DISPERSIVE ESTIMATES FOR PERIODIC DISCRETE ONE-DIMENSIONAL SCHRÖDINGER OPERATORS

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Abstract. In this paper, we consider the periodic discrete one-dimensional Schrödinger operator

$$(H_V\psi)_n = -(\psi_{n+1}+\psi_{n-1}) + V_n\psi_n, \quad n\in\mathbb{Z},$$

with $V_{n+N}=V_n$, $\forall\, n\in\mathbb{Z}$, for some $N\in\mathbb{N}^*$ and $V=\{V_j\}_{j=0}^{N-1}\in\mathbb{R}$. We show the dispersive estimate

$$\|e^{-\mathrm{i}tH_V}\psi\|_{\ell^\infty} \lesssim \|\psi\|_{\ell^1}(1+|t|)^{-\min\left\{\frac{1}{3},\frac{1}{N+1}\right\}}, \quad \forall\,\psi\in\ell^1(\mathbb{Z}), \quad \forall\, t\in\mathbb{R}.$$

1. Introduction and main result

Consider the periodic discrete one-dimensional Schrödinger operator

$$(H_V\psi)_n = -(\psi_{n+1}+\psi_{n-1}) + V_n\psi_n, \quad n\in\mathbb{Z} \tag{1}$$

with $V_{n+N}=V_n$, $\forall\, n\in\mathbb{Z}$, for some $N\in\mathbb{N}^*$ and $V=\{V_j\}_{j=0}^{N-1}\subset\mathbb{R}$. It is well known that H_V has purely absolutely continuous spectrum (see, e.g., [12]) and its time evolution $e^{-\mathrm{i}tH_V}$ presents ballistic transport (see [5]), i.e., the weighted ℓ^2-norm

$$\left(\sum_{n\in\mathbb{Z}} n^2\left|(e^{-\mathrm{i}tH_V}\psi)_n\right|^2\right)^{\frac{1}{2}}$$

grows linearly with t provided that $\sum_{n\in\mathbb{Z}} n^2|\psi_n|^2<\infty$. In this paper, we are interested in the ℓ^1-ℓ^∞ dispersive estimate of $e^{-\mathrm{i}tH_V}$. In general, the dispersion and ballistic transport for a one-dimensional linear operator are related to the absolutely continuous spectrum, even though there is not yet rigorous argument describing the link between them.

We recall that for the free Schrödinger operator, $-\Delta:\ell^2(\mathbb{Z}^\nu)\to\ell^2(\mathbb{Z}^\nu)$, $\nu\geq 1$,

$$(-\Delta\psi)_n = -\sum_{|m-n|=1}\psi_m, \quad n\in\mathbb{Z}^\nu,$$

the ℓ^1-ℓ^∞ dispersive estimate

$$\|e^{\mathrm{i}t\Delta}\psi\|_{\ell^\infty} \lesssim (1+|t|)^{-\frac{1}{3}}\|\psi\|_{\ell^1}, \quad \forall\,\psi\in\ell^1(\mathbb{Z}^\nu), \tag{2}$$

is well known (see [15, 18]). The estimate (2) implies that, for a given well-localized initial condition ψ, the $\ell^\infty-$norm of $e^{\mathrm{i}t\Delta}\psi$ tends to zero as time goes to infinity, with the "$|t|^{-\frac{1}{3}}$" decay rate. In other words, as time goes, $e^{\mathrm{i}t\Delta}\psi$ does not keep well-localized as ψ.

Y. Mi was supported by China Scholarship Council (CSC)(No. 202006190129).

Research of Z. Zhao is partially supported by NSFC grant (11971233).

For the discrete one-dimensional Schrödinger operator $H : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$,

$$(Hq)_n = -(q_{n+1} + q_{n-1}) + V_n q_n \ , \quad n \in \mathbb{Z} \ ,$$

Pelinovsky-Stefanov [16] have shown that

$$\|e^{-\mathrm{i}tH} P_{ac}\psi\|_{\ell^\infty} \lesssim (1+|t|)^{-\frac{1}{3}} \|\psi\|_{\ell^1} \ , \quad \forall \ \psi \in \ell^1(\mathbb{Z}) \ , \tag{3}$$

for "generic"[1] potentials $(V_n)_{n\in\mathbb{Z}}$ decaying sufficiently fast at infinity. Here P_{ac} denotes the projection on the absolutely continuous part of the spectrum. For the $2-$periodic potential case, we have shown the same estimate [14]. For other related works, one can refer to [1, 4, 6, 11, 13]. For the dispersive estimates for continuous Schrödinger operators, we can refer to [17].

