_{1,0}$(corresponding to real times) is drawn on the figure below.

Thus, the diffeomorphism $g = g_{1,0} = \exp X_{0,1} = \phi(1,x)$ can be written

$$g(x) = \frac{x}{1 - 2i\pi x}.$$

The key point is that the function $H(x) = \exp \frac{1}{x}$ is constant along the orbits of g, that is

$$H(g^{(n)}(x)) = H(g^{(n-1)}(x)) = \dots = H(x) = \dots = H((g^{(-n)}(x))$$

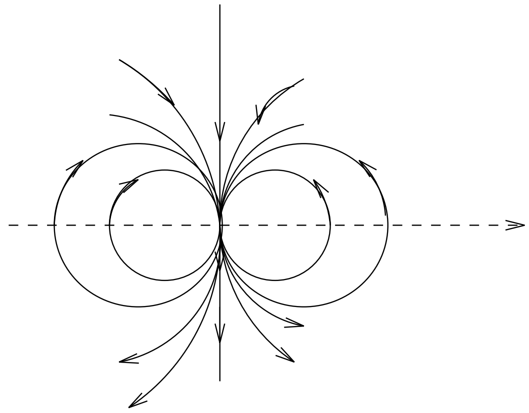


Figure 5.1: Orbits of the flow of $X_{1,0}$

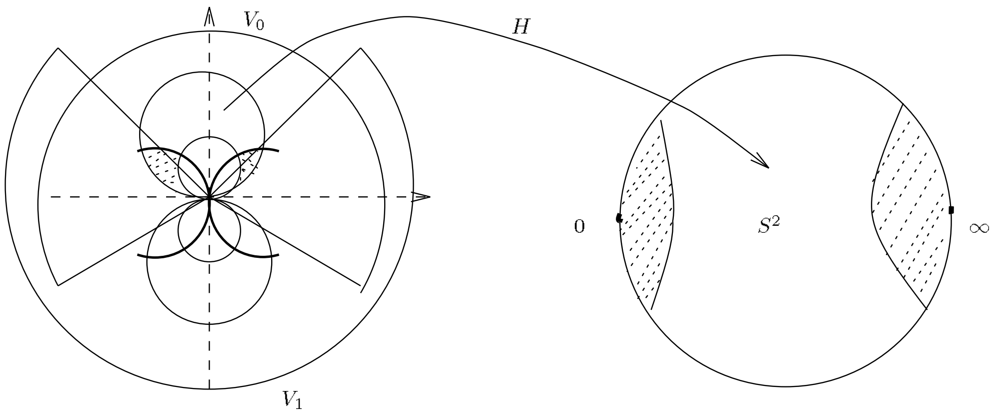


Figure 5.2: Isomorphisms between orbit spaces and S^2

where $g^{(n)} = \underbrace{g \circ \cdots \circ g}_{n\ times}$ and $g^{(-n)} = \underbrace{g^{-1} \circ \cdots \circ g^{-1}}_{n\ times}$.

Let $V_0 = V_{-i}(R,\theta)$(resp. $V_1 = V_i(R,\theta)$) the closed sector bissected by the direction $-i$(resp. i) with radius $R > 0$ and which opening angle is $\pi < 2\theta < 2\pi$. These two sectors gives a covering of the closed disk centered at the origin which radius is R. The application $H : V_0 \setminus \{0\} \to C^*$(resp. $H : V_1 \setminus \{0\} \to C^*$) is onto. Moreover, two points $x, x' \in V_0$(resp. V_1) belong to the same orbit if and only if $H(x) = H(x')$. This fact means that the function H "identifies" the quotient spaces V_0/g and V_1/g to the Riemann sphere $S^2 = C \cup \infty$(the points 0 and ∞ representing simultanously the orbit $\{0\}$ of g).

d beeing a given direction, we shall study the elements of G_d which commute with g, .

1. $Re(d) < 0$

   Let U be a "small" sector bissected by d, that is, which all directions have a negative real part. Then, $H \in \mathcal{A}^{\leq -1}(U)$, and the image of U by H is a neighbourhood of

$0 \in S^2$. Moreover, H is a local diffeomorphism at every point but 0. Consequently, $f \in G(V)$ commutes with g if and only if there exists a local analytic diffeomorphism ϕ of S^2 at 0 such that

$$H \circ f = \phi \circ H \tag{5.1}$$

In fact, if such an equation is satisfied then $f \circ g = H^{-1} \circ \phi \circ H \circ g$ and $g \circ f = g \circ H^{-1} \circ \phi \circ H$. But, $H \circ g = H$ and $g \circ H^{-1} = H^{-1}$; thus, we have $g \circ f = H^{-1} \circ \phi \circ H = f \circ g$. Conversely, if f commutes with g, then $H \circ g \circ f = H \circ f \circ g$, thus $H \circ f = \underbrace{H \circ f \circ H^{-1}}_{\phi} \circ H$.

