DISPERSIVE ESTIMATE FOR TWO-PERIODIC DISCRETE ONE-DIMENSIONAL SCHRÖDINGER OPERATOR

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ABSTRACT. For the two-periodic discrete one-dimensional Schrödinger operator

$$(H\psi)_n = -(\psi_{n+1} + \psi_{n-1}) + V_n\psi_n, \quad n \in \mathbb{Z},$$

with $V_{n+2} = V_n$ for every $n \in \mathbb{Z}$ and $V_0, V_1 \in \mathbb{R}$, we show the dispersive estimate

$$\|e^{-\mathrm{i}tH}\psi\|_{\ell^\infty} \leq 181\left(1 + (V_0 - V_1)^2\right)^{\frac{1}{4}} (1 + |t|)^{-\frac{1}{3}} \|\psi\|_{\ell^1}, \quad \forall\ \psi \in \ell^1(\mathbb{Z}), \quad \forall\ t \in \mathbb{R}.$$

1. Introduction and main result

Consider the two-periodic discrete one-dimensional Schrödinger operator

$$(H\psi)_n = -(\psi_{n+1} + \psi_{n-1}) + V_n\psi_n, \quad n \in \mathbb{Z} \tag{1}$$

with $V_{n+2} = V_n$ for every $n \in \mathbb{Z}$, and $V_0, V_1 \in \mathbb{R}$. It is well known that the spectrum of H is purely absolutely continuous and the time evolution $e^{-\mathrm{i}tH}$ presents ballistic transport (see [4]), i.e., the weighted ℓ^2-norm

$$\left(\sum_{n\in\mathbb{Z}} n^2 \left|(e^{-\mathrm{i}tH}\psi)_n\right|^2\right)^{\frac{1}{2}}$$

grows linearly with t provided that $\sum_{n\in\mathbb{Z}} n^2 |\psi_n|^2 < \infty$. Since ballistic transport means that, for a given well-localized initial condition ψ, $e^{-\mathrm{i}tH}\psi$ does not keep well-localized as ψ, we wander the variation of shape for $e^{-\mathrm{i}tH}\psi$.

For the free Schrödinger operator, $-\Delta : \ell^2(\mathbb{Z}^\nu) \to \ell^2(\mathbb{Z}^\nu)$, $\nu \geq 1$,

$$(-\Delta\psi)_n = -\sum_{|m-n|=1} \psi_m, \quad n \in \mathbb{Z}^\nu, \tag{2}$$

we recall that Stefanov-Kevrekidis [9] have shown: there exists a constant $C > 0$ such that

$$\|e^{\mathrm{i}t\Delta}\psi\|_{\ell^\infty} \leq C(1 + |t|)^{-\frac{1}{3}} \|\psi\|_{\ell^1}, \quad \forall\ \psi \in \ell^1(\mathbb{Z}^\nu). \tag{3}$$

The above inequality is called "dispersive estimate", by which we see that the $\ell^\infty-$norm tends to zero as time goes to infinity, with the "$|t|^{-\frac{1}{3}}$" decay rate.

In general, the dispersion for a one-dimensional linear operator is related to the absolutely continuous spectrum, even though there is not yet rigorous argument

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describing the link between them. For the operator $H : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$,

$$(H\psi)_n = -(\psi_{n+1} + \psi_{n-1}) + V_n\psi_n, \quad n \in \mathbb{Z}, \tag{4}$$

Pelinovsky-Stefanov [7] have shown

$$\|e^{-\mathrm{i}tH} P_{ac}\psi\|_{\ell^\infty} \le C\,(1+|t|)^{-\frac{1}{3}}\,\|\psi\|_{\ell^1}, \quad \forall\, \psi \in \ell^1(\mathbb{Z}),$$

for "generic"[1] pointwise decaying potential $(V_n) \subset \mathbb{R}$ satisfying

$$\sum_n (1+n^2)^{\frac{s}{2}} |V_n| < \infty, \quad s > \frac{5}{2},$$

where P_{ac} means the projection onto the absolutely continuous part of spectrum. Another classical dispersive estimate is given by Komech-Kopylova-Kunze [6], who show that, for $s > \frac{7}{2}$,

$$\|e^{-\mathrm{i}tH} P_{ac}\|_{\ell^2_s \to \ell^2_{-s}} = \mathcal{O}\left(t^{-\frac{3}{2}}\right), \quad t \to \infty, \tag{5}$$

for the operator (4) for "generic"[2] $(V_n) \subset \mathbb{R}$ with finite support, where the weighted ℓ^2-norm is defined by

$$\|q\|_{\ell^2_\sigma} := \left(\sum_{n\in\mathbb{Z}} (1+n^2)^{\frac{\sigma}{2}} |q_n|^2\right)^{\frac{1}{2}}, \quad \sigma \in \mathbb{R}.$$

Other related work has been done by Cuccagna-Tarulli [3] (see also [1]). For the dispersive estimates for continuous Schrödinger operators, we can refer to [8].

Recently, Bambusi-Zhao [2] have considered the quasi-periodic Schrödinger operator $H_\theta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$,

$$(H_\theta\psi)_n = -(\psi_{n+1} + \psi_{n-1}) + V(\theta + n\omega)\psi_n, \quad n \in \mathbb{Z}, \tag{6}$$

where $V \in C^\omega_r(\mathbb{T}^d, \mathbb{R})$ with $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, $d \ge 1$, and $\omega \in DC_d(\gamma,\tau)$ for $\gamma > 0$ and $\tau > d-1$, i.e.,

$$\inf_{j\in\mathbb{Z}} |\langle k, \omega\rangle - j\pi| > \frac{\gamma}{|k|^\tau}, \quad \forall\, k \in \mathbb{Z}^d \setminus \{0\}.$$

It is well known that when the potential function V is sufficiently small, the operator H_θ has purely absolutely continuous spectrum for every $\theta \in \mathbb{T}^d$ and the time evolution $e^{-\mathrm{i}tH_\theta}$ presents ballistic transport (see [11]). As for the dispersion, it is shown that, if $|V|_r$ is sufficiently small, then for every $\theta \in \mathbb{T}^d$:

$$\|e^{-\mathrm{i}tH_\theta}\psi\|_{\ell^\infty} \le C|\ln|V|_r|^{a\ln\ln(3+|t|))^2 d}(1+|t|)^{-\frac{1}{3}}\|\psi\|_{\ell^1}, \quad \forall\, \psi \in \ell^1(\mathbb{Z}),$$

for some absolute constants $C, a > 0$, which implies a $|t|^{-\zeta}-$dispersive decay for any $0 < \zeta < \frac{1}{3}$.

In this paper, the main conclusion is:

Theorem 1.1. *Consider the operator H given in (1). For arbitrary $t \in \mathbb{R}$,*

$$\|e^{-\mathrm{i}tH}\psi\|_{\ell^\infty} \le 181\left(1 + (V_0 - V_1)^2\right)^{\frac{1}{4}}(1+|t|)^{-\frac{1}{3}}\|\psi\|_{\ell^1}, \quad \psi \in \ell^1(\mathbb{Z}). \tag{7}$$

[1] See Definition 1 of [7].

