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Intermittency in passive scalar decay

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Abstract

We consider the decay of a passive scalar in a rough turbulent flow. We show that at large times the scalar intensity is statistically equivalent to the product of two independent random variables. The first factor is a Gaussian variable related to the typical fluctuation at initial conditions. The second factor is essentially the probability that, in a given realization of the flow, tracer particles separate at a very slow rate. Therefore, the large-time scalar distribution is always broader than Gaussian for any initial condition, and large scalar excursions are associated to small relative dispersion events.

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1. Introduction

Intermittency is a relevant aspect of fluid turbulence and the understanding of this phenomenon has attracted much interest in recent years [1]. Roughly speaking, the term intermittency is used to describe situations where rare events have a relatively large probability to occur. Physical quantities that are known to exhibit an intermittent behavior are, e.g., velocity differences, pressure and temperature fluctuations. Here, we will be interested in the intermittency of the single-point statistics of a decaying scalar field—as temperature or impurity concentration—passively transported by an incompressible turbulent flow.

The properties of passive scalar decay may be different for velocity fields that vary smoothly in space and for rough (fully developed turbulent) flows. As for smooth velocity, the problem is considered by Son [2] and Balkovsky and Fouxon [3]. They obtain an exponential decay in time for the scalar concentration. Different moments of the scalar probability function decay at different rates, i.e. the decay is not self-similar in time, and the degree of intermittency increases with time. Another important theoretical contribution to the understanding of the scalar decay is given by Majda et al. [4–6]. His model represents the evolution of a passive scalar diffusing in a random linear shear flow

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with zero correlation time. In that case the decay is self-similar.[1] Despite of its simplicity, the Majda model allows to capture many features of the scalar decay, first of all intermittency. The exact form of high-order moments is found by exploiting a quantum mechanics analogy and the scalar distribution function is consequently shown to be broader than Gaussian, for Gaussian initial conditions. Its tails were later rigorously estimated by Bronski and McLaughlin [7–9] to be stretched exponentials with an exponent depending on the spatial correlation of the initial condition. A recent contribution in the study of scalar decay in linear shear flows is given by Vanden Eijnden [10], who shows that large values of the scalar intensity are a reflection of slowly spreading Lagrangian paths.

In the case of rough velocity fields, the passive scalar decay is treated by Eyink and Xin [11] and Chaves et al. [12] within the context of the Kraichnan model [13]. In particular, in Ref. [12] the asymptotic scalar decay is proved to be self-similar and universal with respect to the random initial condition. These conclusions are extended to a generic self-similar velocity field and supported by numerical simulations on more realistic flows, specifically two-dimensional Navier–Stokes turbulent flows [14,15]. The results obtained in Ref. [12] will be the starting point of our analysis.

The aim of this paper is to prove that, in the limit as the molecular diffusivity tends to zero, the intermittency of a *decaying* passive scalar transported by a *rough* flow is a universal feature. We suppose that the flow is self-similar for distances shorter than the velocity correlation length. Besides, the spatial correlation of the initial scalar field is assumed to be rapidly decaying in space.

The exposition is divided into two parts.

First, we recall the results of Chaves et al. [12], who demonstrate that, at large times, the dominant contribution to the scalar statistics coincides with that given by Gaussian initial conditions.

Second, we show that at large times the scalar intensity is statistically equivalent to the product of two independent random variables. The first factor is a Gaussian variable related to the typical fluctuation at initial conditions. The second factor is essentially the probability that, in a given realization of the flow, tracer particles separate at a very slow rate. Therefore, the large-time scalar distribution is always broader than Gaussian for any initial condition, and large scalar excursions are associated to small relative dispersion events.

It is worth remarking that these results have been proved rigorously by Vanden Eijnden [10] in the special case of linear shear flows and then extended (under different hypotheses from ours) to more general velocity fields, including rough ones.

We remark that the present results are neither valid for smooth flows nor for forced scalar turbulence. Those cases require a separate treatment that is beyond the purpose of this paper.

2. All initial conditions are asymptotically equivalent to Gaussian initial conditions

The evolution of a passive scalar quantity θ diffusing in an incompressible turbulent flow $\mathbf{v}$ is described by the equation

$$\partial_t \theta + \mathbf{v} \cdot \nabla \theta = \kappa \nabla^2 \theta, \quad (1)$$

where κ denotes the molecular diffusivity.