We shall solve the previous equation. For that purpose, we set $\phi(y) = ye^{a+\gamma(y)}$ and $f(x) = \frac{x}{1+r(x)}$, where $\gamma \in C\{y\}$, $\gamma(0) = 0$ and $r \in \mathcal{A}(U)$. Thus, the equation 5.1 can be written

$$\frac{1+r(x)}{x} = \frac{1}{x} + a + \gamma(e^{\frac{1}{x}})$$

that is

$$f(x) = \frac{x}{1 + ax + x\gamma(e^{\frac{1}{x}})}$$

This expression defines an element of $G(U)$; if $a = 0$, that is ϕ tangent to the identity at $0 \in S^2$, then $f \in G^{<0}(U)$. If $\gamma = 0$, then f is an element of the 1-parameter group generated by $X_{1,0}$.

2. $Re(d) > 0$

   This case is similar to the previous one if we change neighbourhoods of 0 into neighbourhoods of $\infty \in S^2$.

3. $Re(d) = 0$

   This time the image by H of any sector bissected by d is the sphere S^2. The equation 5.1 admits a solution $f \in G(U)$ only if ϕ is a global diffeomorphism of S^2, fixing the points 0 and ∞. Consequently f can only be an element of the 1-parameter group of $X_{1,0}$, that is $\exp tX_{1,0}$.

The previous description can be summarized by the fact that the group $G_g^{<0}(U)$ of elements of $G^{<0}(U)$, where U is any sector which directions have negative(resp. positive) real part, commuting with g, is isomorphic to the group of local diffeomorphisms of S^2 which are tangent to Identity, at 0(resp. ∞). Moreover, the group $G_g^{<0}(U)$ of elements of $G^{<0}(U)$, where U is bissected by i(resp. $-i$), commuting with g, is reduced to the Identity.

Let $\mathcal{U}$ the good covering of the disk D_r det from the open sets $U_0 = D_r \setminus \{i\}$ and $U_1 = D_r \setminus \{-i\}$. Let us set

$$U_+ = \{x \in D_r | \; Re(x) > 0\} \text{ and } U_- = \{x \in D_r | \; Re(x) < 0\}.$$

Let

$$f = (f_-, f_+) \in G_g^{<0}(U_-) \times G_g^{<0}(U_+)$$

According to the fundamental theorem, there exists $(h_0, h_1) \in G(U_0) \times G(U_1)$ such that

$$f_- = h_1^{-1} \circ h_0 \text{ on } U_- \text{ and } f_+ = h_0^{-1} \circ h_1 \text{ on } U_+$$

and $Th_0 = Th_1$, where T denotes the Taylor jet application. We shall write $\hat{f}_0$ instead of Tf_0. But, both f_- and f_+ commute with g; thus,

$$h_0 \circ g \circ h_0^{-1} = h_1 \circ g \circ h_1^{-1} \text{ on } U_0 \cap U_1$$

According to the Riemann vanishing singularities theorem, $\hat{h}_0 \circ g \circ \hat{h}_0^{-1}$ is a holomorphic diffeomorphism at the origin, which is formally conjugated to g. Since h_0 and h_1 are defined up to a left composition by a same analytic diffeomorphism tangent to Identity, we have associated, to each $\phi \in Diff_1(S^2, 0, \infty)$, a conjugacy classe, denoted $g(\phi)$, modulo G with formal invariants $k = 1$ and $\lambda = 0$. Moreover, the classes $g(\phi)$ and $g(\phi')$ are identical if and only if ϕ and ϕ' are conjugated by the application $y \mapsto ay$ of $S^2 (a \in C^*)$.

Eventually, we have built infinitely many analytic conjugacy classes of diffeomorphisms formally conjugated to g. Each classe is caracterized by a $\phi \in Diff_1(S^2, 0, \infty)$ defined up to a conjugacy by a global diffeomorphism of $(S^2, 0, \infty)$.

