Reducibility of One-dimensional Quasi-periodic Schrödinger Equations

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Abstract

Consider the time-dependent linear Schrödinger equation

$$\mathrm{i}\dot{q}_n = \epsilon(q_{n+1} + q_{n-1}) + V(x + n\omega)q_n + \delta \sum_{m\in\mathbb{Z}} a_{mn}(\theta + \xi t)q_m, \quad n \in \mathbb{Z},$$

where V is a nonconstant real-analytic function on $\mathbb{T}$, ω satisfies a certain Diophantine condition and $a_{mn}(\theta)$ is real-analytic on $\mathbb{T}^b$, $b \in \mathbb{Z}_+$, decaying with $|m|$ and $|n|$. We prove that, if ϵ and δ are sufficiently small, then for a.e. $x \in \mathbb{T}$ and "most" frequency vectors $\xi \in \mathbb{T}^b$, it can be reduced to an autonomous equation.

Moreover, for this non-autonomous system, "dynamical localization" is maintained in a quasi-periodic time-dependent way.

1 Introduction and Main Results

We consider the one-dimensional Schrödinger equation

$$\mathrm{i}\dot{q}_n = \epsilon(q_{n+1} + q_{n-1}) + V(x + n\omega)q_n + \delta \sum_{m\in\mathbb{Z}} a_{mn}(\theta + \xi t)q_m, \quad n \in \mathbb{Z}, \tag{1.1}$$

where V is a nonconstant real-analytic function on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\omega \in \mathbb{R}$ is a Diophantine number, and for each $m, n \in \mathbb{Z}$, $a_{mn} : \mathbb{T}^b \to \mathbb{R}$ is analytic in a complex neighbourhood $|\mathrm{Im}\theta| < r \leq 1$ of the $b-$torus $\mathbb{T}^b$, satisfying

$$\sup_{|\mathrm{Im}\theta|<r} |a_{mn}(\theta)| < e^{-\rho \max\{|m|,|n|\}}, \quad \rho > 0.$$

We are going to prove that, for ϵ and δ small enough, Eq. (1.1) can be reduced to a constant coefficient equation(independent of t) for "most" value of the parameter ξ, with the corresponding solutions well localized in space all the time. It is stated in the following theorem.

Theorem 1 *There exists a sufficiently small $\epsilon_* = \epsilon_*(V, \omega, r, \rho)$, such that if $0 < \epsilon, \delta < \epsilon_*$, then for a.e. $x \in \mathbb{T}$, one can find a Cantor set $\mathcal{O}_{\epsilon,\delta} = \mathcal{O}_{\epsilon,\delta}(x) \subset \mathbb{T}^b$ with*

$$\mathrm{Leb}(\mathbb{T}^b \setminus \mathcal{O}_{\epsilon,\delta}) \to 0 \quad as \quad \epsilon, \delta \to 0,$$

such that for each $\xi \in \mathcal{O}_{\epsilon,\delta}$ and $\theta \in \mathbb{T}^b$, the equation (1.1) can be analytically reduced to an autonomous equation.

Moreover, given any initial datum $q(0)$ satisfying $|q_n(0)| < ce^{-\rho|n|}$ for some constant $c > 0$, the solution $q(t)$ to the equation (1.1) with $\xi \in \mathcal{O}_{\epsilon,\delta}$ satisfies that for any fixed $d > 0$,

$$\sup_t \sum_{n \in \mathbb{Z}} n^{2d} |q_n(t)|^2 < \infty.$$

Remark 1.1 *The behaviour of solutions for a dynamical equation in the last statement of Theorem 1 is called dynamical localization.*

The Equation (1.1) is a perturbation of an autonomous quasi-periodic Schrödinger equation, whose behavior is determined by the spectral property of the operator $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, defined as

$$(Tq)_n = \epsilon(q_{n+1} + q_{n-1}) + V(x + n\omega)q_n.$$

As shown in [12], for ϵ small enough, the spectrum of this operator is pure point for a.e. $x \in \mathbb{T}$. More precisely, it can be "almost block-diagonalized", which is presented by a KAM scheme(see Proposition 1 for details). The readers can also refer to [6, 10, 15, 22, 23, 26] for other works on the pure point spectrum and localization of quasi-periodic Schrödinger operators, and see [20, 21] for more about dynamical localization.

It is necessary to mention that the KAM theory has been well adapted to Hamiltonian PDE's, especially in the continuous case. Many well-known works have been done for construction of time quasi-periodic solutions(e.g., [3, 14, 16, 17]), for reducibility of non-autonomous equations(e.g., [1, 13]), and for growth of Sobolev norms(e.g., [2, 4, 5, 27, 28]).

However, the KAM technique is not widely applied to the discrete models, especially the case of the dense normal frequencies. A successful application is the model

$$\mathrm{i}\dot{q}_n = \epsilon(q_{n+1} + q_{n-1}) + V(x + n\alpha)q_n + |q_n|^2 q_n, \quad n \in \mathbb{Z}, \tag{1.2}$$

with ϵ sufficiently small, $x \in \mathbb{T}$, and α Diophantine. It is proved that, if $V(x) = \tan \pi x$ [19] or V is a nonconstant real-analytic function on $\mathbb{T}$ [18], the equation (1.2) admits a large family of time quasi-periodic solutions $(q_n(t))_{n\in\mathbb{Z}}$, which are well localized in space. Corresponding to [19], Zhang-Zhao[30] has proved reducibility of the non-autonomous equation

$$\mathrm{i}\dot{q}_n = \epsilon(q_{n+1} + q_{n-1}) + \tan \pi(x + n\alpha)q_n + v_n(\xi t)q_n, \quad n \in \mathbb{Z},$$

for "most" frequencies $\xi \in \mathbb{T}^b$, where $v_n(\cdot)$ is smooth on $\mathbb{T}^b$ and decaying with $|n|$. In the similar sense, the present work corresponds to [18]. Moreover, compared to [18], the gauge invariance of the perturbation is seriously destroyed(see the assumption $(A5)$ in [18]), hence we have to state a new KAM theorem and prove it again.

Apart from the quasi-periodic discrete models above, some systems with random potentials are also studied by other techniques[7, 8, 9, 29].

The remaining part of this paper is organized as follows. In Section 2, we shall present a KAM theorem, which can be directly applied to the equation (1.1). This KAM theorem will be proved in Section 3, by giving the details for one step of the KAM iteration and verifying the convergence. Section 4, which is regarded as an appendix, will present some necessary properties of infinite-dimensional matrices and Hamiltonians.

2 A KAM Theorem

2.1 Statement of KAM theorem

To state a KAM theorem, which can be adapted to the equation (1.1), we start with recalling some necessary definitions of notations, which are related to the Hamiltonian vector field and the Poisson bracket. Some basic properties about this mechanism have been studied in previous works(e.g., [18, 19]).

Given $d, \rho > 0$, let $\ell^1_{d,\rho}(\mathbb{Z})$ be the space of complex valued sequences $q = (q_n)_{n\in\mathbb{Z}}$, with the norm

$$\|q\|_{d,\rho} := \sum_{n\in\mathbb{Z}} |q_n|(1+n^2)^{\frac{d}{2}} e^{\rho|n|} < \infty.$$

For $r, s > 0$, let $\mathcal{D}_{d,\rho}(r,s)$ be the complex b-dimensional neighborhood of $\mathbb{T}^b\times\{0\}\times\{0\}\times\{0\}$ in $\mathbb{T}^b \times \mathbb{R}^b \times \ell^1_{d,\rho}(\mathbb{Z}) \times \ell^1_{d,\rho}(\mathbb{Z})$, i.e.,

$$\mathcal{D}_{d,\rho}(r,s) := \{(\theta, I, q, \bar{q}) : |\mathrm{Im}\theta| < r,\ |I| < s^2,\ \|q\|_{d,\rho} = \|\bar{q}\|_{d,\rho} < s\},$$

where $|\cdot|$ is the ℓ^1-norm of $b-$dimensional complex vectors.

For a real-analytic function $F(\theta, I, q, \bar{q}; \xi)$ on $\mathcal{D} = \mathcal{D}_{d,\rho}(r,s)$, C^1_W(i.e., C^1 in the sense of Whitney) parametrized by $\xi \in \mathcal{O}$, a closed region in $\mathbb{T}^b$, it can be expanded into the Taylor-Fourier series with respect to $\theta, I, q, \bar{q}$:

$$F(\theta, I, q, \bar{q}; \xi) = \sum_{\substack{k\in\mathbb{Z}^b,\, l\in\mathbb{N}^b \\ \alpha,\, \beta}} F_{kl\alpha\beta}(\xi) e^{\mathrm{i}\langle k,\theta\rangle} I^l q^\alpha \bar{q}^\beta,$$

where $I^l := \prod_{j=1}^{b} I_j^{l_j}$, $q^\alpha \bar{q}^\beta = \prod_{(\alpha_n,\beta_n)\neq(0,0)} q_n^{\alpha_n} \bar{q}_n^{\beta_n}$ for $\alpha := (\alpha_n)_{n\in\mathbb{Z}}$, $\beta := (\beta_n)_{n\in\mathbb{Z}}$, $\alpha_n, \beta_n \in \mathbb{N}$, with finite non-vanishing components.