The initial random condition $\theta(\mathbf{r}, 0)$ is taken statistically homogeneous, with zero mean, and with correlation functions rapidly decaying in space.[2]

[1] Remark that, although the flow is smooth even in this case, the Lyapunov exponent is zero. This fact has important consequences on the particle trajectories within the flow. We will come back in the sequel to the relation between the Eulerian (scalar field) statistics and the Lagrangian (fluid particles) one.

[2] Precisely, all the cumulants of the initial scalar field (see Eq. (13)) should be integrable.

If we denote by $L_\theta(t)$ the correlation length of the scalar field $\theta(\mathbf{r}, t)$, the decay will take rise after a characteristic time $T_\theta \simeq L_\theta(0)/v_{\mathrm{rms}}$. In the sequel we will always consider time scales much larger than T_θ.

We take as advecting flow $\mathbf{v}$ an incompressible random field statistically homogeneous in space and stationary in time. Moreover, we assume that the velocity field has self-similar statistics with scaling exponent h ($0 \le h < 1$), in a sense that will be precisely stated later on (see Eq. (4)). Obviously, the self-similarity holds only for distances shorter than the correlation length of the velocity field, L_v, and $L_\theta(t)$ is supposed to be always shorter than L_v. Since in a self-similar flow $L_\theta(t)$ is known to grow as $t^{1/(1-h)}$, this means that we focus on times much larger than T_θ, but, however, shorter than the time $T_v = \mathrm{O}(L_v^{1-h})$ needed for the scalar correlation length to reach the size L_v.

The evolution equation (1) can be solved in terms of particle trajectories satisfying the stochastic differential equation

$$\mathrm{d}\mathbf{r} = \mathbf{v}(\mathbf{r}, t)\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}\boldsymbol{\eta}(t), \tag{2}$$

where $\boldsymbol{\eta}$ is the standard d-dimensional Brownian motion. The particle advection is described by the conditional probability density $p^\kappa(\boldsymbol{\rho}, 0|\mathbf{r}, t)$ that, for a given realization of the velocity field, a tracer particle, being at time t in the position $\mathbf{r}$, were at the initial time in the position $\boldsymbol{\rho}$. Since the stochastic differential equation (2) defines a Markov process, the probability $p^\kappa(\boldsymbol{\rho}, 0|\mathbf{r}, t)$ satisfies the associated Fokker–Planck equation

$$\partial_t p^\kappa + \mathbf{v}\cdot\nabla p^\kappa = \kappa\nabla^2 p^\kappa \tag{3}$$

with initial condition $p^\kappa(\boldsymbol{\rho}, t|\mathbf{r}, t) = \delta(\boldsymbol{\rho} - \mathbf{r})$.

The limiting case $\kappa \to 0$ will be considered here. If the field $\mathbf{v}$ is Lipschitz continuous in $\mathbf{r}$ (i.e. smooth), the solution of Eq. (2) is unique in that limit and the random probability $p^\kappa(\boldsymbol{\rho}, t'|\mathbf{r}, t)$ degenerates to the distribution $\delta(\boldsymbol{\rho} - \mathbf{r}(t'))$ centered at the unique trajectory $\mathbf{r}(t')$. On the contrary, if $\mathbf{v}$ is not Lipschitz continuous (i.e. it is rough), the solution of Eq. (2) fails to be unique and p^κ tends to a non-degenerate probability density as $\kappa \to 0$ (see Refs. [16–18]). Here we are interested in a rough flow. The (broad) limiting probability will henceforth be denoted by $p(\boldsymbol{\rho}, 0|\mathbf{r}, t) = \lim_{\kappa\to 0} p^\kappa(\boldsymbol{\rho}, 0|\mathbf{r}, t)$.