We shall prove that we have obtained all the analytic conjucacy classes which are formally conjugated to g. Let $g' = \hat{h} \circ g \circ \hat{h}^{-1}$, with $\hat{h} \in \hat{G}$ and $g' \in G$. According to the Borel-Ritt theorem, there exist $h_i \in \mathcal{A}(U_i)(i = 0, 1)$ admitting $\hat{h}$ as asymptotic expansion at the origin in the sector of definition. We set $g'_i = h_i^{-1} \circ g' \circ h_i$. Then, we have $g'_i = \psi_i \circ g$ where $\psi_i \in G^{<0}(U_i)$. Thus, according to the proposition below, there exists $f_i \in G^{<0}(U_i)$ such that $g'_i = f_i \circ g \circ f_i^{-1}(i = 0, 1)$. Eventually, we obtain

$$g' = (h_i \circ f_i) \circ g \circ (h_i \circ f_i)^{-1} \text{ on } U_i$$

The functions $h_i \circ f_i (i = 0, 1)$ are analytic in the sector U_i which opening angle is greater than π and they admit $\hat{h}$ has asymptotic expansion in U_i. Moreover, we could show that, on the intersection $U_0 \cap U_1$, the difference $h_0 \circ f_0 - h_1 \circ f_0$ is an exponentially decreasing function of order 1. Consequenly, $\hat{h}$ is 1-summable in any direction but R^+ and R^-. $\square$

The proposition we have used is a "sectorial normalization theorem" due to Kimura

Proposition 5.2.1 *Let $\psi \in G^{<0}(U_1)$. Let $\psi_p = g^{-p} \circ \psi \circ g^p$ for $p \geq 1$, and $h_p = \psi_p \circ \cdots \circ \psi_1$. Then, the sequence (h_p) converges to an element $h \in G^{<0}(U_1)$ such that, $\psi \circ g = h^{-1} \circ g \circ h$.*

Proof:

We follow the proof of Malgrange [Mal82]. It is convenient to work with the variable $z = 1/x$ and $g(z) = z - 2i\pi$.

Let us consider the domain

$$D = \{z| \ -\frac{3\pi}{2} + \theta \leq arg(z + iR) \leq \frac{\pi}{2} - \theta\}$$

where $0 < \theta < \pi$ and R sufficiently large. Then, we have $g^p(D) = D_p \subset D$. Since $\psi - Id$ is infinitely flat, then, we have, for all integer n and R sufficiently large

$$\sup_{z \in D_p} |\psi(z) - z| \leq \frac{A}{p^n}$$

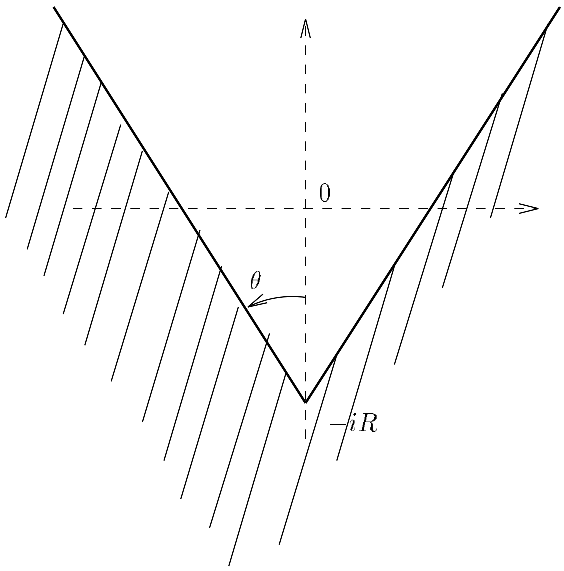


Figure 5.3: Domain D

where A is a suitable constant. Then, by a immediate calculus, we obtain

$$\sup_{z \in D_q} |\psi_p(z) - z| \leq \frac{A}{(p+q)^n}$$

But, the series

$$\sum_{p \geq 0} \frac{1}{(p+q)^n}$$

converges if $n \geq 2$ and its sum is bounded by $\frac{C}{q^{n-1}}$, where C is a constant. Consequently, the sequence $h_p = \psi_p \circ \cdots \circ \psi_1$ is defined and converges, on D_q for q sufficiently large, to a diffeomorphism which is $(n-1)$-flat relatively to Identity. Moreover, we have

$$g \circ h_p = g^{-(p-1)} \circ (\psi \circ g)^p = h_{p-1} \circ \psi \circ g$$

□

We complete this proposition by stating a similar proposition on U_0, but, this time, we set $\psi_p = g^p \circ \psi \circ g^{-p}$. In fact, the key point is that g^{-1} is a contraction on sectors bissected by iR^+ and which opening angle is smaller than 2π, whereas, in the propostion we have proved, g was a contraction on sectors bissected by iR^- with an opening angle smaller than 2π. Moreover, if $\psi \in G^{<-k}(U)$, then, $h \in G^{<-k}(U)$.