With $|\partial_\xi F_{kl\alpha\beta}| := \sum_{j=1}^b \left|\partial_{\xi_j} F_{kl\alpha\beta}\right|$ and $|F_{kl\alpha\beta}|_{\mathcal{O}} := \sup_{\xi\in\mathcal{O}} (|F_{kl\alpha\beta}| + |\partial_\xi F_{kl\alpha\beta}|)$, let

$$\|F(\theta, I, q, \bar{q})\|_{\mathcal{O}} := \sum_{k,l,\alpha,\beta} |F_{kl\alpha\beta}|_{\mathcal{O}}\, e^{|k||\mathrm{Im}\theta|} |I^l|\, |q^\alpha| |\bar{q}^\beta|.$$

Define the weighted norm of F as $\|F\|_{\mathcal{D},\mathcal{O}} := \sup_{\mathcal{D}} \|F(\theta, I, q, \bar{q})\|_{\mathcal{O}}$. For the Hamiltonian vector field $X_F = (\partial_I F, -\partial_\theta F, (-\mathrm{i}\partial_{q_n} F)_{n\in\mathbb{Z}}, (\mathrm{i}\partial_{\bar{q}_n} F)_{n\in\mathbb{Z}})$ on $\mathcal{D}\times\mathcal{O}$, its norm is defined by

$$\begin{aligned} \|X_F\|_{\mathcal{D},\mathcal{O}} &:= \sup_{\mathcal{D}} \frac{1}{s} \sum_{n\in\mathbb{Z}} \left(\|\partial_{q_n} F(\theta, I, q, \bar{q})\|_{\mathcal{O}} + \|\partial_{\bar{q}_n} F(\theta, I, q, \bar{q})\|_{\mathcal{O}}\right)(1+n^2)^{\frac{d}{2}} e^{|n|\rho} \\ &\quad + \sum_{j=1}^b \|\partial_{I_j} F\|_{\mathcal{D},\mathcal{O}} + \frac{1}{s^2} \sum_{j=1}^b \|\partial_{\theta_j} F\|_{\mathcal{D},\mathcal{O}}. \end{aligned} \tag{2.1}$$

Obviously, this norm depends on d, ρ, r, s.

Given two real-analytic functions F and G on $\mathcal{D}$, let $\{\cdot,\cdot\}$ denote the Poisson bracket of such functions, i.e.,

$$\{F,G\} = \langle \partial_I F, \partial_\theta G\rangle - \langle \partial_\theta F, \partial_I G\rangle + \mathrm{i}\sum_{n\in\mathbb{Z}} \left(\partial_{q_n} F \cdot \partial_{\bar{q}_n} G - \partial_{\bar{q}_n} F \cdot \partial_{q_n} G\right).$$

In the remaining part of this paper, all constants labeled with c, c_0, c_1, $\cdots$ are positive and *independent of the iteration step.*

We consider the Hamiltonian

$$H = \langle \xi, I\rangle + \langle Tq, \bar{q}\rangle + P(\theta, q, \bar{q}; \xi),$$

real-analytic on some suitable domain of $(\theta, I, q, \bar{q})$, real-analytic parametrized by $x \in \mathbb{T}$, and C^1_W parametrized by $\xi \in \mathcal{O}$. We assume that

- T is the symmetric matrix defined by the quasi-periodic Schödinger operator with pure point spectrum. More precisely,

$$T_{mn}(x) = \begin{cases} V(x+m\omega), & m = n \\ \epsilon, & m-n = \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

  with V and ω as in the equation (1.1), and ϵ smaller than some critical value ϵ_0, which will be given in Proposition 1 later.

- P can be expanded as $P(\theta, q, \bar{q}; \xi) = \sum_{k\in\mathbb{Z}^b} \langle P^k(\xi)\, q,\, \bar{q}\rangle e^{\mathrm{i}\langle k,\theta\rangle}$, with

$$|P^k_{mn}|_{\mathcal{O}} \le \varepsilon e^{-\rho\max\{|m|,\,|n|\}} e^{-r|k|} \tag{2.2}$$

  for some positive $0 < \varepsilon \ll 1$ and $r, \rho > 0$.

Remark 2.1 *Compared to [18], the gauge invariance of the perturbation is seriously destroyed(see the assumption* $(A5)$ *in [18]), hence we have to state a new KAM theorem and prove it again.*

For $\rho_0 := \frac{\rho}{2}$, $r_0 := \frac{r}{2}$ and any $d, s > 0$, H is real-analytic on the domain $\mathcal{D}_0 := \mathcal{D}_{d,\rho_0}(r_0, s)$, C^1_W parametrized by $\xi \in \mathcal{O}_0 := \mathcal{O}$. The decay estimate (2.2) implies that there exists a numerical constant c_0 such that $\|X_P\|_{\mathcal{D}_0,\mathcal{O}_0} \le c_0(\rho^{-1} + r^{-1})\varepsilon =: \varepsilon_0$, in view of the definition given in (2.1). This provides us the necessary smallness condition for the perturbation argument.

Theorem 2 *There exists a positive ε_*, only depends on V, ω, r and ρ, such that if $\|X_P\|_{\mathcal{D}_0,\mathcal{O}_0} < \varepsilon_0 \le \varepsilon_*$, then for a.e. $x \in \mathbb{T}$, there exists a Cantor set $\mathcal{O}_{\varepsilon_0} = \mathcal{O}_{\varepsilon_0}(x) \subset \mathcal{O}_0$ with $Leb(\mathcal{O}_0 \setminus \mathcal{O}_{\varepsilon_0}) \to 0$ as $\varepsilon_0 \to 0$, such that the following holds.*

There exists an analytic symplectic diffeomorphism $\Psi : \mathcal{D}_{d,0}(r_0/2, s) \to \mathcal{D}_0$, which is C^1_W parametrized by $\xi \in \mathcal{O}_{\varepsilon_0}$, satisfying

$$\|\Psi - Id\|_{\mathcal{D}_{d,0}(r_0/2,\,s),\,\mathcal{O}_{\varepsilon_0}} \to 0^1 \quad as \quad \varepsilon_0 \to 0,$$

such that $H \circ \Psi = \langle \xi, I\rangle + \langle \tilde{\Omega}(\xi) q, \bar{q}\rangle$, independent of θ.

[1]Here the norm $\|\cdot\|_{\mathcal{D}_{d,0}(r_0/2,\,s),\,\mathcal{O}_{\varepsilon_0}}$ is the operator norm.

It is obvious that the Hamiltonian equation

$$\dot{\theta} = \frac{\partial H}{\partial I}, \quad \dot{I} = -\frac{\partial H}{\partial \theta}, \quad \mathrm{i}\dot{q}_n = \frac{\partial H}{\partial \bar{q}_n}, \quad \mathrm{i}\dot{\bar{q}}_n = -\frac{\partial H}{\partial q_n}, \quad n \in \mathbb{Z},$$

is related to Eq. (1.1). So, with ϵ and δ small enough, Theorem 2 can be applied to it. The reducibility statement is obtained directly. Moreover, if $q(0)$ satisfies $|q_n(0)| < ce^{-\rho|n|}$, $q(t) \in \ell^1_{d,0}(\mathbb{Z})$ is a $b-$frequency time quasi-periodic solution to (1.1). Hence,

$$\sup_t \left(\sum_{n\in\mathbb{Z}} n^{2d} |q_n(t)|^2 \right)^{\frac{1}{2}} < c \cdot \sup_t \sum_{n\in\mathbb{Z}} |n|^d |q_n(t)| < +\infty.$$

Therefore, Theorem 1 is a corollary of Theorem 2.