The many-particle probability density function (PDF) $\mathcal{P}_{2n}(\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_{2n}, 0|\mathbf{r}_1, \ldots, \mathbf{r}_{2n}, t)$ that $2n$ tracer particles, that at time t lie in the positions $\mathbf{r}_1, \ldots, \mathbf{r}_{2n}$, were at the initial time $t = 0$ in the positions $\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_{2n}$, respectively, is expressible in terms of the random probability density $p(\boldsymbol{\rho}, 0|\mathbf{r}, t)$ as $\langle p(\boldsymbol{\rho}_1, 0|\mathbf{r}_1, t), \ldots, p(\boldsymbol{\rho}_{2n}, 0|\mathbf{r}_{2n}, t)\rangle_v$. For the sake of simplicity, we will hereafter indicate by an underlined bold type the set of $2n$ vectors in d dimensions that specify the particle configuration, we will write, e.g. $\mathcal{P}_{2n}(\underline{\boldsymbol{\rho}}, 0|\underline{\mathbf{r}}, t)$.[3]

We assume the following scaling behavior for $\mathcal{P}_{2n}$:

$$\mathcal{P}_{2n}(\lambda\underline{\boldsymbol{\rho}}, 0|\lambda\underline{\mathbf{r}}, \lambda^{1-h}t) = \lambda^{-(2n-1)d}\mathcal{P}_{2n}(\underline{\boldsymbol{\rho}}, 0|\underline{\mathbf{r}}, t) \quad (0 \le h < 1), \tag{4}$$

which is expected to hold for rough self-similar flows, in the limit $\kappa \to 0$. It is worth recalling that, under that assumption, Lagrangian particles separate super-diffusively according to a generalization of the Richardson law: $\langle|\underline{\mathbf{r}}(t)|^\zeta\rangle \propto t^{\zeta/(1-h)}$ [1].

The propagator $\mathcal{P}_{2n}$ relates the scalar correlation functions at every time t to those at the initial time

$$C_{2n}(\underline{\mathbf{r}}, t) \equiv \langle\theta(\mathbf{r}_1, t), \ldots, \theta(\mathbf{r}_N, t)\rangle = \int \mathcal{P}_{2n}(\underline{\boldsymbol{\rho}}, 0|\underline{\mathbf{r}}, t)C_{2n}(\underline{\boldsymbol{\rho}}, 0)\,\mathrm{d}\underline{\boldsymbol{\rho}}. \tag{5}$$

[3] Due to the velocity field homogeneity, the Lagrangian particle statistics actually depends only on their relative separations, the absolute position of the center of mass being irrelevant. $\mathcal{P}_{2n}$ is therefore, a function of only $2n-1$ independent space vectors. In the sequel, all the integrations over the configuration space are to be understood as performed over the set of independent coordinates; the volume element $\mathrm{d}\underline{\boldsymbol{\rho}}$ has thus a dimension $(2n-1)d$.

The single-point moments of the scalar PDF are therefore

$$C_{2n}(\underline{\mathbf{0}}) = \langle \theta^{2n} \rangle(t) = \int \mathcal{P}_{2n}(\underline{\boldsymbol{\rho}}, 0|\underline{\mathbf{0}}, t) C_{2n}(\underline{\boldsymbol{\rho}}, 0)\, \mathrm{d}\underline{\boldsymbol{\rho}}, \tag{6}$$

where $\underline{\mathbf{0}} = (\mathbf{0}, \ldots, \mathbf{0})$, and the symbol $\langle \cdot \rangle$ denotes independent averages over the velocity field and over the initial condition $\langle \cdot \rangle \equiv \langle \langle \cdot \rangle_0 \rangle_v$. Exploiting the rescaling of $\mathcal{P}_{2n}$ (Eq. (4)) with $\lambda = t^{-1/(1-h)}$, Eq. (6) can be eventually recasted as

$$\langle \theta^{2n} \rangle(t) = \int \mathcal{P}_{2n}(\underline{\boldsymbol{\rho}}, 0|\underline{\mathbf{0}}, 1) C_{2n}(t^{1/(1-h)}\underline{\boldsymbol{\rho}}, 0)\, \mathrm{d}\underline{\boldsymbol{\rho}}. \tag{7}$$