Recently, Bambusi and the second author [3] have considered the quasi-periodic Schrödinger operator $H_\theta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$,

$$(H_\theta\psi)_n = -(\psi_{n+1} + \psi_{n-1}) + v(\theta + n\omega)\psi_n, \quad n \in \mathbb{Z}, \tag{4}$$

where $v \in C^\omega(\mathbb{T}^d, \mathbb{R})$ with $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, $d \geq 1$, and $\omega \in DC_d(\gamma, \tau)$ for $\gamma > 0$ and $\tau > d - 1$, i.e.,

$$\inf_{j\in\mathbb{Z}} |\langle k, \omega\rangle - j\pi| > \frac{\gamma}{|k|^\tau}, \quad \forall \ k \in \mathbb{Z}^d \setminus \{0\}.$$

It is well known that the spectrum of H_θ is usually a Cantor set for every $\theta \in \mathbb{T}^d$. When the potential function v is sufficiently small, it has purely absolutely continuous spectrum . Moreover, the time evolution $e^{-\mathrm{i}tH_\theta}$ presents ballistic transport (see [22]). As for the dispersion, it is shown that, if v is sufficiently small, then for every $\theta \in \mathbb{T}^d$:

$$\|e^{-\mathrm{i}tH_\theta}\psi\|_{\ell^\infty} \lesssim |\ln |v|_r|^{a \ln\ln(3+|t|))^{2d}} (1+|t|)^{-\frac{1}{3}} \|\psi\|_{\ell^1}, \quad \forall \ \psi \in \ell^1(\mathbb{Z}),$$

for some absolute constant $a > 0$, which implies a $|t|^{-\zeta}-$dispersive decay for any $0 < \zeta < \frac{1}{3}$.

In this paper, the main conclusion is:

Theorem 1.1. *For the operator H_V given in (1), there exists a constant $C_V > 0$, depending on V, such that, for any $t \in \mathbb{R}$,*

$$\|e^{-\mathrm{i}tH_V}\psi\|_{\ell^\infty} < C_V \|\psi\|_{\ell^1} (1+|t|)^{-\min\left\{\frac{1}{3}, \frac{1}{N+1}\right\}}, \quad \forall \ \psi \in \ell^1(\mathbb{Z}).$$

As usual, from this estimate in Theorem 1.1, one can deduce Strichartz estimates via [10] as well as decay for the solution of the nonlinear periodic discrete Schrödinger equation

$$\mathrm{i}\dot{q}_n = -(q_{n+1} + q_{n-1}) + V_n q_n \pm |q_n|^{p-1} q_n, \quad n \in \mathbb{Z}, \tag{5}$$

with $V = \{V_j\}_{j=0}^{N-1} \subset \mathbb{R}$ as in (1), provided p is large enough. If we just focus on dispersive decay in ℓ^∞, we can give the following result for $p \geq \max\{5, N+3\}$ (indeed $\frac{1}{p-2} \leq \min\{\frac{1}{3}, \frac{1}{N+1}\}$ is necessary).

Corollary 1.1. *Under the assumptions of Theorem 1.1, consider Eq. (5) with $p \geq \max\{5, N+3\}$. There exist $C > 0$ and $\delta > 0$ s.t., if the initial datum $q(0)$ fulfills $\|q(0)\|_{\ell^1(\mathbb{Z})} < \delta$, then the solution of Eq. (5) fulfills*

$$\|q(t)\|_{\ell^\infty} \leq C(1+|t|)^{-\min\left\{\frac{1}{3}, \frac{1}{N+1}\right\}}. \tag{6}$$

[1] See Definition 1 of [16].

The proof of Corollary 1.1 is similar to that of Corollary 2 of [3].

The remaining part of paper will be organised as follows. In Section 2, we make some analysis on the periodic Jacobi matrix, and show the $\mathcal{C}^1-$regularity of its eigenvectors and the transversality of its eigenvalues. In Section 3, after getting the explicit form of the time evolution $e^{-\mathrm{i}tH_V}$ via Fourier transform, we apply Van der Corput lemma (Lemma A.1 in Appendix A) to estimate the oscillatory integrals in $e^{-\mathrm{i}tH_V}$, which gives the dispersive estimate.

2. Periodic Jacobi matrix

A straightforward way to establish the dispersive estimate is to get the explicit expression of the time evolution $e^{-\mathrm{i}tH_V}$ via Fourier transform as in [14, 18]. As the period N grows, the expression becomes more and more complicated. To realize such an expression, some properties of the periodic Jacobi matrix, which will be defined below, are needed.