[2] See Definition 5.1 of [6].

Remark 1.1. *For the case $V_0 = V_1$, H is equivalent to $-\Delta$ given in (2) with $\nu = 1$. Hence, for proving Theorem 1.1, we always assume that $V_0 \neq V_1$. In view of (7), we see that the variation of the 2−periodic potential does not essentially affect the $|t|^{-\frac{1}{3}}$−dispersive decay, while the coefficient in front grows with $|V_0 - V_1|$ and goes to ∞ as $|V_0 - V_1| \to \infty$.*

Remark 1.2. *The method employed in this manuscript is mainly by Fourier transform. However, it is not suitable for exploiting the weighted ℓ^2 dispersive estimate as Komech-Kopylova-Kunze [6]. Firstly, it is non-trivial to verify the conditions in their theorems (for example "generic" and "finitely supported" etc) for periodic potentials. Secondly, their techniques such as Laplace transformation and Puiseux expansion of resolvent still have some differences from our proof. Therefore, we are not able to achieve an easy corollary similar to (5), even though we believe that it should be true.*

As usual, from this estimate in Theorem 1.1, one can deduce Strichartz estimates via [5] as well as decay for the solution of the non-linear two-periodic discrete Schrödinger equation

$$\mathrm{i}\dot{q}_n = -(q_{n+1} + q_{n-1}) + V_n q_n \pm |q_n|^{p-1} q_n, \quad n \in \mathbb{Z}, \tag{8}$$

provided p is large enough. Here we concentrate just on dispersive decay in ℓ^∞ and give the result for $p \geq 5$ (indeed $\frac{1}{p-2} \leq \frac{1}{3}$ is necessary).

Corollary 1.1. *Under the assumptions of Theorem 1.1, consider Eq.* (8) *with $p \geq 5$. There exist $C > 0$ and $\delta > 0$ s.t., if the initial datum $q(0)$ fulfills $\|q(0)\|_{\ell^1(\mathbb{Z})} < \delta$, then the solution of Eq.* (8) *fulfills*

$$\|q(t)\|_{\ell^\infty} \leq C(1 + |t|)^{-\frac{1}{3}}. \tag{9}$$

The remaining part of paper is organised as follows. In Section 2, by Fourier transform, we get the explicit form of the time evolution $e^{-\mathrm{i}tH}$, which is represented by the eigenvectors and eigenvalues of the 2×2 matrix

$$A(\theta) = \begin{pmatrix} V_0 & -2\cos(\theta) \\ -2\cos(\theta) & V_1 \end{pmatrix}, \quad \theta \in \mathbb{T}.$$

Then, we show the $\mathcal{C}^1$−regularity of its eigenvectors and the $\mathcal{C}^3$−transversality of its eigenvalues (w.r.t. θ) in Section 3, with a technical lemma shown in Section 4. By applying Van der Corput lemma (Lemma A.1 in Appendix A), we estimate the oscillatory integrals in $e^{-\mathrm{i}tH}$ in Section 3.3, hence the dispersive estimate is shown.

2. Fourier transform and explicit form of $e^{-\mathrm{i}tH}$

We focus on the 2−periodic time-dependent Schrödinger equation

$$\mathrm{i}\dot{q}_n = -(q_{n+1} + q_{n-1}) + V_n\, q_n, \quad n \in \mathbb{Z}, \tag{10}$$

with $q(0) \in \ell^1(\mathbb{Z})$. By the Fourier transform

$$(q_n)_{n\in\mathbb{Z}} \mapsto G(\theta) := \sum_{n\in\mathbb{Z}} q_n e^{\mathrm{i}n\theta}, \quad \theta \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z},$$

the above equation is transformed into

$$\mathrm{i}\partial_t G(\theta,t)=\sum_{n\in\mathbb{Z}}[-(q_{n+1}(t)+q_{n-1}(t))+V_n q_n(t)]e^{\mathrm{i}n\theta}.$$

Decompose $G(\theta,t)$ as $G(\theta,t)=G_0(\theta,t)+G_1(\theta,t)$, with

$$G_j(\theta,t):=\sum_{k\in\mathbb{Z}}q_{2k+j}(t)e^{\mathrm{i}(2k+j)\theta},\quad j=0,1.$$

We find that

$$\begin{aligned}\mathrm{i}\partial_t G_0(\theta,t) &= \mathrm{i}\sum_{k\in\mathbb{Z}}\dot{q}_{2k}(t)e^{\mathrm{i}\cdot 2k\theta}\\ &= \sum_{k\in\mathbb{Z}}\left[-(q_{2k-1}(t)+q_{2k+1}(t))+V_0 q_{2k}(t)\right]e^{\mathrm{i}\cdot 2k\theta}\\ &= -(e^{-\mathrm{i}\theta}+e^{\mathrm{i}\theta})G_1(\theta,t)+V_0\,G_0(\theta,t)\\ &= -2\cos(\theta)\,G_1(\theta,t)+V_0\,G_0(\theta,t)\end{aligned}$$

and similarly,

$$\mathrm{i}\partial_t G_1(\theta,t)=-2\cos(\theta)\,G_0(\theta,t)+V_1\,G_1(\theta,t)$$

It is exactly the 2−dimensional system

(11)
$$\mathrm{i}\partial_t\begin{pmatrix}G_0(\theta,t)\\ G_1(\theta,t)\end{pmatrix}=A(\theta)\begin{pmatrix}G_0(\theta,t)\\ G_1(\theta,t)\end{pmatrix},\quad A(\theta):=\begin{pmatrix}V_0 & -2\cos(\theta)\\ -2\cos(\theta) & V_1\end{pmatrix}.$$

For the Hermitian matrix $A(\theta)$, there exists an 2×2 orthonormal matrix

$$U(\theta)=\begin{pmatrix}U_{00}(\theta) & U_{01}(\theta)\\ U_{10}(\theta) & U_{11}(\theta)\end{pmatrix}$$

analytically depending on θ, such that

$$U^*(\theta)A(\theta)U(\theta)=\Lambda(\theta):=\begin{pmatrix}\lambda_0(\theta) & 0\\ 0 & \lambda_1(\theta)\end{pmatrix},$$

with $\lambda_0(\theta)$ and $\lambda_1(\theta)$ two real eigenvalues of $A(\theta)$. Hence, for Eq. (11), its solution is

$$\begin{pmatrix}G_0(\theta,t)\\ G_1(\theta,t)\end{pmatrix}=U(\theta)\begin{pmatrix}e^{-\mathrm{i}\lambda_0(\theta)t} & 0\\ 0 & e^{-\mathrm{i}\lambda_1(\theta)t}\end{pmatrix}U^*(\theta)\begin{pmatrix}G_0(\theta,0)\\ G_1(\theta,0)\end{pmatrix}.$$