From Eq. (7) it is clear that the time dependence of $\langle \theta^{2n} \rangle$ is completely determined by the spatial properties of the initial condition. To know the large-time expression of the single-point moments, we should therefore investigate the behavior of the initial correlation functions as the particle separations become infinite. Two different classes of initial conditions can be identified according to whether the Corrsin integral [19] $J_0 = \int C_2(\mathbf{r}, t)\, \mathrm{d}\mathbf{r}$ is vanishing or not [12]. J_0 is conserved by the evolution equation (1) and therefore, its value is set by the initial correlation function. We explicitly consider here the case $J_0 \neq 0$. When $J_0 = 0$, it can be shown that for the realizability of the initial condition the integral $J_1 = \int r^{1-h} C_2(\mathbf{r}, 0)\, \mathrm{d}\mathbf{r}$ cannot vanish [11]. For a discussion of the $J_0 = 0$ case we refer the reader to Ref. [12]. For the sake of clarity, let us start from the case $2n = 2$

$$\langle \theta^2 \rangle(t) = \int \mathcal{P}_2(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, 0|\mathbf{0}, \mathbf{0}, 1) C_2(t^{1/(1-h)}\boldsymbol{\rho}_{12}, 0)\, \mathrm{d}\boldsymbol{\rho}_{12}, \tag{8}$$

where $\boldsymbol{\rho}_{12} = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$. In the large-time limit, $C_2(t^{1/(1-h)}\boldsymbol{\rho}_{12}, 0)$ tends to $J_0 t^{-d/(1-h)} \delta(\boldsymbol{\rho}_{12})$ and the variance of the scalar PDF assumes the asymptotic expression

$$\langle \theta^2 \rangle(t) \simeq J_0 t^{-d/(1-h)} \mathcal{P}_2(\mathbf{0}, \mathbf{0}, 0|\mathbf{0}, \mathbf{0}, 1). \tag{9}$$

In the case $2n = 4$ it is convenient to consider the cumulant expansion

$$C_4(\underline{\boldsymbol{\rho}}, 0) = K_4(\underline{\boldsymbol{\rho}}) + K_2(\boldsymbol{\rho}_{12}) K_2(\boldsymbol{\rho}_{34}) + K_2(\boldsymbol{\rho}_{13}) K_2(\boldsymbol{\rho}_{24}) + K_2(\boldsymbol{\rho}_{14}) K_2(\boldsymbol{\rho}_{23}), \tag{10}$$

where the second-order cumulant K_2 coincides with the correlation C_2 for zero mean initial conditions. As a consequence of their definition, the cumulants of a joint probability distribution tend to zero, when any two of the points on which they depend become infinitely far apart [21]. The large-time expression of $C_4(t^{1/(1-h)}\underline{\boldsymbol{\rho}}, 0)$ is thus

$$J_0^{(4)} t^{-3d/(1-h)} \delta(\boldsymbol{\rho}_{12}) \delta(\boldsymbol{\rho}_{13}) \delta(\boldsymbol{\rho}_{14}) + J_0^2 t^{-2d/(1-h)} [\delta(\boldsymbol{\rho}_{12}) \delta(\boldsymbol{\rho}_{34}) + \delta(\boldsymbol{\rho}_{13}) \delta(\boldsymbol{\rho}_{24}) + \delta(\boldsymbol{\rho}_{14}) \delta(\boldsymbol{\rho}_{23})], \tag{11}$$

where $J_0^{(4)} = \int K_4(\underline{\boldsymbol{\rho}})\, \mathrm{d}\underline{\boldsymbol{\rho}}$. It is then immediately seen that the large t contribution to $\langle \theta^4 \rangle$ is dominated by the two-particle terms in Eq. (11)

$$\langle \theta^4 \rangle > (t) \simeq \frac{3 J_0^2}{t^{2d/(1-h)}} \int \mathcal{P}_4(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_2, 0|\underline{\mathbf{0}}, 1)\, \mathrm{d}\boldsymbol{\rho}_1\, \mathrm{d}\boldsymbol{\rho}_2. \tag{12}$$

This explicitly shows that, at large t, the value of $\langle \theta^4 \rangle(t)$ for a generic homogeneous initial field is equivalent to that given by a Gaussian condition with the same value of the Corrsin integral. This conclusion holds for all moments, and therefore the large-time one-point statistics is only determined by the Gaussian contribution from the initial condition.