Fix $N \geq 3$. We consider the periodic Jacobi matrix

$$A(\theta) = \begin{pmatrix} V_0 & -e^{-\mathrm{i}\theta} & 0 & \cdots & 0 & -e^{\mathrm{i}\theta} \\ -e^{\mathrm{i}\theta} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -e^{-\mathrm{i}\theta} \\ -e^{-\mathrm{i}\theta} & 0 & \cdots & 0 & -e^{\mathrm{i}\theta} & V_{N-1} \end{pmatrix}, \quad \theta \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \tag{7}$$

with $V = \{V_j\}_{j=0}^{N-1} \subset \mathbb{R}$. More precisely, the entries of matrix $A(\theta)$ are

$$A_{mn}(\theta) = \begin{cases} V_n, & n = m \\ -e^{-\mathrm{i}\theta}, & n = m-1 \text{ or } (m,n) = (N-1, 0) \\ -e^{\mathrm{i}\theta}, & n = m+1 \text{ or } (m,n) = (0, N-1) \\ 0, & \text{otherwise} \end{cases}, \quad m, n = 0, \cdots, N-1.$$

In particular, for $N = 3$,

$$A(\theta) = \begin{pmatrix} V_0 & -e^{-\mathrm{i}\theta} & -e^{\mathrm{i}\theta} \\ -e^{\mathrm{i}\theta} & V_1 & -e^{-\mathrm{i}\theta} \\ -e^{-\mathrm{i}\theta} & -e^{\mathrm{i}\theta} & V_2 \end{pmatrix}. \tag{8}$$

We will give some properties needed in the sequel in this section. The relation between $A(\theta)$ and the time evolution $e^{-\mathrm{i}tH_V}$ will be shown in Section 3 (see, (26) – (29)). For more properties of the periodic Jacobi matrix in a more general form, we can refer to [2, 9, 20, 21].

For the Hermitian $A(\theta)$, there is an $N\times N$ orthonormal matrix $U(\theta)$,

$$U(\theta)=\begin{pmatrix} U_{0,0}(\theta) & U_{0,1}(\theta) & \cdots & U_{0,N-1}(\theta)\\ U_{1,0}(\theta) & U_{1,1}(\theta) & \cdots & U_{0,N-2}(\theta)\\ \vdots & \vdots & \vdots & \vdots\\ U_{N-1,0}(\theta) & \cdots & U_{N-1,N-2}(\theta) & U_{N-1,N-1}(\theta)\end{pmatrix} \tag{9}$$

such that

$$U^*(\theta)A(\theta)U(\theta)=\Lambda(\theta):=\operatorname{diag}\{\lambda_j(\theta)\}_{j=0}^{N-1}, \tag{10}$$

with $\{\lambda_j(\theta)\}_{j=0}^{N-1}$ the set of eigenvalues of $A(\theta)$. We note that the eigenvalues and the eigenvectors of $A(\theta)$ are analytic with respect to θ (see [8] and P.195 of [7]).

Lemma 2.1. *The characteristic polynomial of $A(\theta)$ satisfies that*

$$\det\,(\lambda I-A(\theta))=P(\lambda)-2\cos(N\theta),$$

with $P(\lambda)$ a monic real polynomial of degree N.

Proof. Let $\{C_{m,n}(\lambda,\theta)\}_{m,n=0}^{N-1}$ be the elements of the matrix $\lambda I-A(\theta)$. Then

$$\det\,(\lambda I-A(\theta))=\sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\prod_{m=0}^{N-1}C_{m,\sigma_m}(\lambda,\theta). \tag{11}$$

Here, the sum is computed over all permutations σ of the set $0,1,\cdots,N-1$, σ_m denotes the value in the m^{th} position after the reordering σ, S_N is the set of all such permutations, and $\operatorname{sgn}(\sigma)$ denotes the signature of σ. For any permutation σ, we have $\sum_{m=0}^{N-1}\sigma_m=\sum_{m=0}^{N-1}m$. Note that there are only 3 nonzero elements in each column of $A(\theta)$. More precisely,

$$\begin{cases} C_{0,n}\equiv 0, & \text{if}\ \ n\neq 0,\ 1\ \text{ or }\ N-1\\ C_{m,n}\equiv 0,\ \ 1\le m\le N-2, & \text{if}\ \ n\neq m,\ m-1\ \text{ or }\ m+1\\ C_{N-1,n}\equiv 0 & \text{if}\ \ n\neq 0,\ N-2\ \text{ or }\ N-1\end{cases}.$$

We focus on the nonzero products $\prod_{m=0}^{N-1}C_{m,\sigma_m}(\lambda,\theta)$. For the permutation with $\sigma_0=N-1$,