2.2 KAM scheme for the quasi-periodic Schrödinger operator

Before the analysis on the Hamiltonian H, we first consider its ingredient, the quasi-periodic Schrödinger operator $T = T(x) : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, defined as

$$(Tq)_n := \epsilon(q_{n+1} + q_{n-1}) + V(x + n\omega)q_n, \quad n \in \mathbb{Z},$$

with V and ω as in the equation (1.1). It is well-known from [12] that if ϵ is sufficiently small, then for a.e. $x \in \mathbb{T}$, the spectrum of $T(x)$ is pure point. Related to the diophantine frequency ω, there exist $\tilde{\tau} > 1$ and $\tilde{\gamma} > 0$ such that

$$\inf_{k\in\mathbb{Z}} |n\omega - k| \geq \frac{\tilde{\gamma}}{|n|^{\tilde{\tau}}}, \quad n \neq 0.$$

The non-constant real-analytic potential V is a smooth function in the Gevrey class

$$\sup_{x\in\mathbb{R}/\mathbb{Z}} |\partial^m V(x)| \leq CL^m m!, \quad m \geq 0,$$

for some C, $L > 0$, and satisfying the transversality condition

$$\max_{0\leq m\leq\tilde{s}} |\partial_\varphi^m (V(x+\varphi) - V(x))| \geq \tilde{\xi} > 0, \quad \forall x, \forall\varphi,$$
$$\max_{0\leq m\leq\tilde{s}} |\partial_x^m (V(x+\varphi) - V(x))| \geq \tilde{\xi} \inf_{k\in\mathbb{Z}} |\varphi - k|, \quad \forall x, \forall\varphi,$$

for some $\tilde{\xi}$, $\tilde{s} > 0$. Clearly, the case $V(x) = \cos 2\pi x$ is included.

With any $N_0 \geq 1$, $\rho_0 = N_0^{-1}$, any $\varepsilon_0 \leq \epsilon e^{\rho_0}$, and $M_0 \geq \max\left\{2^{\tilde{s}+4} C \frac{L^{\tilde{s}+1}(\tilde{s}+1)!}{\tilde{\xi}}, 2\tilde{\tau}, 8\right\}$, one can define the following sequences for $\nu \in \mathbb{N}$:

$$\begin{aligned} &M_{\nu+1} = M_\nu^{\tilde{s}M_\nu^3}, \quad a_\nu = \frac{1}{\tilde{\tau}} M_\nu^{-3\tilde{s}M_\nu^3}, \quad \varepsilon_{\nu+1} = \varepsilon_\nu^{\frac{1}{2}\varepsilon_\nu^{-a_\nu/2}}, \\ &N_{\nu+1} = \varepsilon_\nu^{-a_\nu}, \qquad \rho_{\nu+1} = \varepsilon_\nu^{a_\nu}, \qquad \sigma_{\nu+1} = \frac{1}{3}\rho_\nu. \end{aligned} \tag{2.3}$$

which have already been defined in [18] and originally defined in [12].

Proposition 1 *There exists a constant $\epsilon_0 = \epsilon_0(C, L, \tilde{\xi}, \tilde{s}, \tilde{\gamma}, \tilde{\tau})$ such that if $0 < \epsilon < \epsilon_0$ then the following holds.*

Fixed $x \in \mathbb{R}/\mathbb{Z}$. There exists a sequence of orthogonal matrices U_ν, $\nu \in \mathbb{N}$, with

$$|(U_\nu - I_{\mathbb{Z}})_{mn}| \leq \varepsilon_0^{\frac{1}{2}} e^{-\frac{3}{2}\sigma_\nu |m-n|}, \tag{2.4}$$

such that $U_\nu^* T U_\nu = D_\nu + Z_\nu$, where

- *D_ν is a symmetric matrix which can be block-diagonalized via an orthogonal matrix Q_ν with*

$$(Q_\nu)_{mn} = 0 \ \ if \ \ |m-n| > N_\nu. \tag{2.5}$$

  More precisely, there is a disjoint decomposition $\bigcup_j \Lambda_j^\nu = \mathbb{Z}$ such that

$$\tilde{D}^\nu = Q_\nu^* D_\nu Q_\nu = \prod_j \tilde{D}^\nu_{\Lambda_j^\nu}{}^2 \ \ with \ \ \sharp\Lambda_j^\nu \leq M_\nu, \ \ \mathrm{diam}\Lambda_j^\nu \leq M_\nu N_\nu, \ \ \forall j.^3$$

  Moreover, there exists a full-measure subset $\mathcal{X} \subset \mathbb{R}/\mathbb{Z}$ such that if we fix $x \in \mathcal{X}$, then for each $k \in \mathbb{Z}$, there is a $\nu_0(k)$ such that $\Lambda^{\nu+1}(k) = \Lambda^\nu(k)$, $\forall \nu \geq \nu_0(k)$.

- *Z_ν is a symmetric matrix, and*

$$|(Z_\nu)_{mn}| \leq \varepsilon_\nu e^{-\rho_\nu |m-n|}. \tag{2.6}$$

Remark 2.2 *In this paper, we consider this model for fixed $x \in \mathbb{T}$. From now on, we shall not report the parameter x explicitly, if this dependence is irrelevant.*

3 Proof of KAM Theorem

Now we start the KAM iteration for the Hamiltonian $H_0 := H = \langle \xi, I \rangle + \langle Tq, \bar{q} \rangle + P_0$, real-analytic on $\mathcal{D}_0 = \mathcal{D}_{d,\rho_0}(r_0, s)$, C^1_W parametrized by $\xi \in \mathcal{O}_0$, with $\|X_{P_0}\|_{\mathcal{D}_0, \mathcal{O}_0} \leq \varepsilon_0$.

Suppose that we have arrived at the ν^{th} step of the KAM iteration, $\nu = 0, 1, 2, \cdots$. We consider the real-analytic Hamiltonian $H_\nu = \mathcal{N}_\nu + P_\nu$ on $\mathcal{D}_\nu := \mathcal{D}_{d,\rho_\nu}(r_\nu, s)$ with

$$\mathcal{N}_\nu(\xi) = \langle \xi, I \rangle + \langle (T + W_\nu(\xi))q, \bar{q} \rangle, \quad P_\nu(\xi) = \sum_{k \in \mathbb{Z}^b} \langle P_\nu^k(\xi) q, \bar{q} \rangle e^{\mathrm{i}\langle k, \theta \rangle}$$

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[2]For readers' convenience, we represent some notations for infinite-dimensional matrix. Given an infinite-dimensional matrix D, with $D_{mn} \in \mathbb{C}$ the $(m,n)^{\text{th}}$ entry, for a subset $\Lambda \subset \mathbb{Z}$, we define $\Lambda^\perp := \mathbb{Z} \backslash \Lambda$,

$$\mathbb{C}^\Lambda := \{n \in \mathbb{C}^{\mathbb{Z}} : n_i = 0 \ \text{ if } \ i \notin \Lambda\}, \quad D_\Lambda := \begin{cases} D_{mn}, & m, n \in \Lambda \\ \delta_{mn}, & \text{otherwise} \end{cases}.$$

Then $D_\Lambda : \mathbb{C}^\Lambda + \mathbb{C}^{\Lambda^\perp} \to \mathbb{C}^\Lambda + \mathbb{C}^{\Lambda^\perp}$, acts as $\mathbb{R}^\Lambda \hookrightarrow \mathbb{C}^{\mathbb{Z}} \xrightarrow{D} \mathbb{C}^{\mathbb{Z}} \xrightarrow{\perp\text{proj}} \mathbb{C}^\Lambda$ on the first component and as the identity on the second component. When there is no risk for confusion, we will use D_Λ also to denote its first component.

[3]The disjoint decomposition defines an equivalence relation $m \sim n$ on the integers and, for each $n \in \mathbb{Z}$, we denote its equivalence class by $\Lambda^\nu(n)$.

which is C^1_W parametrized by $\xi \in \mathcal{O}_\nu$, and satisfies $\|X_{P_\nu}\|_{\mathcal{D}_\nu, \mathcal{O}_\nu} \le \varepsilon_\nu$,

$$|(W_\nu)_{mn}|_{\mathcal{O}_\nu} \le \begin{cases} p_\nu e^{-\sigma_\nu \max\{|m|,|n|\}}, & |m|,|n| \le N_\nu \\ 0, & \text{otherwise} \end{cases}, \tag{3.1}$$

$$|(P^k_\nu)_{mn}|_{\mathcal{O}_\nu} \le \varepsilon_\nu e^{-\rho_\nu \max\{|m|,|n|\}} e^{-r_\nu |k|}, \quad \forall k \in \mathbb{Z}^b, \tag{3.2}$$

with some $0 < p_\nu \ll 1$, $0 < r_\nu < r_0$, and ρ_ν, σ_ν, ε_ν defined as in (2.3).