Then, for $n=2k_*+j$ with $k_*\in\mathbb{Z}$ and $j=0$ or 1, the solution of Eq. (10) satisfies

$$\begin{aligned}q_n(t) &= \frac{1}{2\pi}\int_{\mathbb{T}}G_j(\theta,t)e^{-\mathrm{i}(2k_*+j)\theta}\,d\theta\\ &= \frac{1}{2\pi}\int_{\mathbb{T}}\sum_{l_1,l_2=0,1}e^{-\mathrm{i}\lambda_{l_1}(\theta)t}U_{jl_1}(\theta)U^*_{l_1l_2}(\theta)G_{l_2}(\theta,0)e^{-\mathrm{i}(2k_*+j)\theta}\,d\theta\\ &= \frac{1}{2\pi}\sum_{l_1,l_2=0,1}\int_{\mathbb{T}}e^{-\mathrm{i}\lambda_{l_1}(\theta)t}U_{jl_1}(\theta)U^*_{l_1l_2}(\theta)\sum_{k\in\mathbb{Z}}q_{2k+l_2}(0)e^{\mathrm{i}(2(k-k_*)+l_2-j)\theta}\,d\theta.\end{aligned}$$

By the condition $q(0)\in\ell^1(\mathbb{Z})$, we can commute the order of integration and summation. Summarizing the above demonstration, we get

Proposition 2.1. *Given $\psi \in \ell^1(\mathbb{Z})$, $t \in \mathbb{R}$, for $n = 2k_* + j$ with $k_* \in \mathbb{Z}$ and $j = 0$ or 1, we have*

$$\left(e^{-\mathrm{i}tH}\psi\right)_n = \frac{1}{2\pi} \sum_{l_1,l_2=0,1} \sum_{k\in\mathbb{Z}} \psi_{2k+l_2} \int_{\mathbb{T}} U_{jl_1}(\theta) U^*_{l_1 l_2}(\theta) e^{\mathrm{i}\left[-\lambda_{l_1}(\theta)t + (2(k-k_*)+l_2-j)\theta\right]} \, d\theta.$$

3. Properties of eigenvalues and eigenvectors

By Proposition 2.1, we see that the dispersive estimate is determined by the properties of eigenvalues and eigenvectors of matrix $A(\theta)$ with respect to $\theta \in \mathbb{T}$.

3.1. **Transversality of eigenvalues for $A(\theta)$.** For the Hermitian matrix $A(\theta)$ given in (11), it is easy to calculate that

$$\begin{aligned} \lambda_0(\theta) &= \frac{-(V_0+V_1) + \sqrt{(V_0-V_1)^2 + 16\cos^2(\theta)}}{2}, \\ \lambda_1(\theta) &= \frac{-(V_0+V_1) - \sqrt{(V_0-V_1)^2 + 16\cos^2(\theta)}}{2}. \end{aligned}$$

So for $j = 0, 1$, the derivatives of eigenvalues w.r.t. θ satisfy that

$$\left|\lambda_j^{(k)}(\theta)\right| = 2 \left| \left(\sqrt{\frac{(V_0-V_1)^2}{16} + \cos^2(\theta)} \right)^{(k)} \right|, \quad k \geq 1. \tag{12}$$

By the following lemma, we can get a lower bound for $|\lambda_j''| + |\lambda_j'''|$.

Lemma 3.1. *For any $a > 0$, we have that*

$$f_a(\theta) := \left| \frac{d^2}{d\theta^2} \sqrt{\cos^2(\theta) + a} \right| + \left| \frac{d^3}{d\theta^3} \sqrt{\cos^2(\theta) + a} \right| \geq \frac{1}{4\sqrt{a+1}}, \quad \forall\ \theta \in \mathbb{T}.$$

Moreover, the subset

$$\left\{ \theta \in \mathbb{T} : \left| \frac{d^2}{d\theta^2} \sqrt{\cos^2(\theta) + a} \right| < \frac{1}{8\sqrt{a+1}} \right\}$$

consists of at most 8 mutually disjoint subintervals.

We postpone the proof of this technical lemma to Section 4. Now, with Lemma 3.1, we get immediately the following estimate.

Corollary 3.1. *The eigenvalues of $A(\theta)$ satisfy*

$$|\lambda_j''(\theta)| + |\lambda_j'''(\theta)| \geq \frac{2}{\sqrt{16 + (V_0-V_1)^2}}, \quad j = 0, 1, \quad \forall\ \theta \in \mathbb{T}. \tag{13}$$

Moreover, for $j = 0, 1$,

$$\Theta_j := \left\{ \theta \in \mathbb{T} : |\lambda_j''(\theta)| < \frac{1}{\sqrt{16 + (V_0-V_1)^2}} \right\} \tag{14}$$

consists of at most 8 mutually disjoint subintervals.

Remark 3.1. *The dispersive estimate is usually related to the transversality of kernel function in the oscillatory integral. We will see in the proof of Proposition 3.1 the necessity of the lower bound of the second or third derivative of λ_j for getting the "$t^{-\frac{1}{3}}$" asymptotic decay estimate by applying Van der Corput lemma. Moreover, the number of segments on which the second or third derivative is bounded from below is related to the coefficient in front of the decay.*

3.2. C^1 **property of eigenvectors for** $A(\theta)$**.** Since $U(\theta)$ is an orthonormal matrix for any $\theta \in \mathbb{T}$, we have immediately that

$$|U_{lm}(\theta)U^*_{mn}(\theta)| \le \frac{|U_{lm}(\theta)|^2 + |U^*_{mn}(\theta)|^2}{2} \le 1, \quad \forall\ l, m, n = 0, 1.$$

More precisely, for $\alpha := V_1 - V_0$, by a straightforward calculation, we get the eigenvector of $A(\theta)$ corresponding to $\lambda_0(\theta)$, i.e., the first column of $U(\theta)$ is

$$\begin{pmatrix} U_{00}(\theta) \\ U_{10}(\theta) \end{pmatrix} = \frac{1}{\sqrt{\left(\alpha + \sqrt{\alpha^2 + 16\cos^2(\theta)}\right)^2 + 16\cos^2(\theta)}} \begin{pmatrix} \alpha + \sqrt{\alpha^2 + 16\cos^2(\theta)} \\ 4\cos(\theta) \end{pmatrix},$$

and the eigenvector corresponding to $\lambda_1(\theta)$, i.e., the second column of $U(\theta)$ is

$$\begin{pmatrix} U_{01}(\theta) \\ U_{11}(\theta) \end{pmatrix} = \frac{1}{\sqrt{\left(\alpha - \sqrt{\alpha^2 + 16\cos^2(\theta)}\right)^2 + 16\cos^2(\theta)}} \begin{pmatrix} \alpha - \sqrt{\alpha^2 + 16\cos^2(\theta)} \\ 4\cos(\theta) \end{pmatrix}.$$

