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An analytical approach to chaos in Lorenz-like systems. A class of dynamical equations

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Abstract. – The mechanism responsible for the emergence of chaotic behavior has been singled out analytically within a class of three-dimensional dynamical systems which generalize the well-known E. N. Lorenz 1963 system. The dynamics in the phase space has been reformulated in terms of a first-exit-time problem. Chaos emerges due to discontinuous solutions of a transcendental problem ruling the time for a particle to cross a potential wall. Numerical results point toward the genericity of the mechanism.

Many chaotic phenomena in physical sciences and engineering can be satisfactorily described by low-dimensional autonomous dynamical systems. Among them the most celebrated example is the well-known E. N. Lorenz model [1, 2]

$$\begin{cases} \dot{x} &= \sigma(y-x)\,, \\ \dot{y} &= -y+x+(r-1)(1-z)x\,, \\ \dot{z} &= -b(z-xy)\,, \end{cases} \tag{1}$$

here rewritten using rescaled variables x, y, z in the case $r > 1$. Besides it attracted the attention of the scientific community mainly on account of its ability to illustrate how a simple model can produce very rich and varied forms of dynamics, depending on the control parameters in the equations. In particular, chaotic behavior in its configuration space around three fixed unstable points $(0,0,0)$ and $(\pm 1, \pm 1, 1)$ may take place for r larger than a critical value, function of σ and b [1, 2].

Starting from the original work of Lorenz [1], a huge literature has grown on the subject (see, *e.g.*, refs. [3–6] for recent literature), almost independently of its somewhat doubtful success in satisfactorily describing the Rayleigh-Bénard hydrodynamical problem [7].

Quite surprisingly, the rigorous proof of the existence of a strange attractor for the Lorenz system has been given just very recently [8, 9]. The prove is based on rigorous techniques of dynamical mathematics, rigorous on the one hand but very far from physical intuition on the other hand.

Our aim here is to combine analytical approaches to chaos with the very familiar concept of classical mechanics to single out the origin of chaotic behavior for a class of Lorenz-like systems. Specifically, we shall reformulate the Lorenz model dynamics in the phase space in terms of more familiar ideas such as mechanical properties of particles moving in one-dimensional potential fields subjected to viscous forcing. Such reformulation will be the starting point to generalize the original Lorenz system to a whole class of Lorenz-like systems. Among them we shall select a particular piecewise linearized model through which analytical results can be obtained. As we shall see, the dynamics generated by this model can be mapped into a first-exit-time problem. This will allow us to identify analytically the origin of the system chaotic behavior. Finally, the robustness of the identified mechanism will be tested numerically for other fully nonlinear generalized Lorenz-like systems.

Let us start our analysis by noticing that, far from the initial transient, system (1) is equivalent to the (integral-) differential equation [10]

$$\ddot{x} = -(\sigma + 1)\dot{x} - \frac{\partial \mathcal{U}}{\partial x}\,. \tag{2}$$

Equation (2) can be interpreted as a customary classical dynamical equation of motion for a (unit mass) particle subjected to a viscous force $-(\sigma+1)\dot{x}$ in the potential field $\mathcal{U}(x,t)$. Here,

$$\mathcal{U}(x,t) = \frac{b}{2\sigma}U(x) + \left(1 - \frac{b}{2\sigma}\right)U_t(x) \tag{3}$$

is a potential field resulting from a weighted average (in the chaotic regime $b < 2\sigma$) of a constant quartic potential $U(x) \equiv \sigma(r-1)(x^2-1)^2/4$ and a time-dependent quadratic one $U_t(x) \equiv \sigma(r-1)[x^2-1]_b(x^2-1)/2$.

We use the notation

$$[f]_b \equiv b\int_0^\infty \mathrm{d}\tau\, e^{-b\tau}\, f(t-\tau) \tag{4}$$

to indicate the steady-state response $u(t)$ of the linear system $\dot{u}(t) + bu(t) = bf(t)$ to the stationary forcing term $f(t)$. The integral (4) clearly represents the memory of $f(t)$ (at the time t), *i.e.* its exponentially weighted past evolution. Thus, the potential $U_t(x)$ depends on time through the exponential memory of the past motion.