- if $\sigma_{N-1}=0$, then $\sum_{m=1}^{N-2}\sigma_m=\sum_{m=1}^{N-2}m$. Then we have

  $$\sharp\{1\le m\le N-2:\sigma_m=m-1\}=\sharp\{1\le m\le N-2:\sigma_m=m+1\}=K$$

  with some $0\le K\le N-2$. It means, in this case,

$$\begin{aligned}\prod_{i=0}^{N-1}C_{i,\sigma_i} &= C_{0,N-1}\cdot C_{N-1,0}\cdot\prod_{\substack{1\le m\le N-2\\ \sigma_m=m-1}}C_{m,\sigma_m}\cdot\prod_{\substack{1\le m\le N-2\\ \sigma_m=m+1}}C_{m,\sigma_m}\cdot\prod_{\substack{1\le m\le N-2\\ \sigma_m=m}}C_{m,\sigma_m}\\ &= (-e^{\mathrm{i}\theta})(-e^{-\mathrm{i}\theta})\cdot(-e^{\mathrm{i}\theta})^K\cdot(-e^{-\mathrm{i}\theta})^K\prod_{\substack{1\le m\le N-2\\ \sigma_m=m}}(\lambda-V_m)\\ &= \prod_{\substack{1\le m\le N-2\\ \sigma_m=m}}(\lambda-V_m). \qquad (12)\end{aligned}$$

- if $\sigma_{N-1} \neq 0$, then, to keep the conservation $\sum_{m=0}^{N-1} \sigma_m = \sum_{m=0}^{N-1} m$, the unique way is $\sigma_m = m-1$, $1 \leq m \leq N-1$. In this case,

$$\prod_{i=0}^{N-1} C_{i,\sigma_i} = C_{0,N-1} \cdot \prod_{1 \leq m \leq N-1} C_{m,m-1} = (-e^{\mathrm{i}\theta})^N. \tag{13}$$

For the permutation with $\sigma_0 \neq N-1$,

- if $\sigma_{N-1} = 0$, then, to keep the conservation $\sum_{m=0}^{N-1} \sigma_m = \sum_{m=0}^{N-1} m$, the unique way is $\sigma_m = m+1$, $0 \leq m \leq N-2$. In this case,

$$\prod_{i=0}^{N-1} C_{i,\sigma_i} = C_{N-1,0} \cdot \prod_{1 \leq m \leq N-1} C_{m,m+1} = (-e^{-\mathrm{i}\theta})^N. \tag{14}$$

- if $\sigma_{N-1} \neq 0$, then, by the conservation $\sum_{m=0}^{N-1} \sigma_m = \sum_{m=0}^{N-1} m$, we have

$$\sharp\{1 \leq m \leq N-1 : \sigma_m = m-1\} = \sharp\{0 \leq m \leq N-2 : \sigma_m = m+1\} = K$$

with some $0 \leq K \leq N-1$. In this case,

$$\begin{aligned} \prod_{i=0}^{N-1} C_{i,\sigma_i} &= \prod_{\substack{1 \leq m \leq N-1 \\ \sigma_m = m-1}} C_{m,\sigma_m} \cdot \prod_{\substack{0 \leq m \leq N-2 \\ \sigma_m = m+1}} C_{m,\sigma_m} \cdot \prod_{\substack{0 \leq m \leq N-1 \\ \sigma_m = m}} C_{m,\sigma_m} \\ &= (-e^{\mathrm{i}\theta})^K \cdot (-e^{-\mathrm{i}\theta})^K \prod_{\substack{0 \leq m \leq N-1 \\ \sigma_m = m}} (\lambda - V_m) \\ &= \prod_{\substack{0 \leq m \leq N-1 \\ \sigma_m = m}} (\lambda - V_m). \end{aligned} \tag{15}$$

To sum up with (12) – (15), we see that, in the summation (11),

- all $\theta-$dependent terms come from (13) and (14), with the sum equals to

$$(-1)^{N-1} \cdot (-1)^N e^{-\mathrm{i}N\theta} + (-1)^{N-1} \cdot (-1)^N e^{\mathrm{i}N\theta} = -2\cos(N\theta);$$

- the sum of (12) and (15) is a monic real polynomial of degree N, denoted by $P(\lambda)$. $\square$

Lemma 2.2. *The eigenvalues of $A(\theta)$ satisfy that*

$$\sup_{\theta \in \mathbb{T}} |\lambda_j'(\theta)| \leq 2, \quad j = 0, 1, \cdots, N-1.$$