5.3 Normal forms of resonant vectors fields in C^2

After having classified resonant diffeomorphisms, we may want to classify the germs, at the origin $(0,0) \in C^2$, of analytic differential equations of the form

$$\begin{cases} \frac{dx}{dz} = & P(x,y) \\ \frac{dy}{dz} = & Q(x,y) \end{cases} \tag{5.2}$$

where P and Q are both analytic in a neighbourhood of the origin $(0,0) \in C^2$. We shall assume that $P(0,0) = Q(0,0) = 0$; that is, 0 is a singular(or equilibrium) point. Thus, the above system can be rewritten under the following form:

$$\frac{X}{dt} = AX + f(X) \tag{5.3}$$

where A is a complex 2×2-matrix, $X \in C^2$, $f(0) = 0$, and $Df(0) = 0$. For the simplicity of the exposion, we shall assume that A is diagonal.

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

with $(\lambda, \mu) \in C^2$.

Its is well known(see [SDD93], for an overview on normal forms), that if $\frac{\lambda}{\mu} \in C \setminus R$, then the system 5.3 is analytically conjugated to its linear part, that is to the system

$$\frac{Y}{dt} = AY$$

Nevertheless, if this arithmetical condition is not satisfied, then the discussion becomes harder.

As for the diffeomorphisms case we shall be interested only in the resonant cases. We distinguish two cases of resonance:

1. Degenerated case[RM82]
   It occurs when one and only one of the eigenvalues of A is zero.

2. Non degenerated case[RM83]
   It occurs when λ and μ are both non zero, and are such that

$$\frac{\lambda}{\mu} \in Q^-$$

From the geometric point of view, and by reference to the real case, in the degenerated case, we have a **saddle-node**, whereas in the non-degenerated case, we have **resonant saddle**.

As in the case of resonant diffeomorphisms, the formal normal forms are parametrized by a finite number of parameters. The normalizing transformations, that is the change of coordinates which transforms a resonant differential equation to its normal form, are formal Gevrey series. We can show that they are k-summable.

Chapter 6

Singular perturbations, Ducks

The theory of Gevrey asymptotic expansion is very efficient, as we have shown, in the study of singulatities of ordinary differential equations. Here, we were concerned with singulatities in the the variable. Nevertheless, this theory can used in the study of certain problems in which arise singularities in the parameter: these are named the singular perturbations problems.

6.1 Invertible case

6.1.1 Main result

A main result in this direction is due to Sibuya:

Theorem 6.1.1 *Let n be a fixed positive integer. We consider the differential equation*

$$\epsilon^\sigma \frac{dy}{dx} = F(x,\epsilon,y) = f(x,\epsilon) + A(x,\epsilon)y + \sum_{|p|\geq 2} f_p(x,\epsilon)y^p \qquad (6.1)$$

where $\sigma \in N^*$, $(x,\epsilon) \in C^2$, $y \in C^n$, A is a complex $n \times n$-matrix valued function, the f_p's are complex n-dimensional vector valued functions.

We assume that F is analytic in a neighbourhood of $(0,0,0)$, that $f(0,0) = 0$ and that $A(0,0)$ is invertible. Then,

1. *the equation 6.1 admits a unique formal solution of the form*

   $$y = \hat{f}(x) = \sum_{n\geq 0} a_n(x)\epsilon^n$$

   where the a_n's are holomorphic functions on a same disk D of the complex plane;

2. *even if it means shrinking the disk D, the a_n's satisfy to the Gevrey estimate of order $\frac{1}{\sigma}$:*

   $$\|a_n\| \leq C(n!)^{\frac{1}{\sigma}} A^n$$

   for suitable constants $A, C > 0$, and $\|a_n\| = \sup_{x\in D} |a_n(x)|$.

In other words, the equation 6.1 admits a formal Gevrey series of order $\frac{1}{\sigma}$ as solution.