Choose some $r_{\nu+1}$ such that $0 < r_{\nu+1} < r_\nu$, and let $J_\nu := \left[\frac{5}{2}\varepsilon_\nu^{-\frac{a_\nu}{2}}\right]$. For $j = 0, 1, \cdots, J_\nu$, we define the quantities at each KAM sub-step as

$$\rho^{(j)}_\nu = (1 - \frac{j}{2J_\nu})\rho_\nu, \quad r^{(j)}_\nu = r_\nu - \frac{j(r_\nu - r_{\nu+1})}{J_\nu},$$

and $\mathcal{D}^{(j)}_\nu = \mathcal{D}_{d,\rho_{\nu+1}}(r^{(j)}_\nu, s)$, $\varepsilon^{(j)}_\nu = \varepsilon_\nu^{\frac{j}{5}+1}$. Our goal is to construct a set $\mathcal{O}_{\nu+1} \subset \mathcal{O}_\nu$ and a finite sequence of maps

$$\Phi^{(j)}_\nu : \mathcal{D}^{(j)}_\nu \to \mathcal{D}^{(j-1)}_\nu, \quad j = 1, 2, \cdots, J_\nu,$$

so that the Hamiltonian transformed into the $(\nu+1)^{\text{th}}$ KAM cycle

$$H_{\nu+1} := H_\nu \circ \Phi^{(1)}_\nu \circ \cdots \circ \Phi^{(J_\nu)}_\nu = \mathcal{N}_{\nu+1} + P_{\nu+1}$$

is real-analytic on $\mathcal{D}_{\nu+1} = \mathcal{D}^{(J_\nu)}_\nu$ and C^1_W parametrized by $\xi \in \mathcal{O}_{\nu+1}$, with

$$\mathcal{N}_{\nu+1}(\xi) = \langle \xi, I\rangle + \langle (T + W_{\nu+1}(\xi))q, \bar{q}\rangle, \quad P_{\nu+1}(\xi) = \sum_{k\in\mathbb{Z}^b} \langle P^k_{\nu+1}(\xi) q, \bar{q}\rangle e^{\mathrm{i}\langle k,\theta\rangle}$$

satisfies (3.1) and (3.2) with new quantities, and

$$\|X_{P_{\nu+1}}\|_{\mathcal{D}_{\nu+1}, \mathcal{O}_{\nu+1}} \le \varepsilon^{(J_\nu)}_\nu \le \varepsilon_\nu^{\frac{1}{2}\varepsilon_\nu^{-a_\nu/2}} = \varepsilon_{\nu+1}.$$

3.1 Construction of $\mathcal{O}_{\nu+1}$

In view of (2.4) and (3.1), there exists a constant $c_1 > 0$ such that

$$|\,(U^*_\nu W_\nu U_\nu)_{mn}\,|_{\mathcal{O}_\nu} \le c_1\, p_\nu \sigma_\nu^{-2} \cdot e^{-\sigma_\nu \cdot \max\{|m|,|n|\}},$$

by a simple application of Lemma 4.1. Define the truncation $\hat{W}_\nu$ as

$$(\hat{W}_\nu)_{mn} := \begin{cases} (U^*_\nu W_\nu U_\nu)_{mn}\,, & |m|,|n| \le N_\nu \\ 0, & \text{otherwise} \end{cases}. \tag{3.3}$$

It follows that

$$\left|\left(U^*_\nu W_\nu U_\nu - \hat{W}_\nu\right)_{mn}\right|_{\mathcal{O}_\nu} \le \varepsilon_\nu e^{-\rho_\nu \max\{|m|,|n|\}} \tag{3.4}$$

under the assumption

(C1) $c_1\, p_\nu \sigma_\nu^{-2} \cdot e^{-(\sigma_\nu - \rho_\nu)N_\nu} \le \varepsilon_\nu$.

Let $K_{\nu+1} := N_{\nu+1} - (M_\nu + 1)N_\nu$ and

$$\Lambda^\nu := \bigcup \{\Lambda_j^\nu : \Lambda_j^\nu \cap [-(K_{\nu+1} + N_\nu),\ K_{\nu+1} + N_\nu] \neq \emptyset\} \subset [-N_{\nu+1},\ N_{\nu+1}].$$

According to Q_ν in Proposition 1, define

$$\tilde{D}^\nu_{\Lambda^\nu} := \prod_{\Lambda_j^\nu \subset \Lambda^\nu} \tilde{D}^\nu_{\Lambda_j^\nu}, \quad \tilde{W}_\nu := Q_\nu^* \hat{W}_\nu Q_\nu. \tag{3.5}$$

In view of (2.5) and (3.3), we have $(\tilde{W}_\nu)_{mn} \equiv 0$ if $|m|$ or $|n| > 2N_\nu$.

Since both of $\tilde{D}^\nu_{\Lambda^\nu}$ and $\tilde{W}_\nu$ are Hermitian, there is an orthogonal matrix O_ν such that

$$O_\nu^* (\tilde{D}^\nu_{\Lambda^\nu} + \tilde{W}_\nu) O_\nu = \mathrm{diag}\{\mu_j^\nu\}_{j \in \Lambda^\nu},$$

where $\{\mu_j^\nu\}_{j\in\Lambda^\nu}$ are eigenvalues of $\tilde{D}^\nu_{\Lambda^\nu} + \tilde{W}_\nu$. Due to the block-diagonal structure of $\tilde{D}^\nu_{\Lambda^\nu} + \tilde{W}_\nu$, we also have

$$(O_\nu)_{mn} \equiv 0 \ \text{ if } \ |m-n| > 2(M_\nu + 2)N_\nu. \tag{3.6}$$

Indeed, $\tilde{D}^\nu_{\Lambda^\nu} + \tilde{W}_\nu$ can be expressed as a product of smaller blocks

$$\tilde{D}^\nu_{\Lambda^\nu} + \tilde{W}_\nu = (\tilde{D}^\nu_{\Lambda'_\nu} + \tilde{W}_\nu) \cdot \prod_{\substack{\Lambda_j^\nu \cap [-2N_\nu, 2N_\nu] = \emptyset \\ \Lambda_j^\nu \subset \Lambda^\nu}} \tilde{D}^\nu_{\Lambda_j^\nu}$$

where $\Lambda'_\nu := \bigcup\{\Lambda_j^\nu : \Lambda_j^\nu \cap [-2N_\nu, 2N_\nu] \neq \emptyset\}$ with $\mathrm{diam}\Lambda'_\nu \leq 2(M_\nu + 2)N_\nu$. The diagonalization of $\tilde{D}^\nu_{\Lambda^\nu} + \tilde{A}_\nu$ is exactly the diagonalization of $(\tilde{D}^\nu_{\Lambda'_\nu} + \tilde{W}_\nu)$ and $\tilde{D}^\nu_{\Lambda_j^\nu}$.

As for the eigenvalues of $\tilde{D}^\nu_{\Lambda^\nu} + \tilde{W}_\nu$, it is well-known that $\{\mu_n^\nu\}_{n\in\Lambda^\nu}$ C^1_W-smoothly depend on ξ and there exist orthonormal eigenvectors ψ_n^ν corresponding to μ_n^ν, C^1_W-smoothly depending on ξ (see e.g. [11]). In fact, $\mu_n^\nu = \langle (\tilde{D}_{\Lambda^\nu} + \tilde{W}_\nu)\psi_n^\nu, \bar{\psi}_n^\nu \rangle$ and

$$\partial_{\xi_j} \mu_n^\nu = \langle (\partial_{\xi_j}(\tilde{D}_{\Lambda^\nu} + \tilde{W}_\nu))\psi_n^\nu, \bar{\psi}_n^\nu \rangle, \quad j = 1, \cdots, b.$$

By the construction of $\tilde{W}_\nu$, we have $\partial_{\xi_j} \tilde{W}_\nu = Q_\nu^* (\partial_{\xi_j} \hat{W}_\nu) Q_\nu$, with $\hat{W}_\nu$ the truncation of $U_\nu^* W_\nu(\xi) U_\nu$. Since D_ν, U_ν and Q_ν are all independent of ξ,

$$\sup_{\xi \in \mathcal{O}_\nu} |\partial_\xi \mu_n^\nu| \leq c \sup_{\substack{\xi\in\mathcal{O}_\nu \\ m,n}} |\partial_\xi (W_\nu)_{mn}| \leq c p_\nu. \tag{3.7}$$

Now we defined the new parameter set $\mathcal{O}_{\nu+1} \subset \mathcal{O}_\nu$ as

$$\mathcal{O}_{\nu+1} := \left\{ \xi \in \mathcal{O}_\nu : |\langle k, \xi\rangle + \mu_m^\nu - \mu_n^\nu| > \frac{\varepsilon_\nu^{\frac{1}{80}}}{|k|^\tau N_{\nu+1}^4}, \quad k \neq 0, \quad m, n \in \Lambda^\nu. \right\} \tag{3.8}$$

for some $\tau \geq b$. These inequalities are famous small-divisor conditions for controlling the solutions of the linearized equations. Since (3.7) implies

$$|\partial_\xi (\langle k, \xi\rangle - \mu_m^\nu + \mu_n^\nu)| \geq |k| - c p_\nu = O(|k|),$$

combing with $\tau \geq b$, we can get

$$\mathrm{Leb}(\mathcal{O}_\nu \setminus \mathcal{O}_{\nu+1}) \leq c \sum_{k\neq 0} \frac{\varepsilon_\nu^{\frac{1}{80}}}{|k|^{\tau+1}} \sim \varepsilon_\nu^{\frac{1}{80}}. \tag{3.9}$$

3.2 Homological equation and its approximate solution

From now on, to simplify notations, the subscripts (or superscripts) "ν" of quantities at the ν^{th} step are neglected, and the corresponding quantities at the $(\nu+1)^{\text{th}}$ step are labeled with "+". In addition, we still use the superscript (j) to distinguish quantities at various sub-steps.