One can check that, given x from (2), the variables y and z are obtained from the relations $y = \dot{x}/\sigma + x$ and $z = b/(2\sigma)x^2 + (1 - b/(2\sigma))[x^2]_b$, respectively. In this formulation it is, for instance, evident that x and y are synchronizing coordinates while this is not for z [3, 11].

The above description immediately leads to the following considerations highlighting the role of memory in the route to chaos. For $U_t \equiv 0$, the particle motion stops after some time in one of the two minima of $U(x)$. This because of the viscous term $(\sigma+1)\dot{x}$. On the contrary, nontrivial behaviors, including chaotic ones, may take place due to the statistical balance between dissipation and energy exchanges produced by the memory-dependent potential U_t. The bistable character of U plays a crucial role for the emergence of chaos. Indeed, initially very close trajectories starting in the same cell may undergo completely different evolutions if, at a certain time, one has sufficiently energy to cross the peak in $x = 0$, while this is not for the other. This is the first clue that chaos arises from the unpredictability of the instants when particles pass through the barrier in $x = 0$. We shall give in the sequel the analytical proof of this heuristic argument together with the reason at the origin of such unpredictability. To do that, let us give a further reformulation of system (1). This will make it possible to generalize and simplify (1) with the final goal to deal with chaos analytically.

By suitably scaling the time: $t \mapsto \tau \equiv [(r-1)/2]^{1/2}\, t$, eq. (2) can be recast in the form

$$\ddot{x} + \eta\dot{x} + (x^2-1)\, x = -\alpha \left[x^2-1\right]_\beta\, x, \tag{5}$$

which further highlights the role of memory in the dynamics [12] (η, α and β are related to the original parameters σ, b and r).

A more expressive form of eq. (5) is

$$\ddot{x} + \eta\dot{x} + \Big(q(x) + \alpha\,[q(x)]_\beta\Big)\,\Phi'(x) = 0\,, \tag{6}$$

where $\Phi(x) = 1/2\, x^2$ and $q(x) = x^2 - 1 = 2\Phi(x) - 1$. This equation describes the motion of a (unit mass) particle subjected to a viscous force $-\eta\dot{x}$ and interacting with a fixed potential field $\Phi(x)$ through a "dynamically varying charge" $q_t(x) = q(x) + \alpha\,[q(x)]_\beta$. It is constituted by a fixed "core" charge, a "locally acquired" charge, related to the local potential, and an exponentially vanishing "memory" charge, continuously depending on the previous instantaneous charge history. This scheme can be used to mimic the instantaneous effective charge of a particle moving in a background (structured) particle bath. Indeed, the coupling of $[q(x)]_\beta$ with the background potential $\Phi(x)$ yields an endogenous forcing term which allows a self-sustained unceasing motion, even in the presence of friction, and causes a corresponding unceasing inner transfer of the bath charge.

At this point the Lorenz system (1) can be easily generalized in the form

$$\begin{cases} \dot{x} &=& \sigma(y-x)\,, \\ \dot{y} &=& -y + x + (r-1)(1-z)\Phi'(x)\,, \\ \dot{z} &=& -b\left[z - \frac{1}{2}q'(x)(y-x) - q(x) - 1\right]\,, \end{cases} \tag{7}$$

from which (asymptotically in time) eq. (6) follows after some algebraic manipulations.