By a straightforward calculation, we get

$$\begin{aligned} & |(U_{00}(\theta)U^*_{00}(\theta))'|,\ |(U_{01}(\theta)U^*_{10}(\theta))'|,\ |(U_{10}(\theta)U^*_{01}(\theta))'|,\ |(U_{11}(\theta)U^*_{11}(\theta))'| \\ = & \frac{8|\alpha| \cdot |\sin(\theta)\cos(\theta)|}{(\alpha^2 + 16\cos^2(\theta))^{\frac{3}{2}}}, \\ & |(U_{00}(\theta)U^*_{01}(\theta))'|,\ |(U_{10}(\theta)U^*_{00}(\theta))'|,\ |(U_{01}(\theta)U^*_{11}(\theta))'|,\ |(U_{11}(\theta)U^*_{10}(\theta))'| \\ = & \frac{2\alpha^2 |\sin(\theta)|}{(\alpha^2 + 16\cos^2(\theta))^{\frac{3}{2}}}, \end{aligned}$$

then we calculate their integrals:

$$\begin{aligned} \int_{\mathbb{T}} |(U_{00}(\theta)U^*_{00}(\theta))'| d\theta &= 2 - \frac{2|\alpha|}{\sqrt{\alpha^2 + 16}}, \\ \int_{\mathbb{T}} |(U_{00}(\theta)U^*_{01}(\theta))'| d\theta &= \frac{8}{\sqrt{\alpha^2 + 16}}. \end{aligned}$$

To summarise, we have

Lemma 3.2. *For $l, m, n = 0, 1$,*

$$\sup_{\theta \in \mathbb{T}} |U_{lm}(\theta)U^*_{mn}(\theta)| + \int_{\mathbb{T}} \left|(U_{lm}(\theta)U^*_{mn}(\theta))'\right| d\theta \le 3.$$

3.3. **Proof of Theorem 1.1.** With the estimates obtained for the eigenvalues and eigenvectors of $A(\theta)$, we have that

Proposition 3.1. *For $j, l_1, l_2 = 0, 1$, for any $t \in \mathbb{R}$,*

$$\left| \int_{\mathbb{T}} U_{jl_1}(\theta) U^*_{l_1 l_2}(\theta) e^{\mathrm{i}\left[-\lambda_{l_1}(\theta)t + (2(k-k_*)+l_2-j)\theta\right]}\, d\theta \right| < \frac{284\left(16+(V_0-V_1)^2\right)^{\frac{1}{4}}}{(1+|t|)^{\frac{1}{3}}}.$$

Proof. For $|t| \le 1$, since $2^{\frac{1}{3}}(1+|t|)^{-\frac{1}{3}} \ge 1$, we have, by Lemma 3.2,

$$\left| \int_{\mathbb{T}} U_{jl_1}(\theta) U^*_{l_1 l_2}(\theta) e^{\mathrm{i}\left[-\lambda_{l_1}(\theta)t + (2(k-k_*)+l_2-j)\theta\right]}\, d\theta \right| \le 6\pi \le 6\pi \cdot 2^{\frac{1}{3}}(1+|t|)^{-\frac{1}{3}}. \tag{15}$$

Now we assume that $|t| > 1$, which implies that

$$2^{\sigma}(1+|t|)^{-\sigma} > |t|^{-\sigma} \quad \text{for } \sigma = \frac{1}{2} \text{ and } \frac{1}{3}.$$

In view of (13) in Corollary 3.1, we deduce that, for $l_1 = 0, 1$, for any $\theta \in \mathbb{T}$,

$$|\lambda''_{l_1}(\theta)| \ge \frac{1}{\sqrt{16+(V_0-V_1)^2}} \quad \text{or} \quad |\lambda'''_{l_1}(\theta)| \ge \frac{1}{\sqrt{16+(V_0-V_1)^2}}.$$

According to the definition of Θ_j given in (14), we apply Van der Corput lemma (Corollary A.1 in Appendix A) for $k = 3$ on each segment of Θ, and get

$$\begin{aligned}
&\left| \int_{\Theta_{l_1}} U_{jl_1}(\theta) U^*_{l_1 l_2}(\theta) e^{\mathrm{i}\left[-\lambda_{l_1}(\theta)t + (2(k-k_*)+l_2-j)\theta\right]}\, d\theta \right| \\
\le\ & 8 \cdot 18 \cdot 2^{\frac{1}{3}} \left(16+(V_0-V_1)^2\right)^{\frac{1}{6}} (1+|t|)^{-\frac{1}{3}}.
\end{aligned} \tag{16}$$

recalling that Θ_{l_1} consists of at most 8 segments (hence $\mathbb{T} \setminus \Theta_{l_1}$ consists of at most 9 segments). Then, by applying Van der Corput lemma for $k = 2$, on each segment of $\mathbb{T} \setminus \Theta_{l_1}$, we get

$$\begin{aligned}
&\left| \int_{\mathbb{T}\setminus\Theta_{l_1}} U_{jl_1}(\theta) U^*_{l_1 l_2}(\theta) e^{\mathrm{i}\left[-\lambda_{l_1}(\theta)t + (2(k-k_*)+l_2-j)\theta\right]}\, d\theta \right| \\
\le\ & 9 \cdot 8 \cdot 2^{\frac{1}{2}} \left(16+(V_0-V_1)^2\right)^{\frac{1}{4}} (1+|t|)^{-\frac{1}{2}}.
\end{aligned} \tag{17}$$

Thus, combining (16) and (17), we have, for $|t| > 1$,

$$\begin{aligned}
&\left| \int_{\mathbb{T}} U_{jl_1}(\theta) U^*_{l_1 l_2}(\theta) e^{\mathrm{i}\left[-\lambda_{l_1}(\theta)t + (2(k-k_*)+l_2-j)\theta\right]}\, d\theta \right| \\
\le\ & \left(8 \cdot 18 \cdot 2^{\frac{1}{3}} + 9 \cdot 8 \cdot 2^{\frac{1}{2}}\right) \left(16+(V_0-V_1)^2\right)^{\frac{1}{4}} (1+|t|)^{-\frac{1}{3}} \\
<\ & 284 \left(16+(V_0-V_1)^2\right)^{\frac{1}{4}} (1+|t|)^{-\frac{1}{3}}.
\end{aligned} \tag{18}$$

Then Proposition 3.1 is shown by combining (15) and (18). ∎

Let us go back to the expression of $e^{-\mathrm{i}tH}$ given in Proposition 2.1. Since

$$\sum_{l_2=0,1} \sum_{k\in\mathbb{Z}} |\psi_{2k+l_2}| = \|\psi\|_{\ell^1},$$

we have, for every $n \in \mathbb{Z}$,

$$\begin{aligned} |(e^{-\mathrm{i}tH}\psi)_n| &\le \frac{1}{2\pi}\sum_{l_1,l_2=0,1}\sum_{k\in\mathbb{Z}}|\psi_{2k+l_2}|\left|\int_{\mathbb{T}} U_{jl_1}(\theta)U^*_{l_1l_2}(\theta)e^{\mathrm{i}\left[-\lambda_{l_1}(\theta)t+(2(k-k_*)+l_2-j)\theta\right]}\,d\theta\right| \\ &\le \frac{284}{\pi}\left(16+(V_0-V_1)^2\right)^{\frac14}\|\psi\|_{\ell^1}(1+|t|)^{-\frac13} \\ &\le \frac{568}{\pi}\left(1+(V_0-V_1)^2\right)^{\frac14}\|\psi\|_{\ell^1}(1+|t|)^{-\frac13}. \end{aligned}$$

Thus Theorem 1.1 is proved.