Let us now turn to show that the contribution stemming from the second-order cumulants is the leading one for all $\langle \theta^{2n} \rangle(t)$. In general, one should consider the cumulant expansion for $C_{2n}(\underline{\boldsymbol{\rho}}, 0)$ (see, e.g., Ref. [20])

$$C_{2n}(\underline{\boldsymbol{\rho}}, 0) = \sum \frac{1}{q!} \prod_{p=1}^{q} K^{(\boldsymbol{\mu}^{(p)})}, \tag{13}$$

where the summation is performed over all ordered set of vectors $\boldsymbol{\mu}^{(p)} \in \mathbb{N}^{2n}$, $\mu_i^{(p)} \in \{0, 1\}$, with as sum the vector $(1, 1, \ldots, 1) \in \mathbb{N}^{2n}$. In that representation, $K^{(\boldsymbol{\mu}^{(p)})}$ denotes the cumulant $K_m(\boldsymbol{\rho}_{i_1}, \boldsymbol{\rho}_{i_2}, \ldots, \boldsymbol{\rho}_{i_m})$ of order $m = \sum \mu_i^{(p)}$ and depending on the particles positions $\boldsymbol{\rho}_{i_1}, \ldots, \boldsymbol{\rho}_{i_m}$, where the indexes $i_1, \ldots, i_m$ are identified by the non-zero components of $\boldsymbol{\mu}^{(p)}$. If K_m is integrable, one obtains the large-time expression

$$K_m(t^{1/(1-h)}\boldsymbol{\rho}_{i_1}, t^{1/(1-h)}\boldsymbol{\rho}_{i_2}, \ldots, t^{1/(1-h)}\boldsymbol{\rho}_{i_m}) \simeq J_0^{(m)} t^{-(m-1)d/(1-h)} \delta(\boldsymbol{\rho}_{i_1} - \boldsymbol{\rho}_{i_2})\delta(\boldsymbol{\rho}_{i_1} - \boldsymbol{\rho}_{i_3}) \cdots \delta(\boldsymbol{\rho}_{i_1} - \boldsymbol{\rho}_{i_m}) \tag{14}$$

with[4] $J_0^{(m)} = \int K_m(\underline{\boldsymbol{\rho}})\, \mathrm{d}\underline{\boldsymbol{\rho}}$. The initial condition correlation functions are reasonably assumed to decay rapid enough to guarantee the integrability of all the cumulants. Every integral in the sum (13) is thus convergent by hypothesis. The generic addendum in that sum results from the product of q ($q \le n$) cumulants (whose orders have as sum $2n$) and so leads a large-time contribution proportional to $t^{-(2n-q)d/(1-h)}$. It follows that the $(2n-1)!!$ terms like $K_2(t^{1/(1-h)}\boldsymbol{\rho}_{i_1,i_2}) \cdots K_2(t^{1/(1-h)}\boldsymbol{\rho}_{i_{2n-1},i_{2n}})$ (corresponding to $q = n$) are the dominant ones in the limit of large t. The ratio between the non-Gaussian contribution from the initial condition and the two-particle one decays as $t^{-d/(1-h)}$ with a proportionality constant depending on the initial condition and on the order of the cumulant. Consequently, the large-time expression of the scalar single-point moments for a generic n reads[5] (see Ref. [12])

$$\langle \theta^{2n} \rangle(t) \simeq \frac{J_0^n}{t^{nd/(1-h)}} (2n-1)!! \int \mathcal{P}_{2n}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_n, \boldsymbol{\rho}_n, 0|\underline{\mathbf{0}}, 1)\, \mathrm{d}\boldsymbol{\rho}_1, \ldots, \mathrm{d}\boldsymbol{\rho}_n. \tag{15}$$

It could be already inferred from Eq. (15) that in general $\theta(\mathbf{0}, t)$ is not normally distributed at large times. Indeed, the condition for Gaussianity is that the integral appearing in Eq. (15) should be proportional to a constant to the nth power for all n. The only known occurrence of this factorization is when the Lagrangian particles are independent and $\mathcal{P}_{2n}$ equals the product of $2n$ terms of the type $\mathcal{P}_1(\boldsymbol{\rho}, 0|\mathbf{r}, t)$. Within the framework of the Kraichnan model this is exactly verified only for a diffusive velocity field [12].