As already observed, the chaotic behavior of Lorenz's system essentially depends on the unpredictability of the instants when x change its sign: as long as it keeps constant sign, the system evolution is certainly nonlinear, and nevertheless not chaotic at all. This fact suggests a slight modification of the original form, in order to single out analytically the origin of chaos without being faced with the difficulties arising from nonlinear problems. We thus set in eqs. (6) and (7) $\Phi(x) = |x|$ and $q(x) = \Phi(x) - 1$ obtaining

$$\ddot{x} + \eta\dot{x} + \Big\{|x| - 1 + \alpha\,[|x|-1]_\beta\Big\}\ \mathrm{sgn}(x) = 0 \tag{8}$$

and the corresponding piecewise linearized system

$$\begin{cases} \dot{x} &=& \sigma(y-x)\,, \\ \dot{y} &=& -y + x + (r-1)(1-z)\ \mathrm{sgn}(x)\,, \\ \dot{z} &=& -bz + b\ \mathrm{sgn}(x)\frac{x+y}{2}\,, \end{cases} \tag{9}$$

where $\mathrm{sgn}(x) \equiv |x|/x$.

Our assumptions for Φ and q correspond in eq. (2) to $U(x) = \sigma(r-1)(|x|-1)^2/2$ and $U_t(x) = \sigma(r-1)[|x|-1]_\beta(|x|-1)$. The fundamental bistable character of U is thus maintained.

In order to solve eq. (8), we exploit the fact that it is left invariant under the transformations $x \mapsto -x$, $\tau \mapsto \tau$. We can thus focus on one of the two regions $x < 0$ and $x > 0$. Let us consider, *e.g.*, $x > 0$. Being eq. (8) a second-order integral-differential equation, it can

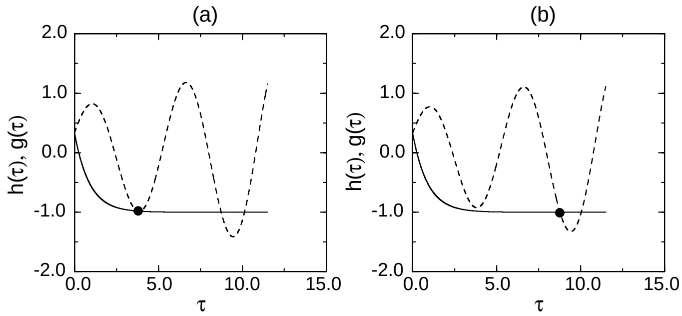

Fig. 1 – Graphical interpretation of the discontinuous character of τ_1 for small changes of the initial conditions. The full line represents $g(\tau)$, the dashed lines $h(\tau)$ for the parameters $\alpha = 6.50$, $\beta = 0.19$, $\eta = 0.78$ and for initial conditions: (a) $\dot{\xi}_0 = 2.26$, $\ddot{\xi}_0 = -2.00$; (b) $\dot{\xi}_0 = 2.30$, $\ddot{\xi}_0 = -2.00$. Bullets denote the first intersection between g and h defining the first collision time against the wall: (a) $\tau_1 = 3.91$; (b) $\tau_1 = 8.76$.

be reduced to an equivalent third-order linear differential equation by applying the operator $(\mathrm{d}/\mathrm{d}\tau + \beta)$. The result reads [13]

$$\frac{\mathrm{d}^3\xi}{\mathrm{d}\tau^3} + (\beta+\eta)\frac{\mathrm{d}^2\xi}{\mathrm{d}\tau^2} + (1+\beta\eta)\frac{\mathrm{d}\xi}{\mathrm{d}\tau} + \beta(1+\alpha)\xi = 0\,, \tag{10}$$

where $\xi \equiv x - 1$.

Notice that when the particles cross the barrier in $x = 0$, the evolution described by eq. (10) has to restart with new initial conditions corresponding to an elastic collision against a rigid wall posed in $x = 0$.

It can be checked that, in the case of chaotic motion, the general solution of eq. (10) can always be written in the form

$$\xi(\tau) = e^{\lambda_r \tau}\left(C_1 \cos(\lambda_i \tau) + C_2 \sin(\lambda_i \tau)\right) + C_3 e^{-\lambda_0 \tau}\,, \tag{11}$$

with $\lambda_0, \lambda_r, \lambda_i \geq 0$ and C_1, C_2 and C_3 real coefficients determined from the initial conditions on ξ, $\dot{\xi}$ and $\ddot{\xi}$.