With $\mathcal{O}_+$ defined as in (3.8), we have

Proposition 2 *There exist two real-analytic Hamiltonians*

$$F = \sum_{k\in\mathbb{Z}^b\setminus\{0\}} \langle F^k q,\, \bar{q}\rangle e^{\mathrm{i}\langle k,\theta\rangle}, \quad \grave{P} = \sum_{k\in\mathbb{Z}^b} \langle \grave{P}^k q,\, \bar{q}\rangle e^{\mathrm{i}\langle k,\theta\rangle},$$

and a Hermitian matrix W', all of which are C^1_W-parametrized by $\xi\in\mathcal{O}_+$, such that

$$\{\mathcal{N}, F\} + P = \langle W' q,\, \bar{q}\rangle + \grave{P}. \tag{3.10}$$

Moreover, for ε sufficiently small,

$$|F^k_{mn}|_{\mathcal{O}_+} \le \varepsilon^{\frac{4}{5}} |k|^{2\tau+1} e^{-|k|r} e^{-\rho\max\{|m|,\,|n|\}}, \tag{3.11}$$

$$|\grave{P}^k_{mn}|_{\mathcal{O}_+} \le \varepsilon^{\frac{7}{5}} |k|^{2\tau+1} e^{-|k|r} e^{-\rho^{(1)}\max\{|m|,\,|n|\}}, \tag{3.12}$$

$$|W'_{mn}|_{\mathcal{O}_+} \le \left\{ \begin{array}{cl} \varepsilon e^{-\rho\max\{|m|,\,|n|\}}, & |m|, |n| \le N_+ \\ 0, & \textit{otherwise} \end{array} \right. . \tag{3.13}$$

Proof of Proposition 2: We accomplish the proof with the following procedures.

- **Approximate linearized equations**

First of all, we try to construct a Hamiltonian $F = \sum_{k\in\mathbb{Z}^b\setminus\{0\}} \langle F^k q, \bar{q}\rangle e^{\mathrm{i}\langle k,\theta\rangle}$, such that

$$\{\mathcal{N}, F\} + P = \langle P^0 q, \bar{q}\rangle. \tag{3.14}$$

By a straightforward calculation and simple comparison of coefficients, the equation (3.14) is equivalent to

$$(\langle k,\xi\rangle I_{\mathbb{Z}} - (T+W))F^k + F^k(T+W) = \mathrm{i}P^k, \quad \forall k\ne 0. \tag{3.15}$$

We can instead consider the equation

$$\left(\langle k,\xi\rangle I_{\mathbb{Z}} - (D+\hat{W})\right)\hat{F}^k + \hat{F}^k(D+\hat{W}) = \mathrm{i}\hat{P}^k, \tag{3.16}$$

where D and $\hat{W}$ are defined in the previous subsection, and for $k\ne 0$,

$$\hat{P}^k_{mn} = \left\{ \begin{array}{cl} (U^* P^k U)_{mn}, & |m|, |n| \le K_+ \\ 0, & \text{otherwise} \end{array} \right. . \tag{3.17}$$

By (3.2) and (2.4), combining with Lemma 4.1, there exists $c_2>0$ such that

$$|(U^* P^k U)_{mn}|_{\mathcal{O}} \le c_2(\sigma-\rho)^{-2}\varepsilon e^{-\rho\max\{|m|,\,|n|\}} e^{-|k|r}. \tag{3.18}$$

This means

$$|(U^* P^k U - \hat{P}^k)_{mn}|_{\mathcal{O}} \le \frac{1}{4}\varepsilon^{\frac{7}{5}} e^{-\rho^{(1)}\max\{|m|,\,|n|\}} e^{-|k|r} \tag{3.19}$$

under the assumption that

(C2) $c_2(\sigma-\rho)^{-4}e^{-(\rho-\rho^{(1)})K_+} \leq \frac{1}{4}\varepsilon^{\frac{2}{5}}$.

Equation (3.16) provides us with an approximate solution to (3.15), with the error estimated later.

Consider the equation

$$\left(\langle k,\xi\rangle I_\Lambda - (\tilde{D}_\Lambda+\tilde{W})\right)\tilde{F}^k + \tilde{F}^k(\tilde{D}_\Lambda+\tilde{W}) = \mathrm{i}\tilde{P}^k, \tag{3.20}$$

where $\tilde{D}_\Lambda$, $\tilde{W}$ are defined as in (3.5) via the orthogonal matrix Q, and $\tilde{P}^k := Q^*\hat{P}^kQ$. Note that $Q_{mn}=0$ if $|m-n|>N$, then by (3.17), we have

$$\tilde{P}^k_{mn}\equiv 0, \ \text{ if } \ |m| \ \text{ or } \ |n| > K_+ + N.$$

Thus, recalling that $\Lambda := \bigcup\{\Lambda_j : \Lambda_j\cap[-(K_++N), \ K_++N]\neq\emptyset\}$, solutions of these finite-dimensional equations satisfy

$$\tilde{F}^k_{mn}\equiv 0, \ \text{ if } \ m \ \text{ or } \ n\notin\Lambda.$$

Then, in view of the facts

$$\left(\langle k,\xi\rangle I_{\mathbb{Z}} \pm (\tilde{D}+\tilde{W})\right)\tilde{F}^k = \left(\langle k,\xi\rangle I_\Lambda \pm (\tilde{D}_\Lambda+\tilde{W})\right)\tilde{F}^k, \quad \tilde{F}^k(\tilde{D}+\tilde{W}) = \tilde{F}^k(\tilde{D}_\Lambda+\tilde{W}),$$

they are also solutions of

$$\left(\langle k,\xi\rangle I_{\mathbb{Z}} - (\tilde{D}+\tilde{W})\right)\tilde{F}^k + \tilde{F}^k(\tilde{D}+\tilde{A}) = \mathrm{i}\tilde{P}^k,$$

which is equivalent to Equation (3.16) since D can be block-diagonalized by the orthogonal matrix Q.

Finally, we can focus on the equation

$$(\langle k,\xi\rangle - \mu_m + \mu_n)\check{F}^k_{mn} = \mathrm{i}(O^*\tilde{P}^kO)_{mn},$$

for $k\neq 0$ and $m,n\in\Lambda$, which is transformed from (3.20) by diagonalizing $\tilde{D}_\Lambda+\tilde{W}$ via the orthogonal matrix O. Obviously, these equations can be solved in $\mathcal{O}_+$ defined in (3.8). Hence, (3.16) is solved with $\hat{F}^k = QO\check{F}^kO^*Q^*$.

Let $F^k := U\hat{F}^kU^*$, then we obtain a Hamiltonian $F = \sum_{k\neq 0}\langle F^kq,\bar{q}\rangle e^{\mathrm{i}\langle k,\theta\rangle}$. It is easy to see that $\bar{F} = F$, by noting $\left(F^{(-k)}\right)^* = F^k$.

- **Estimates for coefficients of F**

By the construction above, one sees that

$$F^k_{mn} = \mathrm{i}\sum_{\mathcal{F}_0}\frac{U_{mn_1}Q_{n_1n_2}O_{n_2n_3}O^*_{n_3n_4}Q^*_{n_4n_5}\hat{P}^k_{n_5n_6}Q_{n_6n_7}O_{n_7n_8}O^*_{n_8n_9}Q^*_{n_9n_{10}}U^*_{n_{10}n}}{\langle k,\xi\rangle - \mu_{n_3}+\mu_{n_8}},$$

where the summation notation $\mathcal{F}_0$ denotes

$$\left\{\begin{array}{l} n_1\in\mathbb{Z}, \ \ |n_2-n_1|\leq N, \ \ |n_3-n_2|, |n_4-n_3|\leq 2(M+2)N, \ \ |n_5-n_4|\leq N, \\ n_{10}\in\mathbb{Z}, \ \ |n_9-n_{10}|\leq N, \ \ |n_8-n_9|, |n_7-n_8|\leq 2(M+2)N, \ \ |n_6-n_7|\leq N \end{array}\right\}$$

by virtue of the structure of Q and O, i.e, (2.5) and (3.6). Then, by (3.18) and Lemma 4.1,

$$\sup_{\xi\in\mathcal{O}_+}|F^k_{mn}(\xi)|\le c(\varepsilon^{-\frac{1}{80}}|k|^\tau N_+^4)(\sigma-\rho)^{-4}M^4N^8e^{(4M+10)N\rho}\varepsilon e^{-|k|r}e^{-\rho\max\{|m|,|n|\}}.$$

Here we have applied the property of the orthogonal matrices Q and O, and used the factor $e^{(4M+10)N\rho}$ to recover the exponential decay.