4. Proof of Lemma 3.1

Section 4 is devoted to the proof of Lemma 3.1. By a direct calculation, we get

$$\frac{d^2}{d\theta^2}\sqrt{\cos^2(\theta)+a} = \frac{-\cos^4(\theta)-a\cos^2(\theta)+a\,\sin^2(\theta)}{(\cos^2(\theta)+a)^{\frac32}}, \tag{19}$$

$$\frac{d^3}{d\theta^3}\sqrt{\cos^2(\theta)+a} = \frac{\sin(\theta)\cos(\theta)\left(4a^2+3a\sin^2(\theta)+5a\cos^2(\theta)+\cos^4(\theta)\right)}{(\cos^2(\theta)+a)^{\frac52}}. \tag{20}$$

Given any $\varepsilon>0$, $\left|\frac{d^2}{d\theta^2}\sqrt{\cos^2(\theta)+a}\right|<\varepsilon$ is equivalent to

$$\left(-\cos^4(\theta)-2a\cos^2(\theta)+a\right)^2-\varepsilon^2\left(\cos^2(\theta)+a\right)^3<0.$$

For the above 8−degree polynomial of $\cos(\theta)$, there are at most 8 zeroes and hence at most 4 subintervals of $\cos(\theta)$ such that the above inequality holds. Since the cosine function is strictly monotonic on $(0,\pi)$ and $(\pi,2\pi)$, we see that there are at most 8 subintervals of $\theta\in\mathbb{T}$ such that $\left|\frac{d^2}{d\theta^2}\sqrt{\cos^2(\theta)+a}\right|<\varepsilon$.

We notice that

$$\begin{aligned} f_a(\theta) &= \frac{|\cos^4(\theta)+a\cos^2(\theta)-a\,\sin^2(\theta)|}{(\cos^2(\theta)+a)^{\frac32}} \\ &\quad + \frac{|\sin(\theta)\cos(\theta)|\cdot|4a^2+3a\sin^2(\theta)+5a\cos^2(\theta)+\cos^4(\theta)|}{(\cos^2(\theta)+a)^{\frac52}}. \end{aligned}$$

is a π−periodic even function. So it is sufficient to focus on the interval $[0,\frac{\pi}{2}]$, on which we have

$$\begin{aligned} \left|\frac{d^3}{d\theta^3}\sqrt{\cos^2(\theta)+a}\right| &= \frac{d^3}{d\theta^3}\sqrt{\cos^2(\theta)+a} \\ &= \frac{\sin(\theta)\cos(\theta)\left(4a^2+3a\sin^2(\theta)+5a\cos^2(\theta)+\cos^4(\theta)\right)}{(\cos^2(\theta)+a)^{\frac52}}. \end{aligned}$$

As for the second derivative, there exists $\theta_*\in[\frac{\pi}{4},\frac{\pi}{2}]$ with

$$\cos^4(\theta_*)+a\cos^2(\theta_*)-a\sin^2(\theta_*)=0 \tag{21}$$

such that

$$\left|\frac{d^2}{d\theta^2}\sqrt{\cos^2(\theta)+a}\right| = \begin{cases} -\dfrac{d^2}{d\theta^2}\sqrt{\cos^2(\theta)+a}, & \theta\in[0,\theta_*] \\ \dfrac{d^2}{d\theta^2}\sqrt{\cos^2(\theta)+a}, & \theta\in[\theta_*,\frac{\pi}{2}] \end{cases}$$
$$= \begin{cases} \dfrac{\cos^4(\theta)+a\cos^2(\theta)-a\sin^2(\theta)}{(\cos^2(\theta)+a)^{\frac{3}{2}}}, & \theta\in[0,\theta_*] \\ \dfrac{-\cos^4(\theta)-a\cos^2(\theta)+a\sin^2(\theta)}{(\cos^2(\theta)+a)^{\frac{3}{2}}}, & \theta\in[\theta_*,\frac{\pi}{2}] \end{cases}.$$

Indeed, Equation (21) can be written as $\cos^4(\theta_*)+2a\cos^2(\theta_*)-a=0$, which gives a unique solution $\theta_*\in[0,\frac{\pi}{2}]$ such that

$$\cos^2(\theta_*)=\sqrt{a^2+a}-a=\frac{a}{\sqrt{a^2+a}+a}<\frac{1}{2}.$$

This means $\theta_*\in(\frac{\pi}{4},\frac{\pi}{2}]$ and

$$\cos^4(\theta)+2a\cos^2(\theta)-a\begin{cases} >0, & \theta\in[0,\theta_*) \\ <0, & \theta\in(\theta_*,\frac{\pi}{2}] \end{cases}.$$

Lemma 4.1. *For every $a>0$, we have*

$$f_a(\theta)\geq\frac{1}{\sqrt{\cos^2(\theta)+a}},\quad\forall\,\theta\in\left[0,\frac{\pi}{4}\right]. \tag{22}$$

Proof. On $x\in[0,\frac{\pi}{4}]\subset[0,\theta_*)$, we have

$$\begin{aligned} f_a(\theta) &= \frac{\cos^4(\theta)+a\cos^2(\theta)-a\sin^2(\theta)}{(\cos^2(\theta)+a)^{\frac{3}{2}}} \\ &\quad+\frac{\sin(\theta)\cos(\theta)\left(4a^2+3a\sin^2(\theta)+5a\cos^2(\theta)+\cos^4(\theta)\right)}{(\cos^2(\theta)+a)^{\frac{5}{2}}}. \end{aligned}$$

Hence $(\cos^2(\theta)+a)^{\frac{5}{2}}f_a(\theta)=L_1(\theta)+L_2(\theta)+L_3(\theta)$ with

$$L_1(\theta) := -a^2\sin^2(\theta)+a^2\cos^2(\theta)+4a^2\sin(\theta)\cos(\theta)$$
$$= a^2\left(2\sin(2\theta)+\cos(2\theta)\right), \tag{23}$$
$$L_2(\theta) := 2a\cos^4(\theta)+5a\sin(\theta)\cos^3(\theta)-a\sin^2(\theta)\cos^2(\theta)+3a\sin^3(\theta)\cos(\theta)$$
$$= a\left(2\sin(2\theta)+\cos(2\theta)+\frac{1}{4}\sin(4\theta)+\frac{3}{8}\cos(4\theta)+\frac{5}{8}\right), \tag{24}$$
$$L_3(\theta) := \cos^6(\theta)+\sin(\theta)\cos^5(\theta)$$
$$= \sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)\cos^5(\theta). \tag{25}$$