The instant τ_1 at which the first particle collision against the wall in $x = 0$ occurs is thus defined by the equation $\xi(\tau_1) = -1$. From (11), τ_1 is thus the smallest positive solution of the transcendental equation

$$C_1\, e^{\lambda_r \tau_1} \cos(\lambda_i \tau_1) \;+\; C_2\, e^{\lambda_r \tau_1} \sin(\lambda_i \tau_1) \;+\; C_3\, e^{-\lambda_0 \tau_1} = -1\,. \tag{12}$$

Geometrically, we can interpret τ_1 as the first intersection of $g(\tau) \equiv -C_3\, e^{-\lambda_0 \tau} - 1$, with $h(\tau) \equiv C_1\, e^{\lambda_r \tau} \cos(\lambda_i \tau) + C_2\, e^{\lambda_r \tau} \sin(\lambda_i \tau)$. The function g is a decreasing exponential and h an oscillating function with growing amplitude. It is thus easily understood why even a little modification of initial conditions can produce a discontinuous variation of τ_1 (see fig. 1). As we shall see, the same reason applies also for the class of systems (7). The dependence of τ_1 on the initial conditions $\dot{\xi}_0$ and $\ddot{\xi}_0$ ($\xi_0 = -1$) is implicitly contained in eq. (12). Coefficients C_1, C_2 and C_3 are indeed linearly related to the initial conditions. The way to describe the global behavior of τ_1 in terms of the pair $(\dot{\xi}_0, \ddot{\xi}_0)$, although simple, results quite lengthy. We thus confine our attention on the corresponding graphical shape presented in fig. 2. From the figure it appears that τ_1 shows sensitivity with respect to the initial conditions only in a limited subset

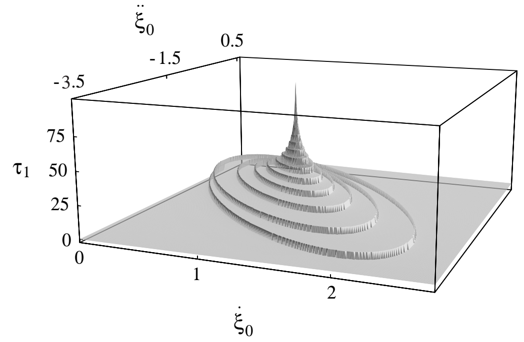

Fig. 2 – The behavior of the first collision time, τ_1, defined by eq. (12), *vs.* the initial conditions $\dot{\xi}_0$ and $\ddot{\xi}_0$ for $\alpha = 6.50$, $\beta = 0.19$, $\eta = 0.78$.

of the half-plane $\dot{\xi}_0 \geq 0$. As we shall see, in the chaotic regime the system is quickly attracted inside this region. It will thus be clear why chaotic behaviors can arise also for apparently simple nonlinearities as isolated (noneliminable) discontinuities (see, *e.g.*, refs. [14–18] for other examples of piecewise linearized systems displaying varied forms of chaotic behavior).

Notice that the same behavior showed in fig. 2 for τ_1 holds for the n-th collision time τ_n as a function of the $(n-1)$-th initial conditions $(\dot{\xi}_0^{(n-1)}, \ddot{\xi}_0^{(n-1)})$.

For $(\dot{\xi}_0, \ddot{\xi}_0) = (\lambda_0, -\lambda_0^2)$ one has $\tau_1 \to \infty$ (the peak in fig. 2). In this case, it is possible to show that the system configuration, described by $\boldsymbol{\xi} \equiv (\xi(\tau), \dot{\xi}(\tau), \ddot{\xi}(\tau))$, exactly lies on the stable manifold $\mathcal{W}^{\mathrm{s}} \equiv \{t\, \boldsymbol{w}_3 | \, t\rangle - 1\}$ with $\boldsymbol{w}_3 \equiv (1, -\lambda_0, \lambda_0^2)$ and its motion is an exponential decay on the fixed point $\boldsymbol{\xi} = \mathbf{0}$.