To estimate $|\partial_{\xi_j}F^k_{mn}|$, we need to differentiate both sides of (3.20) with respect to ξ_j, $j=1,2,\cdots,b$. Then we obtain the equation about $\partial_{\xi_j}\tilde{F}^k$

$$(\langle k,\xi\rangle I_\Lambda-(\tilde{D}_\Lambda+\tilde{W}))(\partial_{\xi_j}\tilde{F}^k)-(\partial_{\xi_j}\tilde{F}^k)(\tilde{D}_\Lambda+\tilde{W})=\breve{P}^k_{\xi_j},$$

which can also be solved by diagonalizing $\tilde{D}_\Lambda+\tilde{A}$ via O as above, where

$$\breve{P}^k_{\xi_j}:=\mathrm{i}\partial_{\xi_j}\tilde{R}^k+\tilde{F}^k(\partial_{\xi_j}\tilde{W})-(\partial_{\xi_j}(\langle k,\xi\rangle I-\tilde{W}))\tilde{F}^k.$$

We get the formulation

$$\partial_{\xi_j}F^k_{mn}=\sum_{\mathcal{F}_1}\frac{U_{mn_1}Q_{n_1n_2}O_{n_2n_3}O^*_{n_3n_4}(\breve{P}^k_{\xi_j})_{n_4n_5}O_{n_5n_6}O^*_{n_6n_7}Q^*_{n_7n_8}U^*_{n_8n}}{\langle k,\xi\rangle-\mu_{n_3}+\mu_{n_6}},$$

with $\mathcal{F}_1$ denotes

$$\left\{\begin{array}{ll} n_1\in\mathbb{Z}, & |n_2-n_1|\le N,\quad |n_3-n_2|,\,|n_4-n_3|\le 2(M+2)N,\\ n_8\in\mathbb{Z}, & |n_7-n_8|\le N,\quad |n_6-n_7|,\,|n_5-n_6|\le 2(M+2)N\end{array}\right\}.$$

By the decay property of $\hat{R}^k$ and $\partial_{\xi_j}\hat{A}$, we have that

$$\sup_{\xi\in\mathcal{O}_+}|(\breve{P}^k_{\xi_j})_{mn}|\le c(\varepsilon^{-\frac{1}{80}}|k|^{\tau+1}N_+^4)(\sigma-\rho)^{-4}M^4N^8e^{(4M+11)N\rho}\varepsilon e^{-|k|r}e^{-\rho\max\{|m|,|n|\}}.$$

Thus there exists $c_3>0$ such that

$$\begin{aligned}&\sup_{\xi\in\mathcal{O}_+}(|F^k_{mn}|+|\partial_\xi F^k_{mn}|)\\ \le\ & c_3(\varepsilon^{-\frac{1}{40}}|k|^{2\tau+1}N_+^8)(\sigma-\rho)^{-6}M^8N^{14}e^{(8M+20)N\rho}\varepsilon e^{-\rho\max\{|m|,|n|\}}e^{-|k|r}\\ \le\ & \varepsilon^{\frac{4}{5}}|k|^{2\tau+1}e^{-|k|r}e^{-\rho\max\{|m|,|n|\}},\end{aligned}$$

under the assumption

(C3) $c_3(\sigma-\rho)^{-6}N_+^8M^8N^{14}e^{(8M+20)N\rho}\varepsilon^{\frac{7}{40}}\le 1.$

- **Estimate for the error**

Let W' be the truncation of P^0, satisfying

$$W'_{mn}=\left\{\begin{array}{ll}P^0_{mn}, & |m|,|n|\le N_+\\ 0, & \text{otherwise}\end{array}\right.,$$

and with $\grave{W} := W - U\hat{W}U^*$, $\grave{Z} := UZU^*$, let $\grave{P} = \sum_{k\in\mathbb{Z}^b} \langle \grave{P}^k q, \bar{q}\rangle e^{\mathrm{i}\langle k,\theta\rangle}$, satisfying

$$\grave{P}^0 := P^0 - W', \quad \grave{P}^k := (P^k - U\hat{P}^k U^*) - \mathrm{i}(\grave{W} + \grave{Z})F^k + \mathrm{i}F^k(\grave{W} + \grave{Z}),$$

Then we obtain The equation (3.10).

By (3.2), we have (3.13) holds and

$$|\grave{P}^0_{mn}|_{\mathcal{O}_+} \le \varepsilon e^{-\rho\max\{|m|,\,|n|\}} \le \varepsilon^{\frac{7}{5}} e^{-\rho^{(1)}\max\{|m|,\,|n|\}}, \tag{3.21}$$

under the assumption

(C4) $e^{-(\rho-\rho^{(1)})N_+} \le \varepsilon^{\frac{2}{5}}$.

As for the case $k \neq 0$ in (3.12), by (3.19) and **(C2)**, combining with Lemma 4.1,

$$\Big|\Big(P^k - U\hat{P}^k U^*\Big)_{mn}\Big|_{\mathcal{O}} = \Big|\Big(U(U^*P^kU - \hat{P}^k)U^*\Big)_{mn}\Big|_{\mathcal{O}} \le \frac{1}{4}\varepsilon^{\frac{7}{5}} e^{-\rho^{(1)}\max\{|m|,\,|n|\}} e^{-|k|r}. \tag{3.22}$$

In view of (2.6) and (3.4),

$$|\grave{W}_{mn}|_{\mathcal{O}} \le c(\sigma-\rho)^{-2}\varepsilon e^{-\rho\max\{|m|,\,|n|\}}, \quad |\grave{Z}_{mn}| \le c(\sigma-\rho)^{-2}\varepsilon e^{-\rho|m-n|}.$$

Then, by applying Lemma 4.1 again, there exists $c_4 > 0$ such that

$$\begin{aligned}
&\Big|\Big(F^k(\grave{W} + \grave{Z})\Big)_{mn}\Big|_{\mathcal{O}_+}, \quad \Big|\Big((\grave{W} + \grave{Z})F^k\Big)_{mn}\Big|_{\mathcal{O}_+} \\
\le\ & c_4(\sigma-\rho)^{-2}(\rho-\rho^{(1)})^{-1}\varepsilon^{\frac{9}{5}}|k|^{2\tau+1}e^{-|k|r}e^{-\rho^{(1)}\max\{|m|,\,|n|\}} \\
\le\ & \frac{1}{4}\varepsilon^{\frac{7}{5}}|k|^{2\tau+1}e^{-|k|r}e^{-\rho^{(1)}\max\{|m|,\,|n|\}},
\end{aligned} \tag{3.23}$$

provided that

(C5) $c_4(\sigma-\rho)^{-2}(\rho-\rho^{(1)})^{-1}\varepsilon^{\frac{2}{5}} \le \frac{1}{4}$.

Thus, we can obtain the estimate for $\grave{P}^k$ by putting (3.21) – (3.23) together. ■

Let $\mathcal{D}_i := \mathcal{D}_{d,\rho_+}(r^{(1)} + \frac{i}{4}(r - r^{(1)}),\ \frac{i}{4}s)$, $i = 1, 2, 3, 4$. Lemma 4.1 in [18] shows

Lemma 3.1 *There is a constant $c_5 > 0$ such that*

$$\begin{aligned}
\|X_F\|_{\mathcal{D}_3,\,\mathcal{O}_+} &\le c_5(r - r^{(1)})^{-(2\tau+b+1)}(\rho-\rho_+)^{-2}\varepsilon^{\frac{4}{5}}, \\
\|X_{\grave{P}}\|_{\mathcal{D}_3,\,\mathcal{O}_+} &\le c_5(r - r^{(1)})^{-(2\tau+b+1)}(\rho^{(1)}-\rho_+)^{-2}\varepsilon^{\frac{7}{5}}.
\end{aligned}$$

Moreover, if

(C6) $c_5(r - r^{(1)})^{-(2\tau+b+1)}(\rho^{(1)}-\rho_+)^{-2}\varepsilon^{\frac{1}{20}} \le \frac{1}{3}$,

we have $\|X_F\|_{\mathcal{D}_3,\,\mathcal{O}_+} \le \varepsilon^{\frac{3}{4}}$ and $\|X_{\grave{P}}\|_{\mathcal{D}_3,\,\mathcal{O}_+} \le \varepsilon^{\frac{5}{4}}$.