Note that for $\theta \in [0, \frac{\pi}{4}]$, we have $\cos(\theta) \geq \sin(\theta)$. Then

$$\begin{aligned} L_1(\theta) &= a^2 + 2a^2 \sin(\theta)(2\cos(\theta) - \sin(\theta)) \\ &\geq a^2, \\ L_2(\theta) &\geq 2a\cos^2(\theta) + (3a\sin(\theta)\cos^3(\theta) - a\sin^2(\theta)\cos^2(\theta) + 3a\sin^3(\theta)\cos(\theta)) \\ &\geq 2a\cos^2(\theta), \\ L_3(\theta) &\geq \cos^4(\theta). \end{aligned}$$

Therefore,

$$(\cos^2(\theta) + a)^{\frac{5}{2}} f_a(\theta) \geq a^2 + 2a\cos^2(\theta) + \cos^4(\theta) = (\cos^2(\theta) + a)^2$$

which implies (22). □

Lemma 4.2. *For $0 < a \leq \frac{1}{8}$, we have*

$$f_a(\theta) > \frac{1}{4\sqrt{1+a}}, \quad \forall\, \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right].$$

Proof. For $\theta \in (\frac{\pi}{4}, \theta_*]$,

$$\cos^2(\theta) \geq \sqrt{a^2 + a} - a = \frac{a}{\sqrt{a^2 + a} + a},$$

and we still have that $(\cos^2(\theta) + a)^{\frac{5}{2}} f_a(\theta) = L_1(\theta) + L_2(\theta) + L_3(\theta)$ with $L_1(\theta)$, $L_2(\theta)$, $L_3(\theta)$ defined as in (23)–(25). Note that on $[\frac{\pi}{4}, \frac{\pi}{2}]$, L_1, L_3 are decreasing and $L_2 \geq 0$.

- For L_3 on $(\frac{\pi}{4}, \theta_*]$, we have

$$\begin{aligned} \frac{L_3(\theta)}{(\cos^2(\theta) + a)^{\frac{5}{2}}} &= \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) \frac{\cos^5(\theta)}{(\cos^2(\theta) + a)^{\frac{5}{2}}} \\ &\geq \frac{1}{(1 + \frac{a}{\cos^2(\theta_*)})^{\frac{5}{2}}} \\ &= \frac{1}{(1 + a + \sqrt{a^2 + a})^{\frac{5}{2}}} \\ &\geq \frac{2}{3\sqrt{3} \cdot \sqrt{1+a}} \end{aligned}$$

  since $0 < a \leq \frac{1}{8}$.
- For L_1 on $(\frac{\pi}{4}, \theta_*]$, we have

$$\begin{aligned} L_1(\theta) &\geq L_1(\theta_*) \\ &= \frac{a^2}{\sqrt{a^2 + a} + a} \left(4(a^2 + a)^{\frac{1}{4}} \sqrt{a} + 2a - \sqrt{a^2 + a} - a\right) \\ &= \frac{a^2}{\sqrt{a^2 + a} + a} \left(4(a^2 + a)^{\frac{1}{4}} \sqrt{a} + a - \sqrt{a^2 + a}\right) \\ &\geq 0, \quad \text{if } a \geq \frac{1}{(2 + \sqrt{5})^4 - 1} = \frac{9}{8\sqrt{5}} - \frac{1}{2}. \end{aligned}$$

Hence, for $\frac{9}{8\sqrt{5}} - \frac{1}{2} \leq a \leq \frac{1}{8}$,

$$f_a(\theta) = \frac{1}{(\cos^2(\theta) + a)^{\frac{5}{2}}} (L_1(\theta) + L_2(\theta) + L_3(\theta)) \geq \frac{2}{3\sqrt{3} \cdot \sqrt{1+a}}.$$

As for the case $0 < a < \frac{9}{8\sqrt{5}} - \frac{1}{2}$, we have

$$\begin{aligned}
\frac{L_3(\theta_*)}{60} + L_1(\theta_*) &\geq \frac{\cos^5(\theta_*)}{60} + \frac{a^2}{\sqrt{a^2+a}+a}\left(4(a^2+a)^{\frac{1}{4}}\sqrt{a} + 2a - \sqrt{a^2+a} - a\right) \\
&= \frac{a^{\frac{5}{2}}}{60(\sqrt{a^2+a}+a)^{\frac{5}{2}}} - \frac{a^2\sqrt{a^2+a}}{\sqrt{a^2+a}+a} \\
&= \frac{a^{\frac{5}{2}} - 60a^2\sqrt{a^2+a}(\sqrt{a^2+a}+a)^{\frac{3}{2}}}{60(\sqrt{a^2+a}+a)^{\frac{5}{2}}} \\
&> 0
\end{aligned}$$

since a is small enough. Then

$$f_a(\theta) \geq \frac{59}{60}\frac{L_3(\theta_*)}{(\cos^2(\theta_*)+a)^{\frac{5}{2}}} \geq \frac{59}{90\sqrt{3}\sqrt{1+a}} > \frac{1}{4\sqrt{1+a}}.$$

For $\theta \in (\theta_*, \frac{\pi}{2}]$, we have

$$\begin{aligned}
f_a(\theta) &= \frac{-\cos^4(\theta) - a\cos^2(\theta) + a\sin^2(\theta)}{(\cos^2(\theta)+a)^{\frac{3}{2}}} \\
&\quad + \frac{\sin(\theta)\cos(\theta)\left(4a^2 + 3a\sin^2(\theta) + 5a\cos^2(\theta) + \cos^4(\theta)\right)}{(\cos^2(\theta)+a)^{\frac{5}{2}}}
\end{aligned}$$

By a straightforward calculation, we get

$$\begin{aligned}
& (\cos^2(\theta)+a)^{\frac{7}{2}} \cdot f_a'(\theta) \\
= \; & -a^2\sin^2(\theta)\left(4a + 3\sin^2(\theta)\right) && (26)\\
& + 2a\cos^2(\theta)\left(2a^2 + 5a\sin^2(\theta) + 6\sin^4(\theta)\right) && (27)\\
& + a^2\sin(\theta)\cos(\theta)\left(4a + 3\sin^2(\theta)\right) + 6a\cos^6(\theta) + 6a\sin(\theta)\cos^5(\theta) && (28)\\
& + a\cos^4(\theta)\left(9a + 14\sin^2(\theta)\right) + 3a\sin(\theta)\cos^3(\theta)\left(3a + \sin^2(\theta)\right) && (29)\\
& + \cos^8(\theta) + \sin(\theta)\cos^7(\theta). && (30)
\end{aligned}$$

A simple observation shows that

- the term $12a\cos^2(\theta)\sin^4(\theta)$ in (27) is decreasing on $(\arctan(\sqrt{2}), \frac{\pi}{2})$,
- the term $3a^2\sin^3(\theta)\cos(\theta)$ in (28) is decreasing on $(\frac{\pi}{3}, \frac{\pi}{2})$,
- all the other terms in (26)–(30) are decreasing on $(\frac{\pi}{4}, \frac{\pi}{2})$.