For $(\dot{\xi}_0, \ddot{\xi}_0) \neq (\lambda_0, -\lambda_0^2)$ the evolution in the phase space ξ, $\dot{\xi}$, $\ddot{\xi}$ consists of both a "rapid" decay towards $\boldsymbol{\xi} = \mathbf{0}$ along the stable manifold $\mathcal{W}^{\mathrm{s}}$ and a "slow" amplified oscillation on the two-dimensional unstable manifold, $\mathcal{W}^{\mathrm{u}}$, generated by $\boldsymbol{w}_1 \equiv (1, \lambda_c, \lambda_c^2)$ and $\boldsymbol{w}_2 \equiv (1, \bar{\lambda}_c, \bar{\lambda}_c^2)$, with $\lambda_c = \lambda_r + i\lambda_i$ and $\bar{\lambda}_c = \lambda_r - i\lambda_i$.

Focusing now on the n-th collision, we thus have from eq. (12) the behavior of τ_n *vs.* the pair $(\dot{\xi}_0^{(n-1)}, \ddot{\xi}_0^{(n-1)})$. Taking the first and the second time derivative of eq. (12), we can relate $\dot{\xi}_0^{(n)}$ and $\ddot{\xi}_0^{(n)}$ to the initial conditions $\dot{\xi}_0^{(n-1)}$ and $\ddot{\xi}_0^{(n-1)}$ (remember that C_1, C_2 and C_3 are linearly related to such initial conditions). The result is a two-dimensional Poincaré map connecting $(\dot{\xi}_0^{(n-1)}, \ddot{\xi}_0^{(n-1)})$ to $(\dot{\xi}_0^{(n)}, \ddot{\xi}_0^{(n)})$. The nonlinear character of the map is entirely contained in the transcendental and discontinuos relation between τ_n and the initial conditions.

Let us now assume that the attraction towards the unstable manifold $\mathcal{W}^{\mathrm{u}}$ is sufficiently fast (the goodness of this approximation is controlled by $\beta + \eta$) for the system to be considered as belonging to $\mathcal{W}^{\mathrm{u}}$ at the collision time against the plane $\xi = -1$. If this is the case, "immediately" after the n-th collision, velocity and acceleration are connected by a straight line in the plane $\dot{\xi}_0^{(n-1)}$-$\ddot{\xi}_0^{(n-1)}$ intersecting the base of the curve shown in fig. 2. The functional dependence of τ_n on $\dot{\xi}_0^{(n-1)}$ and $\ddot{\xi}_0^{(n-1)}$ is thus restricted on this straight line, *i.e.* τ_n is a function of one variable alone, say, $\dot{\xi}_0^{(n-1)}$. The map $\tau_n = \tau_n(\dot{\xi}_0^{(n-1)})$ is shown in fig. 3(a).

Furthermore, the Poincaré map connecting $(\dot{\xi}_0^{(n-1)}, \ddot{\xi}_0^{(n-1)})$ to $(\dot{\xi}_0^{(n)}, \ddot{\xi}_0^{(n)})$ reduces to a one-dimensional map, say, between $\dot{\xi}_0^{(n-1)}$ and $\dot{\xi}_0^{(n)}$. Figure 4(a) shows this map for $\alpha = 6.50$, $\beta = 0.19$, $\eta = 0.78$. Its behavior is evidently chaotic: the discontinuities derive from the analogous ones in the function $\tau_n = \tau_n(\dot{\xi}_0^{(n-1)})$.