Proof. Let $v_j(\theta)$ be the normalized eigenvector corresponding to the eigenvalue $\lambda_j(\theta)$. It is actually the $j^{\text{th}}-$column of $U(\theta)$, i.e.,

$$v_j(\theta) = \begin{pmatrix} U_{0,j}(\theta) \\ U_{1,j}(\theta) \\ \vdots \\ U_{N-1,j}(\theta) \end{pmatrix}.$$

It is well-known that $\lambda_j'(\theta) = \langle A'(\theta)v_j(\theta), v_j(\theta)\rangle$. Since

$$A'(\theta) = \begin{pmatrix} 0 & \mathrm{i}e^{-\mathrm{i}\theta} & 0 & \cdots & 0 & -\mathrm{i}e^{\mathrm{i}\theta} \\ -\mathrm{i}e^{\mathrm{i}\theta} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \mathrm{i}e^{-\mathrm{i}\theta} \\ \mathrm{i}e^{-\mathrm{i}\theta} & 0 & \cdots & 0 & -\mathrm{i}e^{\mathrm{i}\theta} & 0 \end{pmatrix}$$

or, for $N = 3$,

$$A'(\theta) = \begin{pmatrix} 0 & \mathrm{i}e^{-\mathrm{i}\theta} & -\mathrm{i}e^{\mathrm{i}\theta} \\ -\mathrm{i}e^{\mathrm{i}\theta} & 0 & \mathrm{i}e^{-\mathrm{i}\theta} \\ \mathrm{i}e^{-\mathrm{i}\theta} & -\mathrm{i}e^{\mathrm{i}\theta} & 0 \end{pmatrix}$$

and $v_j(\theta)$ is the normalized eigenvector, we have

$$\begin{aligned} |\lambda_j'(\theta)| &= |\langle A'v_j, v_j\rangle| \\ &\leq \left|\mathrm{i}e^{-\mathrm{i}\theta}\left(U_{1,j}\overline{U}_{0,j} + U_{2,j}\overline{U}_{1,j} + \cdots + U_{N-1,j}\overline{U}_{N-2,j} + U_{0,j}\overline{U}_{N-1,j}\right)\right| \\ &\quad + \left|\mathrm{i}e^{\mathrm{i}\theta}\left(\overline{U}_{1,j}U_{0,j} + \overline{U}_{2,j}U_{1,j} + \cdots + \overline{U}_{N-1,j}U_{N-2,j} + \overline{U}_{0,j}U_{N-1,j}\right)\right| \\ &\leq 2\left(|U_{0,j}|^2 + |U_{1,j}|^2 + |U_{2,j}|^2 + \cdots + |U_{N-2,j}|^2 + |U_{N-1,j}|^2\right) \\ &= 2. \quad \square \end{aligned}$$

Proposition 2.1. *There exists a constant $\delta > 0$, depending on V, such that, for $j = 0, 1, \cdots, N-1$,*

$$|\lambda_j''(\theta)| + |\lambda_j'''(\theta)| + \cdots + |\lambda_j^{(N+1)}(\theta)| \geq \delta, \quad \forall\ \theta \in \mathbb{T}. \tag{16}$$

Proof. To show (16), it is sufficient to show that there is no $\theta \in \mathbb{T}$ such that

$$\lambda_j''(\theta) = \lambda_j'''(\theta) = \cdots = \lambda_j^{(N+1)}(\theta) = 0,$$

since $|\lambda_j''(\theta)| + |\lambda_j'''(\theta)| + \cdots + |\lambda_j^{(N+1)}(\theta)|$ is continuous for $j = 0, 1, \cdots, N-1$.

In view of Lemma 2.1, we see that, for all $\theta \in \mathbb{T}$,

$$P(\lambda_j(\theta)) = 2\cos(N\theta), \quad j = 0, 1, \cdots, N-1. \tag{17}$$

Suppose that, for some j, there exists $\theta_* \in \mathbb{T}$ such that

$$\lambda_j''(\theta_*) = \lambda_j'''(\theta_*) = \cdots = \lambda_j^{(N+1)}(\theta_*) = 0.$$