For the Hamiltonian flow Φ_F^t associated with F, we have

Corollary 1 *For ε sufficiently small, we have $\Phi_F^t : \mathcal{D}_2 \to \mathcal{D}_3$, $-1 \le t \le 1$ and moreover,*

$$\|D\Phi_F^t - Id\|_{\mathcal{D}_1} < 2\varepsilon^{\frac{3}{4}}.$$

Let $F^{(1)}$, $W^{(1)}$, $\grave{P}^{(1)}$ be the corresponding quantities in the homological equation (3.10) respectively, which means that we are in the 1^{st} sub-step. Then

$$H^{(1)} := H \circ \Phi^1_{F^{(1)}} = (\mathcal{N} + P) \circ \Phi^1_{F^{(1)}} = \mathcal{N} + \langle W^{(1)}q, \bar{q}\rangle + P^{(1)},$$

with $P^{(1)}$ the same as in some standard arguments(e.g., [18, 19]), and, by Lemma 3.1 and Corollary 1,

$$\|X_{P^{(1)}}\|_{\mathcal{D}^{(1)},\mathcal{O}_+} \le \varepsilon^{\frac{6}{5}} = \varepsilon^{(1)}.$$

As for the decay estimate of $P^{(1)}$, note that

$$\begin{aligned} P^{(1)} \;&=\; \grave{P}^{(1)} + \{P, F^{(1)}\} + \frac{1}{2!}\{\{\mathcal{N}, F^{(1)}\}, F^{(1)}\} + \frac{1}{2!}\{\{P, F^{(1)}\}, F^{(1)}\} + \cdots \\ &\quad + \frac{1}{n!}\{\cdots\{\mathcal{N}, \underbrace{F^{(1)}\}\cdots, F^{(1)}}_{n}\} + \frac{1}{n!}\{\cdots\{P, \underbrace{F^{(1)}\}\cdots, F^{(1)}}_{n}\} + \cdots. \end{aligned}$$

Applying Lemma 4.2 to $\{P, F^{(1)}\} = \sum_{k\in\mathbb{Z}^b} \langle \{P, F^{(1)}\}^k q,\, \bar{q}\rangle e^{\mathrm{i}\langle k,\theta\rangle}$, we can find a constant $c_6 > 0$ such that

$$|\{P, F^{(1)}\}^k_{mn}|_{\mathcal{O}_+} \le c_6\,(r - r^{(1)})^{-(2\tau+b+1)}(\rho - \rho^{(1)})^{-1}\varepsilon^{\frac{9}{5}} e^{-r^{(1)}|k|} e^{-\rho^{(1)}\max\{|m|,|n|\}}$$

according to (3.2) and (3.11). Since $\{\mathcal{N}, F^{(1)}\} = -P + \langle W^{(1)}q, \bar{q}\rangle + \grave{P}^{(1)}$, with $W^{(1)}$ and $\grave{P}^{(1)}$ estimated in Proposition 2, we can apply Lemma 4.2 iteratively, and get

Proposition 3 $P^{(1)} = \sum_{k\in\mathbb{Z}^b} \langle P^{(1)k}(\xi)q,\, \bar{q}\rangle e^{\mathrm{i}\langle k,\theta\rangle}$ *satisfies*

$$|P^{(1)k}_{mn}|_{\mathcal{O}_+} \le \varepsilon^{(1)} e^{-r^{(1)}|k|} e^{-\rho^{(1)}\max\{|m|,|n|\}}$$

under the assumption that

(C7) $c_6\,(r - r^{(1)})^{-(2\tau+b+1)}(\rho - \rho^{(1)})^{-1}\varepsilon^{\frac{1}{5}} \le 1,$

The process above is a sub-step of KAM iteration.

3.3 A succession of symplectic transformations

Suppose that we have arrived at the j^{th} sub-step, $j = 1, \cdots, J$, with $J = \left[\frac{5}{2}\varepsilon^{\frac{a}{2}}\right]$. We encounter the Hamiltonian

$$H^{(j-1)} = H \circ \Phi^1_{F^{(1)}} \circ \cdots \circ \Phi^1_{F^{(j-1)}} = \mathcal{N} + \sum_{i=1}^{j-1} \langle W^{(i)}q, \bar{q}\rangle + P^{(j-1)},$$

with the superscript "(0)" labeling quantities before the 1^{st} sub-step in particular. As demonstrated in Proposition 2, on $\mathcal{O}_+$, the following homological equation

$$\{\mathcal{N}, F^{(j)}\} + P^{(j-1)} = \langle W^{(j)}q, \bar{q}\rangle + \grave{P}^{(j)}, \tag{3.24}$$

can be solved, with $F^{(j)}$, $W^{(j)}$, $\grave{P}^{(j)}$ having properties similar to $F^{(1)}$, $W^{(1)}$, $\grave{P}^{(1)}$ respectively. Then we obtain

$$H^{(j)} = H^{(j-1)} \circ \Phi^1_{F^{(j)}} = \mathcal{N} + \sum_{i=1}^{j} \langle W^{(i)} q, \bar{q} \rangle + P^{(j)}.$$

It can be summarized that

Proposition 4 *Consider the Hamiltonian $H = \mathcal{N} + P$. There exist J symplectic transformations $\Phi^{(j)} = \Phi^1_{F^{(j)}}$, $j = 1, \cdots, J$, generated by the corresponding real-analytic Hamiltonians $F^{(j)}$ respectively, such that*

$$H^{(j)} = H \circ \Phi^{(1)} \circ \cdots \circ \Phi^{(j)} = \mathcal{N} + \sum_{i=1}^{j} \langle W^{(i)} q, \bar{q} \rangle + P^{(j)}.$$

(a) $F^{(j)} = \sum_{k \in \mathbb{Z}^b \setminus \{0\}} \langle F^{(j)k}(\xi) q,\, \bar{q} \rangle e^{\mathrm{i}\langle k, \theta \rangle}$ *satisfies the equation (3.24) on $\mathcal{O}_+$, and*

$$|F^{(j)k}_{mn}|_{\mathcal{O}_+} \leq \varepsilon^{-\frac{1}{4}} \varepsilon^{(j-1)} e^{-r^{(j-1)}|k|} e^{-\rho^{(j-1)} \max\{|m|,\,|n|\}}.$$

(b) $W^{(j)}$ *satisfies that*

$$|W^{(j)}_{mn}|_{\mathcal{O}_+} \leq \left\{ \begin{array}{cc} \varepsilon^{(j-1)} e^{-\rho^{(j-1)} \max\{|m|,\,|n|\}}, & |m|, |n| \leq N_+ \\ 0, & \textit{otherwise} \end{array} \right. . \tag{3.25}$$

(c) $P^{(j)} = \sum_{k \in \mathbb{Z}^b} \langle P^{(j)k}(\xi) q,\, \bar{q} \rangle e^{\mathrm{i}\langle k, \theta \rangle}$ *satisfies*

$$|P^{(j)k}_{mn}|_{\mathcal{O}_+} \leq \varepsilon^{(j)} e^{-r^{(j)}|k|} e^{-\rho^{(j)} \max\{|m|,\,|n|\}}, \tag{3.26}$$

and $\|X_{P^{(j)}}\|_{\mathcal{D}^{(j)}, \mathcal{O}_+} \leq \varepsilon^{(j)}$.

Let $\Phi = \Phi^{(1)} \circ \cdots \circ \Phi^{(J)}$, then $\Phi : \mathcal{D}_+ \times \mathcal{O}_+ \to \mathcal{D} \times \mathcal{O}$. Let $\mathcal{N}_+ = \langle \xi, I \rangle + \langle (T + W_+) q, \bar{q} \rangle$, with $W_+ = W + \sum_{j=1}^{J} W^{(j)}$, and $P_+ = P^{(J)}$. From (3.25) and (3.26), we have

$$\begin{array}{rcl} |(W_+)_{mn}|_{\mathcal{O}_+} & \leq & \left\{ \begin{array}{cc} p_+ e^{-\sigma_+ \max\{|m|,\,|n|\}}, & |m|, |n| \leq N_+ \\ 0, & \text{otherwise} \end{array} \right. , \\ |(P^k_+)_{mn}|_{\mathcal{O}_+} & \leq & \varepsilon_+ e^{-r_+|k|} e^{-\rho_+ \max\{|m|,\,|n|\}}, \quad \forall k \in \mathbb{Z}^b. \end{array}$$

with $p_+ = p + \varepsilon^{\frac{1}{2}}$, $\sigma_+ = \frac{1}{3}\rho$. Till now, one step of KAM iterations has been completed, and the next cycle can be started for H_+.

3.4 Iteration lemma and convergence

Besides the sequences defined in (2.3), we define, with $p_0 = \varepsilon_0^{\frac{1}{2}}$,

$$p_{\nu+1} = p_\nu + \varepsilon_\nu^{\frac{1}{2}}, \quad r_\nu = r_0 \left(1 - \sum_{i=2}^{\nu+1} 2^{-i}\right).$$

Now, we have defined all sequences appearing in the KAM cycles. In the previous work, the assumptions (**C1**) – (**C7**) have been verified for these sequences(refer to the proof of Lemma 5.1 in [18]). So the preceding analysis can be summarized as follows.