Recalling that $a \leq \frac{1}{8}$, we have $\cos^2(\theta_*) = \frac{a}{a+\sqrt{a^2+a}} \leq \frac{1}{4}$, which implies that $(\theta_*, \frac{\pi}{2}) \subset (\frac{\pi}{3}, \frac{\pi}{2})$. Hence $(\cos^2(\theta)+a)^{\frac{7}{2}} \cdot f_a'(\theta)$ is decreasing on $(\theta_*, \frac{\pi}{2})$. Since

$$\left(\cos^2\left(\frac{\pi}{2}\right) + a\right)^{\frac{7}{2}} \cdot f_a'\left(\frac{\pi}{2}\right) = -a^{\frac{11}{2}}(4a+3) < 0,$$

there is at most one local maximum in $(\theta_*, \frac{\pi}{2})$ for $f_a(\theta)$. After simple calculations as following:

$$\begin{aligned}
f_a(\theta_*) &= \frac{\sin(\theta_*)\cos(\theta_*)\left(4a^2+3a\sin^2(\theta_*)+5a\cos^2(\theta_*)+\cos^4(\theta_*)\right)}{(\cos^2(\theta_*)+a)^{\frac{5}{2}}} \\
&= \frac{\sin(\theta_*)\cos(\theta_*)\left(4a^2+3a+2a\cos^2(\theta_*)+\cos^4(\theta_*)\right)}{(\cos^2(\theta_*)+a)^{\frac{5}{2}}} \\
&= \frac{(a^2+a)^{\frac{1}{4}}a^{\frac{1}{2}}}{(a^2+a)^{\frac{5}{4}}(\sqrt{a^2+a}+a)}\left(4a^2+3a+2a(\sqrt{a^2+a}-a)+(\sqrt{a^2+a}-a)^2\right) \\
&= \frac{4a^{\frac{1}{2}}}{a+\sqrt{a^2+a}},
\end{aligned}$$

we can get that for every $\theta \in (\theta_*, \frac{\pi}{2}]$,

$$f_a(\theta) \geq \min\left\{f_a(\theta_*), f_a\left(\frac{\pi}{2}\right)\right\} = \min\left\{\frac{4a^{\frac{1}{2}}}{a+\sqrt{a^2+a}}, \frac{1}{\sqrt{a}}\right\} > \frac{1}{\sqrt{1+a}}. \quad \square$$

Lemma 4.3. *For $a > \frac{1}{8}$, we have*

$$f_a(\theta) > \frac{1}{4\sqrt{1+a}}, \quad \forall\ \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right].$$

Proof. On $\left(\frac{\pi}{4}, \frac{\pi}{2}\right]$, we always have $\sin(\theta) > \cos(\theta)$.

For $\theta \in (\frac{\pi}{4}, \theta_*]$, a direct computation yields that

$$\begin{aligned}
f_a(\theta) &= \frac{\cos^2(\theta)}{\sqrt{\cos^2(\theta)+a}} + \frac{4\sin(\theta)\cos(\theta)}{\sqrt{\cos^2(\theta)+a}} \\
&\quad + \frac{3\sin^3(\theta)\cos(\theta)-3\sin(\theta)\cos^3(\theta)}{(\cos^2(\theta)+a)^{\frac{3}{2}}} - \frac{a\sin^2(\theta)}{(\cos^2(\theta)+a)^{\frac{3}{2}}} \\
&\quad - \frac{3\sin^3(\theta)\cos^3(\theta)}{(\cos^2(\theta)+a)^{\frac{5}{2}}}.
\end{aligned}$$

We can see

$$\begin{aligned}
\frac{\sin(\theta)\cos(\theta)}{\sqrt{\cos^2(\theta)+a}} - \frac{a\sin^2(\theta)}{(\cos^2(\theta)+a)^{\frac{3}{2}}} &= \frac{\sin(\theta)\cos(\theta)(\cos^2(\theta)+a)-a\sin^2(\theta)}{(\cos^2(\theta)+a)^{\frac{3}{2}}} \\
&= \frac{\sin(\theta)}{(\cos^2(\theta)+a)^{\frac{3}{2}}}\left(\cos^3(\theta)+a(\cos(\theta)-\sin(\theta))\right),
\end{aligned}$$

where $\cos^3(\theta)+a(\cos(\theta)-\sin(\theta))$ is decreasing and hence

$$\begin{aligned}
\cos^3(\theta)+a(\cos(\theta)-\sin(\theta)) &> \frac{1}{\sin(\theta_*)}\left[\sin(\theta_*)\cos^3(\theta_*)+a\sin(\theta_*)(\cos(\theta_*)-\sin(\theta_*))\right] \\
&= \frac{1}{\sin(\theta_*)}\left[\sin(\theta_*)\cos(\theta_*)\sqrt{a^2+a}-a\sin^2(\theta_*)\right] \\
&> \frac{1}{\sin(\theta_*)}\left[\cos^2(\theta_*)\sqrt{a^2+a}-a\sin^2(\theta_*)\right] \\
&= 0.
\end{aligned}$$

Moreover,

$$
\begin{aligned}
& \frac{3\sin(\theta)\cos(\theta)}{\sqrt{\cos^2(\theta)+a}} + \frac{3\sin^3(\theta)\cos(\theta) - 3\sin(\theta)\cos^3(\theta)}{(\cos^2(\theta)+a)^{\frac{3}{2}}} - \frac{3\sin^3(\theta)\cos^3(\theta)}{(\cos^2(\theta)+a)^{\frac{5}{2}}} \\
= & \frac{3\sin(\theta)\cos(\theta)}{(\cos^2(\theta)+a)^{\frac{5}{2}}}\left[(\cos^2(\theta)+a)^2 + (\sin^2(\theta)-\cos^2(\theta))(\cos^2(\theta)+a) - \sin^2(\theta)\cos^2(\theta)\right] \\
= & \frac{3\sin(\theta)\cos(\theta)}{(\cos^2(\theta)+a)^{\frac{5}{2}}}(a^2+a) > 0.
\end{aligned}
$$

Since $a > \frac{1}{8}$ means that $\cos^2(\theta_*) = \frac{a}{\sqrt{a^2+a}+a} > \frac{1}{4}$, we have

$$
f_a(\theta) > \frac{\cos^2(\theta_*)}{\sqrt{\cos^2(\theta_*)+a}} > \frac{1}{2\sqrt{1+4a}} > \frac{1}{4\sqrt{1+a}}, \quad \forall\, \theta \in \left(\frac{\pi}{4}, \theta_*\right].
$$