Then, (17) implies that

$$
\begin{aligned}
&(18) & P(\lambda_j(\theta_*)) &= 2\cos(N\theta_*), \\
&(19) & P'(\lambda_j(\theta_*))\lambda_j'(\theta_*) &= -2N\sin(N\theta_*), \\
&(20) & P''(\lambda_j(\theta_*))\lambda_j'(\theta_*)^2 &= -2N^2\cos(N\theta_*), \\
&(21) & P'''(\lambda_j(\theta_*))\lambda_j'(\theta_*)^3 &= 2N^3\sin(N\theta_*), \\
& & &\vdots \\
&(22) & N!\lambda_j'(\theta_*)^N &= \begin{cases} (-1)^{\frac{N+1}{2}}2N^N\sin(N\theta_*), & N \text{ is odd} \\ (-1)^{\frac{N}{2}}2N^N\cos(N\theta_*), & N \text{ is even} \end{cases}, \\
&(23) & 0 &= \begin{cases} (-1)^{\frac{N+1}{2}}2N^{N+1}\cos(N\theta_*), & N \text{ is odd} \\ (-1)^{\frac{N}{2}+1}2N^{N+1}\sin(N\theta_*), & N \text{ is even} \end{cases}.
\end{aligned}
$$

Recalling that $N \geq 3$, (22) and (23) imply that

$$
|\lambda_j'(\theta_*)| = \left(\frac{2}{N!}\right)^{\frac{1}{N}} N > 2. \tag{24}
$$

Indeed, by a direct computation, the above inequality holds for $N = 3, 4, 5, 6$, and for $N \geq 7$,

$$
\begin{aligned}
\frac{N^N}{N!} &= \prod_{k=1}^{N-1}\left(1+\frac{1}{k}\right)^N \cdot \prod_{i=1}^{N-1}\prod_{k=1}^{i}\left(1+\frac{1}{k}\right)^{-1} \\
&= \prod_{k=1}^{N-1}\left(1+\frac{1}{k}\right)^N \cdot \prod_{i=1}^{N-1}\left(1+\frac{1}{i}\right)^{-(N-i)} \\
&= \prod_{i=1}^{N-1}\left(1+\frac{1}{i}\right)^i \\
&> \prod_{i=1}^{N-1}\left(e^1 \cdot \left(1+\frac{1}{i}\right)^{-\frac{1}{2}}\right) \\
&= \frac{e^{N-1}}{\sqrt{N}}
\end{aligned}
$$

which implies that

$$
\frac{2^{\frac{1}{N}}N}{(N!)^{\frac{1}{N}}} > \frac{e^{1-\frac{1}{N}}}{\sqrt[2N]{N}} = e \cdot \left(e\sqrt{N}\right)^{-\frac{1}{N}} > 2.
$$

(24) contradicts with Lemma 2.2, hence Proposition 2.1 is shown. $\square$

With Proposition 2.1, for $j = 0, 1, \cdots, N-1$, we have the decomposition of $\mathbb{T}$ according to the optimal order of transversality of λ_j:

$$
\mathbb{T} = \bigcup_{i=1}^{N}\Theta_j^{(i)}, \quad \Theta_j^{(i)} := \left\{\theta \in \mathbb{T} : \begin{array}{l} |\lambda_j^{(i+1)}(\theta)| \geq \frac{\delta}{N} \\ |\lambda_j^{(i'+1)}(\theta)| < \frac{\delta}{N},\ 1 \leq i' \leq i-1 \text{ if } i \geq 2 \end{array}\right\}. \tag{25}
$$

Since the eigenvalues $\lambda_j(\theta)$ of the matrix $A(\theta)$ is analytic on θ, in view of Lemma 3 of [7], we see that $\Theta_j^{(i)}$ consists of finitely many disjoint subintervals. We denote this number by $M_j^{(i)}$.

3. Proof of Theorem 1.1

3.1. **Expression of time evolution via Fourier transformation.** Since the sharp estimate for the cases $N = 1, 2$ is already known (see [14, 15, 18]), we can focus on the linear $N-$periodic discrete Schrödinger equation with $N \geq 3$:

$$\left\{ \begin{array}{l} \mathrm{i}\dot{q}_n = -(q_{n+1} + q_{n-1}) + V_n\, q_n, \quad n \in \mathbb{Z} \\ q(0) \in \ell^1(\mathbb{Z}) \end{array} \right. . \tag{26}$$

By the Fourier transform

$$(q_n)_{n\in\mathbb{Z}} \mapsto G(\theta) := \sum_{n\in\mathbb{Z}} q_n e^{\mathrm{i}n\theta}, \quad \theta \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z},$$

(26) becomes

$$\mathrm{i}\partial_t G(\theta, t) = \sum_{n\in\mathbb{Z}} [-(q_{n+1}(t) + q_{n-1}(t)) + V_n q_n(t)] e^{\mathrm{i}n\theta}.$$

Decompose $G(\theta, t)$ as $G(\theta, t) = \sum_{j=0}^{N-1} G_j(\theta, t)$, with

$$G_j(\theta, t) := \sum_{k\in\mathbb{Z}} q_{kN+j}(t) e^{\mathrm{i}(kN+j)\theta}, \quad j = 0, 1, \cdots, N-1. \tag{27}$$