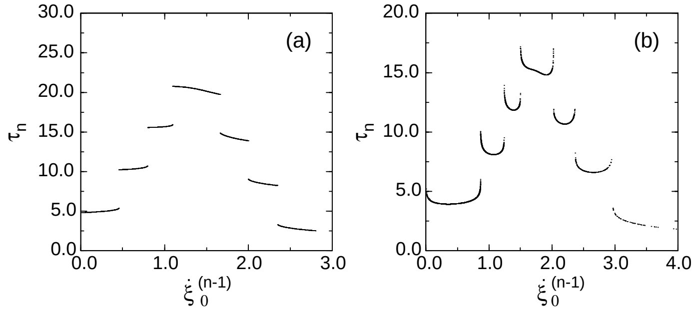

Fig. 3 – The one-dimensional map $\tau_n = \tau_n(\dot{\xi}_0^{(n-1)})$; (a) for the piecewise linearized system (9); (b) for the Lorenz model (1).

The final question to be addressed concerns the relation of our results with the original Lorenz system (1) and more generally with the class of dynamical systems (7). When the evolution of the latter systems can be approximated in $x = 0$ by one-dimensional maps, behaviors similar to those observed for the piecewise linearized system have been found. Specifically, focusing on the Lorenz model (1), the analogous of the maps reported in figs. 3(a) and 4(a) are shown in figs. 3(b) and 4(b). Similar behaviors have been obtained for other choices of Φ and q in (7). These maps have been derived by numerical integration of (7) by a standard Runge-Kutta scheme, whereas we remember that all results relative to the piecewise linearized system have been obtained analytically.

The resemblance of figs. 3(a), 4(a) with figs. 3(b), 4(b) points toward the robustness of the mechanism we have identified as cause of chaos in the linearized system (9).

In conclusion, the origin of chaos for a whole class of three-dimensional autonomous dynamical systems has been singled out analytically exploiting familiar ideas related to mechanical properties of particles moving in one-dimensional potential fields in the presence of dissipation. Chaos is entirely contained in a (transcendental) equation ruling a first-exit-time problem (see fig. 1) whose solutions appear discontinuous for small changes in the initial conditions. More specifically, the chaotic dynamics of the Lorenz system is synthesized in the combination of the step-like first exit time (fig. 3) and the return map for the initial conditions (fig. 4). Results have been obtained analytically for a piecewise linearized model belonging to a more general

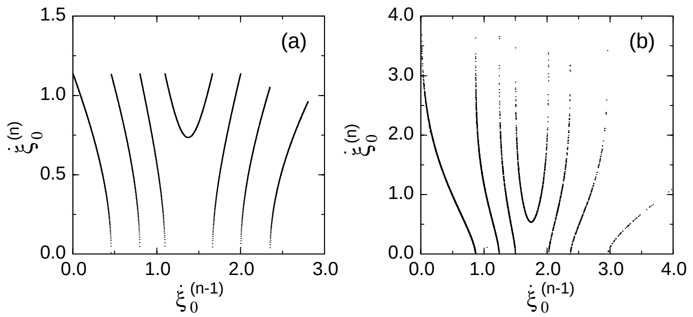

Fig. 4 – The one-dimensional map between $\dot{\xi}_0^{(n)}$ and $\dot{\xi}_0^{(n-1)}$; (a) for the piecewise linearized system (9); (b) for the Lorenz model (1).

class of dynamical systems. We, however, showed numerically that the basic reason for the chaos to emerge applies also for the general case.

Finally, it is under investigation whether or not our mechanism works also for higher-dimensional systems like those investigated, *e.g.*, in refs. [19, 20].

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[12] Equation (5) resembles other examples of chaotic nonlinear nonautonomous equations, as, for instance, the inverse Duffing equation $\ddot{x} + \eta\dot{x} + (x^2 - 1)\ x = A\cos(\Omega t)$, which describes a sinusoidally forced anharmonic oscillator: in our case the motion is self-sustained through the generally nonsinusoidal "autonomous" forcing term $-\alpha\left[x^2 - 1\right]_\beta\ x$.
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