In the next section we present an alternative, more informative demonstration of the super-Gaussian behavior of scalar statistics. It will unveil the relation between large scalar excursions and the depletion of particle dispersion.

3. Gaussian initial conditions give super-Gaussian large-time distributions

We have recalled above the result obtained in Ref. [12] that the large-time scalar PDF for an arbitrary initial condition coincides with that obtained from a Gaussian initial condition with the same value of J_0. We can now focus on the situation where $\theta(\mathbf{r}, 0)$ is a Gaussian random field and then prove that $\theta(\mathbf{r}, t)$ is super-Gaussian. This will demonstrate that the passive scalar decay in a self-similar rough flow is generically intermittent.

The scalar field at time t can be written in the form

$$\theta(\mathbf{r}, t) = \int \theta(\boldsymbol{\rho}, 0) p(\boldsymbol{\rho}, 0|\mathbf{r}, t)\, \mathrm{d}\boldsymbol{\rho}, \tag{16}$$

where $p(\boldsymbol{\rho}, 0|\mathbf{r}, t)$ is the random probability density defined in Section 2: $p(\boldsymbol{\rho}, 0|\mathbf{r}, t) = \lim_{\kappa \to 0} p^\kappa(\boldsymbol{\rho}, 0|\mathbf{r}, t)$. For a given realization of the velocity field the scalar PDF $P(\theta, t)$ may be expressed in the form $P(\theta, t) = \langle \delta(\theta - \theta(\mathbf{r}, t)) \rangle$ by means of the Lagrangian expression of θ (Eq. (16)). The above average is performed over the initial condition

[4] The integral $J_0^{(2)}$ coincides with the Corrsin integral J_0.

[5] Notice that in the case $J_0 = 0$ the scalar single-point moments decay is different: $\langle \theta^{2n} \rangle(t) \propto t^{-n[d/(1-h)+1]}$ [12].

statistics and the realizations of the velocity field. Exploiting the oscillatory representation of the Dirac delta and averaging over the Gaussian initial condition yields

$$P(\theta,t)=\left\langle \frac{1}{(2\pi\langle\theta_0^2\rangle)^{1/2}\sigma_v(\mathbf{r},t)}\,\mathrm{e}^{-\theta^2/2\langle\theta_0^2\rangle\sigma_v^2(\mathbf{r},t)}\right\rangle_v \tag{17}$$

with

$$\sigma_v^2(\mathbf{r},t)=\frac{1}{\langle\theta_0^2\rangle}\int \mathrm{d}\boldsymbol{\rho}\,\mathrm{d}\boldsymbol{\rho}'C_2(\boldsymbol{\rho}-\boldsymbol{\rho}',0)p(\boldsymbol{\rho},0|\mathbf{r},t)p(\boldsymbol{\rho}',0|\mathbf{r},t), \tag{18}$$

and $\langle\theta_0^2\rangle=C_2(\mathbf{0},0)$. If the initial condition has a correlation length L_θ, the random variable $\sigma_v^2(\mathbf{r},t)$, that depends on the specific realization of the flow, can be interpreted as the probability that two tracer particles clumped in $\mathbf{r}$ at time t, were initially at a distance smaller than L_θ

$$\sigma_v^2(\mathbf{r},t)\simeq \mathrm{Prob}\{|\boldsymbol{\rho}-\boldsymbol{\rho}'|<L_\theta\}. \tag{19}$$

The random probability $\sigma_v^2(\mathbf{r},t)$ is therefore inversely related to the dispersion rate of a scalar blob: the larger the $\sigma_v^2(\mathbf{r},t)$, the smaller the rate-of-growth of the blob.