We find that, for $j = 0, 1, \cdots, N-1$,

$$\begin{aligned} \mathrm{i}\partial_t G_j(\theta, t) &= \mathrm{i} \sum_{k\in\mathbb{Z}} \dot{q}_{kN+j}(t) e^{\mathrm{i}(kN+j)\theta} \\ &= \sum_{k\in\mathbb{Z}} [-(q_{kN+j+1}(t) + q_{kN+j-1}(t)) + V_{kN+j} q_{kN+j}(t)]\, e^{\mathrm{i}(kN+j)\theta} \\ &= -(e^{-\mathrm{i}\theta} G_{j+1}(\theta, t) + e^{\mathrm{i}\theta} G_{j-1}(\theta, t)) + V_j\, G_j(\theta, t). \end{aligned} \tag{28}$$

Actually, we can rewrite them as the $N-$dimensional system

$$\mathrm{i}\partial_t \mathcal{G}(\theta, t) = A(\theta)\mathcal{G}(\theta, t), \tag{29}$$

with $\mathcal{G}(\theta, t) := (G_j(\theta, t))_{j=0}^{N-1}$, where $A(\theta)$ is a periodic Jacobi matrix given in (7) or (8). Hence, according to (10), the solution to Eq. (29) is

$$\mathcal{G}(\theta, t) = U(\theta) e^{-\mathrm{i}\Lambda(\theta)t} U^*(\theta)\mathcal{G}(\theta, 0), \quad \text{with} \quad e^{-\mathrm{i}\Lambda(\theta)t} := \mathrm{diag}\{e^{-\mathrm{i}\lambda_j(\theta)t}\}_{j=0}^{N-1}.$$

Then, for $n = k_* N + j$ with $k_* \in \mathbb{Z}$ and $0 \leq j \leq N-1$, we have

$$\begin{aligned} q_n(t) &= \frac{1}{2\pi} \int_{\mathbb{T}} G_j(\theta, t) e^{-\mathrm{i}(k_* N+j)\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} \sum_{l_1, l_2=0}^{N-1} e^{-\mathrm{i}\lambda_{l_1}(\theta)t} U_{j,l_1}(\theta) U^*_{l_1,l_2}(\theta) G_{l_2}(\theta, 0) e^{-\mathrm{i}(k_* N+j)\theta}\, d\theta \\ &= \frac{1}{2\pi} \sum_{l_1, l_2=0}^{N-1} \int_{\mathbb{T}} e^{-\mathrm{i}\lambda_{l_1}(\theta)t} U_{j,l_1}(\theta) U^*_{l_1,l_2}(\theta) \sum_{k\in\mathbb{Z}} q_{kN+l_2}(0) e^{\mathrm{i}((k-k_*)N+l_2-j)\theta}\, d\theta \end{aligned}$$

To summarise, we have

Proposition 3.1. *Given $\psi \in \ell^1(\mathbb{Z})$, $t \in \mathbb{R}$, for $n = k_*N + j$ with $k_* \in \mathbb{Z}$ and $0 \le j \le N-1$, $j \in \mathbb{Z}$, we have*

$$\left(e^{-\mathrm{i}tH_V}\psi\right)_n = \frac{1}{2\pi}\sum_{l_1,l_2=0}^{N-1}\sum_{k\in\mathbb{Z}}\psi_{kN+l_2}\int_{\mathbb{T}} U_{j,l_1}(\theta)U^*_{l_1,l_2}(\theta)e^{\mathrm{i}\left[-\lambda_{l_1}(\theta)t+((k-k_*)N+l_2-j)\theta\right]}\,d\theta.$$

3.2. **Dispersive estimate.** Now let us consider the oscillatory integral in Proposition 3.1, which leads us to the dispersive estimate.

Since $U(\theta)$ is an orthonormal matrix for every $\theta \in \mathbb{T}$, we have immediately that

$$|U_{j,l_1}(\theta)U^*_{l_1,l_2}(\theta)| \le \frac{|U_{j,l_1}(\theta)|^2 + |U^*_{l_1,l_2}(\theta)|^2}{2} \le 1, \quad \forall\ j, l_1, l_2 = 0, 1, \cdots, N-1. \tag{30}$$

Due to the analyticity of the elements of $U(\theta)$, there exists a constant $D > 0$, depending on $V = \{V_j\}_{j=0}^{N-1}$, such that, for j, l_1, $l_2 = 0, 1, \cdots, N-1$,

$$\sup_{\theta\in\mathbb{T}}\left|U_{j,l_1}(\theta)U^*_{l_1,l_2}(\theta)\right| + \int_{\mathbb{T}}\left|(U_{j,l_1}(\theta)U^*_{l_1,l_2}(\theta))'\right|d\theta \le D. \tag{31}$$