For $\theta \in (\theta_*, \frac{\pi}{2}]$,

$$
\begin{aligned}
f_a(\theta) \;=\; & \frac{-\cos^4(\theta) - a\cos^2(\theta) + a\sin^2(\theta)}{(\cos^2(\theta)+a)^{\frac{3}{2}}} \\
& + \frac{\sin(\theta)\cos(\theta)\left(4a^2 + 3a\sin^2(\theta) + 5a\cos^2(\theta) + \cos^2(\theta)\right)}{(\cos^2(\theta)+a)^{\frac{5}{2}}}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& (\cos^2(\theta)+a)^{\frac{5}{2}} f_a(\theta) \\
= \;& a^2\sin^2(\theta) - a^2\cos^2(\theta) + 4a^2\sin(\theta)\cos(\theta) \qquad (31) \\
& + 5a\sin(\theta)\cos^3(\theta) - 2a\cos^4(\theta) + 2a\sin^3(\theta)\cos(\theta) \qquad (32) \\
& + a\sin^3(\theta)\cos(\theta) + a\sin^2(\theta)\cos^2(\theta) + \sin(\theta)\cos^5(\theta) - \cos^6(\theta) \qquad (33)
\end{aligned}
$$

where the term in (31)

$$
\begin{aligned}
& = a^2\sin^2(\theta) - a^2\cos^2(\theta) + 4a^2\sin(\theta)\cos(\theta) \\
& = a^2 + 2a^2\cos(\theta)(2\sin(\theta) - \cos(\theta)) \\
& > a^2 + \frac{\cos^4(\theta)}{32},
\end{aligned}
$$

the term in (32)

$$
\begin{aligned}
& = 5a\sin(\theta)\cos^3(\theta) - 2a\cos^4(\theta) + 2a\sin^3(\theta)\cos(\theta) \\
& > 3a\sin(\theta)\cos^3(\theta) + 2a\sin^3(\theta)\cos(\theta) \\
& > \frac{\cos^4(\theta)}{8} + 2a\left(\cos^4(\theta) + \sin^2(\theta)\cos^2(\theta)\right) \\
& = \frac{\cos^4(\theta)}{8} + 2a\cos^2(\theta),
\end{aligned}
$$

and the term in (33)

$$
\begin{aligned}
& = a\sin^3(\theta)\cos(\theta) + a\sin^2(\theta)\cos^2(\theta) + \sin(\theta)\cos^5(\theta) - \cos^6(\theta) \\
& > \frac{1}{8}\sin^3(\theta)\cos(\theta) + \frac{1}{8}\sin^2(\theta)\cos^2(\theta) \\
& > \frac{\cos^4(\theta)}{4}.
\end{aligned}
$$

So we have

$$(\cos^2(\theta)+a)^{\frac{5}{2}} f_a(\theta) > a^2 + 2a\cos^2(\theta) + \frac{13}{32}\cos^4(\theta) > \frac{13}{32}(\cos^2(\theta)+a)^2,$$

which implies that

$$f_a(\theta) > \frac{13}{32\sqrt{\cos^2(\theta)+a}}, \quad \forall\, \theta \in \left(\theta_*, \frac{\pi}{2}\right]. \quad \square$$

Appendix A. Van der Corput lemma

For readers' convenience, we recall Van der Corput lemma and its corollary which is used in this paper. The proof can be found in Chapter VIII of [10].

Lemma A.1. *Suppose that ψ is real-valued and $\mathcal{C}^k$ in (a,b) for some $k \geq 2$, and*

$$|\psi^{(k)}(x)| \geq 1, \quad \forall\, x \in (a,b). \tag{34}$$

For any $\lambda \in \mathbb{R}^+$, we have

$$\left|\int_a^b e^{\mathrm{i}\lambda\psi(x)}dx\right| \leq (5\cdot 2^{k-1}-2)\lambda^{-\frac{1}{k}}.$$

If the hypothesis (34) in the above lemma is replaced by

$$\text{“}|\psi^{(k)}(x)| \geq c, \quad \forall\, x \in (a,b)\text{”} \tag{35}$$

for some $c > 0$ independent of x, then it is easy to derive from Lemma A.1 that

$$\left|\int_a^b e^{\mathrm{i}\lambda\psi(x)}dx\right| \leq (5\cdot 2^{k-1}-2)c^{-\frac{1}{k}}\lambda^{-\frac{1}{k}}, \quad \forall\, \lambda \in \mathbb{R}_+.$$

Moreover, since (35) also holds for $-\psi$, Lemma A.1 implies that

$$\left|\int_a^b e^{\mathrm{i}\lambda\psi(x)}dx\right| \leq (5\cdot 2^{k-1}-2)c^{-\frac{1}{k}}|\lambda|^{-\frac{1}{k}}, \quad \forall\, \lambda \in \mathbb{R}\setminus\{0\}.$$

Corollary A.1. *Suppose that ψ is real-valued and $\mathcal{C}^k$ in (a,b) for some $k \geq 2$, and that $|\psi^{(k)}(x)| \geq c$ for all $x \in (a,b)$. Let h be $\mathcal{C}^1$ in (a,b). Then*

$$\left|\int_a^b e^{\mathrm{i}\lambda\psi(x)}h(x)dx\right| \leq (5\cdot 2^{k-1}-2)c^{-\frac{1}{k}}\left[|h(b)| + \int_a^b |h'(x)|dx\right]|\lambda|^{-\frac{1}{k}}, \quad \forall\, \lambda \in \mathbb{R}\setminus\{0\}.$$

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References

[1] Bambusi, D.: Asymptotic stability of breathers in some Hamiltonian networks of weakly coupled oscillators. Comm. Math. Phys., **324(2)**, 515–547 (2013).
[2] Bambusi, D., Zhao, Z.: Dispersive estimate for one-dimensional quasi-periodic Schrödinger operator. Preprint.
[3] Cuccagna, S., Tarulli, M.: On asymptotic stability of standing waves of discrete Schrödinger equation in $\mathbb{Z}$. SIAM J. Math. Anal., **41(3)**, 861–885 (2009).

[4] Damanik, D., Lukic, M., Yessen, W.: Quantum dynamics of periodic and limit-periodic Jacobi and block Jacobi matrices with applications to some quantum many body problems. Commun. Math. Phys. **337(3)**, 1535–1561 (2015).
[5] Keel, M., Tao, T.: Endpoint Strichartz estimates. Amer. J. Math. **120**, 955–980 (1998).
[6] Komech, A., Kopylova, E., Kunze, M.: Dispersion estimates for 1D discrete Schrödinger and Klein-Gordon equations. Applicable Analysis **85(12)**, 1487–1508 (2006).
[7] Pelinovsky, D., E., Stefanov, A.: On the spectral theory and dispersive estimates for a discrete Schrödinger equation in one dimension. J. Math. Phys. **49**, 113501 (2018).
[8] Schlag, W.: Dispersive estimates for Schrödinger operators: A survey. *Mathematical aspects of nonlinear dispersive equations*, Ann. of Math. Stud., **163**, Princeton Univ. Press, Princeton, NJ, 2007.
[9] Stefanov, A., Kevrekidis, P. G.: Asymptotic behaviour of small solutions for the discrete nonlinear Schrödinger and Klein-Gordon equations. Nonlinearity **18**, 1841–1857 (2005).
[10] Stein, E. M.: *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*. Princeton University Press, Princeton, NJ, 1993.
[11] Zhao, Z.: Ballistic motion in one-dimensional quasi-periodic discrete Schrödinger equation. Commun. Math. Phys. **347**, 511–549 (2016).

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