Since $\mathbf{v}$ is a random field, $\sigma_v(\mathbf{r},t)$ is characterized by a PDF $H(\sigma_v,t)$, determined by the velocity statistics. The complete scalar PDF is then derived by taking the average over such distribution

$$P(\theta,t)=\int \mathrm{d}\sigma_v H(\sigma_v,t)\frac{1}{(2\pi\langle\theta_0^2\rangle)^{1/2}\sigma_v}\,\mathrm{e}^{-\theta^2/2\langle\theta_0^2\rangle\sigma_v^2}. \tag{20}$$

Remark that, after averaging over the velocity statistics, the dependence on $\mathbf{r}$ is lost due to the statistical homogeneity of $\mathbf{v}$. The latter expression for $P(\theta,t)$ is the convolution between H and the Gaussian distribution with zero mean and variance $\langle\theta_0^2\rangle$. This can be recast in the statement that the random variable θ at any time t can always be expressed as

$$\theta \triangleq \theta_0\sigma_v, \tag{21}$$

where θ_0 is a Gaussian variable with variance $\langle\theta_0^2\rangle$ only depending on the initial condition. The evolution of $P(\theta,t)$ in time is contained in $H(\sigma_v,t)$, which is in turn determined by the Lagrangian statistics of the turbulent advecting flow. An analogous factorization is proved to be at the origin of asymptotic scalar intermittency in linear shear flows [10].

The single-point scalar moments are then easily computed as[6]

$$\langle\theta^{2n}\rangle(t)=(2n-1)!!\langle\theta_0^2\rangle^n\,\langle\sigma_v^{2n}\rangle_v(t). \tag{22}$$

The Jensen inequality [20] insures that $\langle\sigma_v^{2n}\rangle_v\ge\langle\sigma_v^2\rangle_v^n$, and thus

$$\frac{\langle\theta^{2n}\rangle(t)}{[\langle\theta^2\rangle(t)]^n}\ge(2n-1)!!. \tag{23}$$

Therefore, the scalar PDF $P(\theta,t)$ is always broader than Gaussian, except for the case $H(\sigma_v,t)=\delta(\sigma_v-\bar{\sigma}(t))$. Physically, the condition $H(\sigma_v,t)=\delta(\sigma_v-\bar{\sigma}(t))$ is expected to hold only for the special case of a diffusive velocity field. This is in agreement with the results in Eq. (15) for a Kraichnan velocity field with exponent $h=0$ [12]. As an instance of the relation (21), if the distribution function of σ_v^2 is exponential, the scalar PDF has exponential tails

[6] It should be noted that the scalar single-point moments converge to the large-time expression (15) not uniformly in n [10]. Therefore, each moment $\langle\sigma_v^{2n}\rangle(t)$ satisfies the decay law (15) only for times $t\gg \mathrm{O}(n^{(1-h)/2})$.

as in the case of two-dimensional Navier–Stokes flows analyzed in Ref. [12]. In general, we expect the statistics of σ_v^2 to be sensitive to the details of the velocity field.

It should be noted that large deviations of θ correspond to large values of σ_v. According to the physical meaning of σ_v^2 (see Eq. (19)), the passive scalar intermittency is therefore connected to clustering phenomena of the Lagrangian trajectories. This has been already pointed out by Shraiman and Siggia [22] for smooth flows and in a more general context by Vanden Eijnden [10].

4. Conclusions and discussion

We have shown that the large-time one-point statistics of a decaying passive scalar is generically intermittent. The scalar intensity is a random variable equivalent to the product of a Gaussian variable related to the initial condition times a positive random variable that measures the frequency of exceedingly small particle dispersion events. Our conclusions are independent of the choice of random initial conditions, which are only assumed to be homogeneous in space and with rapidly decaying spatial correlation.

It could be of interest to investigate the situation where the velocity field is itself intermittent, as in real three-dimensional flows. Since the previous arguments hold for whatever h, $0 < h < 1$, it is plausible that intermittency is present also in this case. It is, however, unclear whether the decay will still be self-similar in time. This point requires further investigations.

We remark that similar results can be obtained in the case of forced passive scalar transport with Gaussian forcing. In that situation a source term $f(\mathbf{r}, t)$ should be added to the right-hand side of Eq. (1) and the scalar intermittency is proved by following a method similar to the one we have presented. In the forced case the quantity σ_v^2 becomes the average time a fluid spot takes to reach a size larger than the correlation length of the forcing. Note, however, that these results hold only for a Gaussian forcing. We do not expect in the forced case the same degree of universality as in the decaying case. Finally, there are experimental [23] and numerical [24] observations of sub-Gaussian scalar PDFs (i.e., with narrower than Gaussian tails) for forced scalar. There, the sub-Gaussianity of the scalar distribution may stem from a sub-Gaussian statistics of the forcing. Further analysis is needed to clarify this point.