For $|t| \le 1$, since $2^{\frac{1}{N+1}}(1+|t|)^{-\frac{1}{N+1}} \ge 1$, we have, by (30),

$$\left|\int_{\mathbb{T}} U_{j,l_1}(\theta)U^*_{l_1,l_2}(\theta)e^{\mathrm{i}\left[-\lambda_{l_1}(\theta)t+((k-k_*)N+l_2-j)\theta\right]}\,d\theta\right| \le 2\pi\cdot 2^{\frac{1}{N+1}}(1+|t|)^{-\frac{1}{N+1}}. \tag{32}$$

Now we assume that $|t| > 1$, which implies that

$$2^{\zeta}(1+|t|)^{-\zeta} > |t|^{-\zeta} \quad \text{for all} \quad \zeta = \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{N+1}.$$

According to the definition of $\Theta_j^{(i)}$ given in (25), we apply Van der Corput lemma (Lemma A.1 in Appendix A) for $k = i+1$, $i = 1, 2, \cdots, N$ on each subintervals of $\Theta_{l_1}^{(i)}$, and get

$$\left|\int_{\Theta_{l_1}^{(i)}} U_{j,l_1}(\theta)U^*_{l_1,l_2}(\theta)e^{\mathrm{i}\left[-\lambda_{l_1}(\theta)t+((k-k_*)N+l_2-j)\theta\right]}\,d\theta\right| \le C_{l_1,i}(1+|t|)^{-\frac{1}{i+1}}. \tag{33}$$

with the constant defined by

$$C_{l_1,i} := M_{l_1}^{(i)}\cdot D\cdot(5\cdot 2^i - 2)\left(\frac{\delta}{N}\right)^{-\frac{1}{i+1}}\cdot 2^{\frac{1}{i+1}},$$

recalling that $M_{l_1}^{(i)}$ is the number of subintervals in $\Theta_{l_1}^{(i)}$, and D is the upper bound given in (31).

Back to the expression of $e^{-\mathrm{i}tH_V}$ given in Proposition 3.1. Since

$$\sum_{l_2=0}^{N-1}\sum_{k\in\mathbb{Z}}|\psi_{kN+l_2}| = \|\psi\|_{\ell^1},$$

by (32) and (33), we have, for every $t \in \mathbb{R}$,

$$\begin{aligned}\left|\left(e^{-\mathrm{i}tH_V}\psi\right)_n\right| &\le \frac{1}{2\pi}\sum_{l_1,l_2=0}^{N-1}\sum_{k\in\mathbb{Z}}|\psi_{kN+l_2}|\left|\int_{\mathbb{T}} U_{j,l_1}(\theta)U^*_{l_1,l_2}(\theta)e^{\mathrm{i}\left[-\lambda_{l_1}(\theta)t+((k-k_*)N+l_2-j)\theta\right]}\,d\theta\right| \\ &\le C_V\|\psi\|_{\ell^1}(1+|t|)^{-\frac{1}{N+1}}, \quad \forall\ n \in \mathbb{Z},\end{aligned}$$

for some positive constant C_V depending on V. Thus Theorem 1.1 is proved.

Appendix A. Van der corput lemma

For the convenience of readers, we present Van der Corput lemma which is used in this paper. The original statement can be found in many textbooks on Harmonic Analysis (see, e.g., Chapter VIII of [19]).

Lemma A.1. *Suppose that ψ is real-valued and $\mathcal{C}^k$ in (a,b) for some $k \geq 2$, and that $|\psi^{(k)}(x)| \geq c$ for all $x \in (a,b)$. Let h be $\mathcal{C}^1$ in (a,b). Then*

$$\left| \int_a^b e^{\mathrm{i}\lambda\psi(x)} h(x)dx \right| \leq (5 \cdot 2^{k-1} - 2)c^{-\frac{1}{k}} \left[|h(b)| + \int_a^b |h'(x)|dx \right] |\lambda|^{-\frac{1}{k}}, \quad \forall\, \lambda \in \mathbb{R} \setminus \{0\}.$$

The proof of Lemma A.1 is given in Appendix of [3, 14].

Acknowledgements. Z. Zhao would like to thank Prof. D. Bambusi and Prof. D. Damanik for their interests and fruitful discussions which helped to improve the proof. Z. Zhao would also like to thank the support of Visiting Scholars "Nonlinear Mathematical Models and Methods" of Fudan University during his visits.

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