Lemma 3.2 *There exists ε_0 sufficiently small such that the following holds for all $\nu \in \mathbb{N}$.*

(a) *$H_\nu = \mathcal{N}_\nu + P_\nu$ is real-analytic on $\mathcal{D}_\nu$, and C^1_W-parametrized by $\xi \in \mathcal{O}_\nu$, where*

$$\mathcal{N}_\nu = \langle \xi, I\rangle + \langle (T + W_\nu(\xi))q, \bar{q}\rangle, \quad P_\nu = \sum_{k\in\mathbb{Z}^b} \langle P_\nu^k(\xi)q, \bar{q}\rangle e^{\mathrm{i}\langle k,\theta\rangle},$$

satisfying $\|X_{P_\nu}\|_{\mathcal{D}_\nu, \mathcal{O}_\nu} \le \varepsilon_\nu$, and

$$|(W_\nu)_{mn}|_{\mathcal{O}_\nu} \le \begin{cases} p_\nu e^{-\sigma_\nu \max\{|m|,|n|\}}, & |m|,|n| \le N_\nu \\ 0, & \textit{otherwise} \end{cases},$$

$$|(P_\nu^k)_{mn}|_{\mathcal{O}_\nu} \le \varepsilon_\nu e^{-r_\nu|k|} e^{-\rho_\nu \max\{|m|,|n|\}}, \quad \forall k \in \mathbb{Z}^b.$$

(b) *For each ν, there is a symplectic transformation $\Phi_\nu : \mathcal{D}_{\nu+1} \to \mathcal{D}_\nu$ with*

$$\|D\Phi_\nu - Id\|_{\mathcal{D}_{\nu+1}, \mathcal{O}_{\nu+1}} \le \varepsilon_\nu^{\frac{1}{2}},$$

such that $H_{\nu+1} = H_\nu \circ \Phi_\nu$.

Let $\mathcal{O}_{\varepsilon_0} := \cap_{\nu=0}^\infty \mathcal{O}_\nu$. In view of (3.9), it is clear that $\mathrm{Leb}(\mathcal{O}_0 \setminus \mathcal{O}_{\varepsilon_0}) \sim \varepsilon_0^{\frac{1}{80}}$.

Fix $x \in \mathcal{X}$, with $\mathcal{X}$ defined as in Proposition 1. This means that, for each $n \in \mathbb{Z}$, there is a $\nu_0(n)$ such that $\Lambda^{\nu+1}(n) = \Lambda^\nu(n)$, $\forall \nu \ge \nu_0(n)$. In this case, the local decay rate for n will not shrink with ν necessarily(ρ_ν is the global upper bound of the rates for all $n \in \mathbb{Z}$).

Define $\Psi^\nu = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{\nu-1}$, $\nu \in \mathbb{N}$. An induction argument shows that $\Psi^\nu : \mathcal{D}_{\nu+1} \to \mathcal{D}_0$, and $H_0 \circ \Psi^\nu = H_\nu = \mathcal{N}_\nu + P_\nu$. As in standard arguments (e.g. [24, 25]), thanks to Corollary 1, it concludes that H_ν, $\mathcal{N}_\nu$, P_ν, Ψ^ν and W_ν converge uniformly on $\mathcal{D}_{d,0}(\frac{1}{2}r_0, s) \times \mathcal{O}_{\varepsilon_0}$ to, say, H_∞, $\mathcal{N}_\infty$, P_∞, Ψ^∞ and W_∞ respectively, in which case it is clear that $\mathcal{N}_\infty = \langle \xi, I\rangle + \langle (T + W_\infty)q, \bar{q}\rangle$. Since $\|X_{P_\nu}\|_{\mathcal{D}_\nu,\mathcal{O}_\nu} \le \varepsilon_\nu$ with $\varepsilon_\nu \to 0$, it follows that $\|X_{P_\infty}\|_{\mathcal{D}_{d,0}(\frac{1}{2}r_0,0), \mathcal{O}_{\varepsilon_0}} = 0$.

4 Appendix: Decay Property of Matrices and Hamiltonians

Lemma 4.1 (Lemma 2.1 in [18]) *Given two matrices $F = (F_{mn})_{m,n\in\mathbb{Z}}$ and $G = (G_{mn})_{m,n\in\mathbb{Z}}$. Let $K = FG$.*

(1) *If* $|F_{mn}| \le c_F e^{-\rho_F|m-n|}$, $|G_{mn}| \le c_G e^{-\rho_G|m-n|}$ *for some* $c_F,\, c_G,\, \rho_F,\, \rho_G > 0$, *then we have*

$$|K_{mn}| \le c_K e^{-\rho_K|m-n|}$$

for any $0 < \rho_K < \min\{\rho_F,\, \rho_G\}$, *with* $c_K = c \cdot c_F c_G (\min\{\rho_F,\, \rho_G\} - \rho_K)^{-1}$.

(2) *In the cases that:*

- $|F_{mn}| \le c_F e^{-\rho_F \max\{|m|,\,|n|\}}$, $|G_{mn}| \le c_G e^{-\rho_G|m-n|}$,
- $|F_{mn}| \le c_F e^{-\rho_F|m-n|}$, $|G_{mn}| \le c_G e^{-\rho_G \max\{|m|,\,|n|\}}$,
- $|F_{mn}| \le c_F e^{-\rho_F \max\{|m|,\,|n|\}}$, $|G_{mn}| \le c_G e^{-\rho_G \max\{|m|,\,|n|\}}$,

we have $|K_{mn}| \le c_K e^{-\rho_K \max\{|m|,\,|n|\}}$.

Moreover, if $\rho_F \ne \rho_G$, *the conclusions above hold with*

$$\rho_K = \min\{\rho_F,\, \rho_G\}, \quad c_K = c \cdot c_F c_G |\rho_F - \rho_G|^{-1}.$$

It can be adapted to the Hamiltonian:

Lemma 4.2 *Given two Hamiltonians*

$$F = \sum_{k\in\mathbb{Z}^b} \langle F^k(\xi)q, \bar{q}\rangle e^{\mathrm{i}\langle k,\theta\rangle}, \quad G = \sum_{k\in\mathbb{Z}^b} \langle G^k(\xi)q, \bar{q}\rangle e^{\mathrm{i}\langle k,\theta\rangle},$$

C^1_W *parametrized by* $\xi \in \mathcal{O}$, *satisfying*

$$|F^k_{mn}|_{\mathcal{O}} \le c_F |k|^d e^{-r_F|k|} e^{-\rho_F \max\{|m|,\,|n|\}}, \quad |G^k_{mn}|_{\mathcal{O}} \le c_G |k|^d e^{-r_G|k|} e^{-\rho_G \max\{|m|,\,|n|\}}$$

for some $d,\, c_F,\, c_G,\, \rho_F,\, \rho_G,\, r_F,\, r_G > 0$. *Then for* $K = \{F, G\} = \sum_{k\in\mathbb{Z}^b} \langle K^k(\xi)q, \bar{q}\rangle e^{\mathrm{i}\langle k,\theta\rangle}$, *we have*

$$|K^k_{mn}|_{\mathcal{O}} \le c_K e^{-\rho_K \max\{|m|,\,|n|\}} e^{-r_K|k|}$$

for any $0 < \rho_K < \min\{\rho_F,\, \rho_G\}$, $0 < r_K < \min\{r_F,\, r_G\}$, *with*

$$c_K = c \cdot c_F c_G (\min\{\rho_F,\, \rho_G\} - \rho_K)^{-1} (\min\{r_F,\, r_G\} - r_K)^{-(b+d)}.$$

Proof: The matrix element of K^k can be formulated as

$$K^k_{mn} = \sum_{\substack{k_1, k_2 \in \mathbb{Z}^b \\ k_1 + k_2 = k}} \left(F^{k_1} G^{k_2} - G^{k_1} F^{k_2}\right)_{mn}.$$

By Lemma 4.1, we know that $\left|\left(F^{k_1} G^{k_2} - G^{k_1} F^{k_2}\right)_{mn}\right|_{\mathcal{O}}$ is bounded by

$$c \cdot c_F c_G (\min\{\rho_F,\, \rho_G\} - \rho_K)^{-1} |k_1|^d |k_2|^d \left(e^{-r_F|k_1|} e^{-r_G|k_2|} + e^{-r_G|k_1|} e^{-r_F|k_2|}\right) e^{-\rho_K \max\{|m|,\,|n|\}},$$

with the part $e^{-r_F|k_1|} e^{-r_G|k_2|} + e^{-r_G|k_1|} e^{-r_F|k_2|}$ smaller than

$$e^{-(\min\{r_F, r_G\} - r_K)\cdot(|k_1| + |k_2|)} e^{-r_K(|k_1| + |k_2|)}.$$

After the summation on k_1 and k_2, the proof is finished. ■

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