Sibuya gave a "cohomological" proof of this theorem in [Sib90]. It is done in the same way as the proof of the Maillet theorem 4.2.3: we build a good covering $\{V_i\}$ of a punctured disk in the ϵ-plane and funcions $\{f_i\}$ wich are holomorphic in the variables $(x, \epsilon) \in D \times V_i$. These functions are built so as to satisfy the equation 6.1. Then, we estimate the associated 1-cochain. We find that the difference $f_i - f_j$ on the intersection $D \times (V_i \cap V_j)$ is an exponential decreasing function of order $\frac{1}{\sigma}$ in the variable ϵ, uniformely in $x \in D$. Consequently, we obtain that $\hat{f} = T f_i$ is a formal Gevrey series of order $\frac{1}{\sigma}$ which is solution of the equation.

In the case where $\sigma = 1$, this theorem was proved by Canalis-Durand, by computing the coefficients a_n and then by estimating them[CD90].

6.1.2 Application to "delayed bifurcation" phenomenon

According to Martinet[Mar92], this result can be used in order to study the "delayed bifurcation" phenomeon. This is done in the framework of Nonstandard Analysis. This phenomenon arises with the phenomenon of exponential "compression-explosion" of trajectories. Let us consider a $p+1$-dimensional system

$$\begin{cases} \frac{x}{dt} = & F(x, \lambda) \\ \frac{\lambda}{dt} = & \epsilon \end{cases} \tag{6.2}$$

where ϵ is a fixed infinitely small complex number, $x \in C^p$, $\lambda \in C$, and F a holomorphic C^p-function valued. The previous equation leads to the following one

$$\epsilon \frac{dx}{d\lambda} = F(x, \lambda)$$

If we set $\epsilon = 0$, we obtain the "steady curve" $\mathcal{C}_0$ defined by $F(x, \lambda) = 0$. We shall assume that this curve is the graph of an analytic function $x = c_0(\lambda)$. We make some more assumptions:

1. The matrix $\frac{\partial F}{\partial x}(0, 0)$ is invertible,
2. The vector field $F(x, \lambda)\frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial \lambda}$ is transversal to the curve $\mathcal{C}_0$ outside an infinitely small tubular domain of $\mathcal{C}_0$.

According to the above Sibuya theorem, the previous equation admits a formal solution(formal in the variable ϵ, and analytic in λ):

$$\hat{x}(\lambda, \epsilon) = c_0(\lambda) + \sum_{n \geq 1} c_n(\lambda) \epsilon^n$$

The functions c_n are holomorphic and bounded on a same disk D_r(of radius r). Moreover, they satisfy the Gevrey estimates of order 1:

$$\|c_n\| \leq C n! A^n$$

for suitable $C, A > 0$, and $\|a_n\| = \sup_{\lambda \in D_r} |a_n(\lambda)|$.

The quasi-sum of this series, obtained, for instance, with a partial Laplace transform, is 1-precise and is a quasi-solution, that is, is solution of the equation, up to an exponential decreasing function of order 1 of ϵ.

Then, we wonder what does happen when the invertibility of the matrix $\frac{\partial F}{\partial x}(0, 0)$ is not required. In this case, a strange and interesting phenomenon can appear: ducks(canards). We shall not developped this point here, and we invite the reader the consult [Ram] and [CD90].

6.2 Averaging Problems

Let U be an open subset in R^n and $f : U \times R \to R^n$ be a real analytic function in the first variable X and 1-periodic in the second one t. Next, we consider the differential system

$$\frac{dX}{dt} = \epsilon f(X, t) \tag{6.3}$$

where $\epsilon > 0$ is a "small parameter".

The problem is whether such an equation can be conjugated to an autonomous system by means of a change of variables close to identity (analytic in X and ϵ, 1-periodic in t). Generally, this problem does not admit a solution. Nevertheless, it can be shown [Sau92] that there exists a formal series in ϵ, with analytic coefficients

$$\hat{u}(Y, t, \epsilon) = \sum_{n \geq 0} u_n(Y, t)\epsilon^n$$

such that the change of coordinates $X = Y - \epsilon\hat{u}(Y, t, \epsilon)$ transforms the system 6.3 into the system

$$\frac{dY}{dt} = \epsilon\hat{g}(Y, \epsilon)$$

where

$$\hat{g}(Y, \epsilon) = \sum_{n \geq 0} g_n(Y)\epsilon^n$$

is a formal power series in ϵ with analytic coefficients on a same disk.

By estimating the coefficient of these series, Sauzin [Sau92] proved that both were Gevrey of order 1.

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