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References

[1] G. Falkovich, K. Gawędzki, M. Vergassola, Particles and fields in fluid turbulence, Rev. Mod. Phys. 73 (2001) 913–975.
[2] D.T. Son, Turbulent decay of a passive scalar in the Batchelor limit: exact results from a quantum-mechanical approach, Phys. Rev. E 59 (1999) R3811–R3814.
[3] E. Balkovsky, A. Fouxon, Universal long-time properties of Lagrangian statistics in the Batchelor regime and their application to the passive scalar problem, Phys. Rev. E 60 (1999) 4164–4174.
[4] A.J. Majda, The random uniform shear layer: an explicit example of turbulent diffusion with broad tail probability distribution, Phys. Fluids A 5 (1993) 1963–1970.
[5] A.J. Majda, Explicit inertial range renormalization theory in a model of turbulent diffusion, J. Statist. Phys. 73 (1993) 515–542.
[6] R.M. McLaughlin, A.J. Majda, An explicit example with non-Gaussian probability distribution for nontrivial scalar mean and fluctuation, Phys. Fluids 8 (1996) 536–547.

[7] J.C. Bronski, R.M. McLaughlin, Scalar intermittency and the ground state of periodic Schrödinger equations, Phys. Fluids 9 (1996) 181–190.
[8] J.C. Bronski, R.M. McLaughlin, The problem of moments and the Majda model for scalar intermittency, Phys. Lett. A 265 (2000) 257–263.
[9] J.C. Bronski, R.M. McLaughlin, Rigorous estimate of the tails of the probability distribution function for the random linear shear model, J. Statist. Phys. 98 (2000) 897–915.
[10] E. Vanden Eijnden, Non-Gaussian invariant measures for the Majda model of decaying turbulent transport, Commun. Pure Appl. Math. 54 (2001) 1146–1167.
[11] G.L. Eyink, J. Xin, Self-similar decay in the Kraichnan model of a passive scalar, J. Statist. Phys. 100 (2000) 679–741.
[12] M. Chaves, G. Eyink, U. Frisch, M. Vergassola, Universal decay of scalar turbulence, Phys. Rev. Lett. 86 (2001) 2305–2308.
[13] R.H. Kraichnan, Small-scale structure of a scalar field convected by turbulence, Phys. Fluids 11 (1968) 945–963.
[14] J. Paret, P. Tabeling, Intermittency in the two-dimensional inverse cascade of energy: experimental observations, Phys. Fluids 10 (1998) 3126–3136.
[15] G. Boffetta, A. Celani, M. Vergassola, Inverse energy cascade in two-dimensional turbulence: deviations from Gaussian behavior, Phys. Rev. E 61 (2000) R29–R32.
[16] D. Bernard, K. Gawędzki, A. Kupiainen, Slow modes in passive advection, J. Statist. Phys. 90 (1998) 519–569.
[17] W. E, E. Vanden Eijnden, Generalized flows, intrinsic stochasticity, and turbulent transport, Proc. Natl. Acad. Sci. USA 97 (2000) 8200–8205.
[18] Y. Le Jan, O. Raimond, Integration of Brownian vector fields, Ann. Probab. 30 (2002) 826–873.
[19] S. Corrsin, The decay of isotropic temperature fluctuations in an isotropic turbulence, J. Aeronaut. Sci. 18 (1951) 417–423.
[20] A.N. Shiryaev, Probability, Springer, Berlin, 1984.
[21] A.S. Monin, A.M. Yaglom, in: J. Lumley (Ed.), Statistical Fluid Mechanics, vol. 1, MIT Press, Cambridge, MA, 1975.
[22] B. Shraiman, E.D. Siggia, Lagrangian path integrals and fluctuations in random flows, Phys. Rev. E 49 (1994) 2912–2927.
[23] Jayesh, Z. Warhaft, Probability distribution of a passive scalar in grid-generated turbulence, Phys. Rev. Lett. 67 (1991) 3503–3506.
[24] A. Celani, A. Lanotte, A. Mazzino, M. Vergassola, Fronts in passive scalar turbulence, Phys. Fluids 13 (2001) 1768–1783.