Linearly Stable Quasi-periodic Breathers in a Class of Random Hamiltonian Systems*

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Abstract

In this paper, we construct linearly stable quasi-periodic breathers for the Hamiltonian systems in the form

$$\mathrm{i}\dot{q}_n + v_n q_n + \delta |q_n|^2 q_n + \varepsilon_n (q_{n+1} + q_{n-1}) = 0, \quad n \in \mathbb{Z}$$

where $\{v_n\}_{n\in\mathbb{Z}}$ is a family of time independent independent identically distributed (i.i.d) random variables with common distribution $g = dv_n$, $v_n \in [0,1]$ and $|\varepsilon_n| \leq \varepsilon e^{-\varrho|n|}$ with $\varepsilon, \varrho > 0$. We prove that for ε, δ sufficiently small, the equation admits a family of small-amplitude and linear stable, time quasi-periodic solutions for most of the parameters $\{v_n\}_{n\in\mathbb{Z}}$.

1 Introduction and main result

During the past two decades or so, there have been many remarkable results in KAM (Kolmogorov–Arnold–Moser) theory of Hamiltonian partial differential equations achieved either by methods from the finite dimensional KAM theory[4, 13, 16, 17, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 39, 40, 41, 42, 43], or by a Newtonian scheme developed by Craig,Wayne and Bourgain [5, 6, 7, 8, 9, 10, 14], motivated by the construction of quasi-periodic breathers(solutions that are quasi-periodic in time and exponentially localized in space) in infinite dimensional Hamiltonian systems.

In this paper, we seek time quasi-periodic solutions to the non-linear random lattice equation

$$\mathrm{i}\dot{q}_n + v_n q_n + \delta |q_n|^2 q_n + \varepsilon_n (q_{n+1} + q_{n-1}) = 0 \tag{1.1}$$

on $\mathbb{Z} \times [0, \infty)$, where $|\varepsilon_n| \leq \varepsilon e^{-\varrho|n|}$ with $\varepsilon, \varrho > 0$, ε, δ are sufficiently small, and $\{v_n\}_{n\in\mathbb{Z}}$ is a family of time independent independent identically distributed (i.i.d) random variables with common distribution $g(v_n) = dv_n$, $v_n \in [0, 1]$. The probability space is taken to be $[0, 1]^{\mathbb{Z}}$ with measure

$$\prod_{n\in\mathbb{Z}} g(v_n) = \prod_{n\in\mathbb{Z}} dv_n, \quad v_n \in [0, 1]. \tag{1.2}$$

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$V = \{v_n\}_{n\in\mathbb{Z}}$ serves as parameters for the nonlinear equation (1.1).

In view of the previous papers, there are many results related to infinite dimensional Hamiltonian systems. The linear random Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}q = (\varepsilon\Delta + V)q =: Hq \tag{1.3}$$

on $Z^d \times [0, \infty)$ has been studied for several decades, where Δ is the discrete Laplacian:

$$\Delta_{ij} = \begin{cases} 1, & |i-j|_{\ell^1} = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and $V = \{v_j\}_{j\in\mathbb{Z}^d}$, the potential, is a family of time independent i.i.d. bounded random variables. It is well known from the works in [2, 3, 15, 18, 19, 20, 21, 22] etc. that (1.3) has Anderson Localization(A.L.) after the physicist P.Anderson [1], i.e. if $q(0) \in \ell^2(\mathbb{Z}^d)$, for any $\kappa > 0$, one can find R such that

$$\|q(t)\|_{\ell^2(\{\mathbb{Z}\setminus[-R,R]\}^d)} < \kappa, \quad \forall t. \tag{1.4}$$

Since the potential is time independent: $V(j,t) = V(j)$, properties of time evolution can be deduced from the spectral properties of H. Let $\sigma(H)$ be the spectrum of H, which is defined in (1.1), then

$$\sigma(H) = [-2\varepsilon d, 2\varepsilon d] + \text{supp } g, \quad a.s.$$

(the probability can be defined in (1.2) or in more general forms see [12, 38]). If $0 < \varepsilon \ll 1$ then almost surely the spectrum of H is (dense) pure point, $\sigma(H) = \sigma_{pp}(H)$, with exponentially localized eigenfunctions ϕ_j, $j \in \mathbb{Z}^d$. Given $q(0) \in \ell^2(\mathbb{Z}^d)$, we decompose $q(0)$ as $q(0) = \sum_{j\in\mathbb{Z}^d} a_j\phi_j$. So

$$q(t) = \sum_{j\in\mathbb{Z}^d} a_j\phi_j e^{-\mathrm{i}\lambda_j t},$$

where λ_j are the eigenvalues for the eigenfunctions ϕ_j. Thus $q(t)$ is almost-periodic in time and satisfies the upper bound in (1.4).

Craig and Wayne [14] retrieved the origination of the KAM method - Newtonian iteration method together with the Lyapunov-Schmidt decomposition which involves the Green's function analysis and the control of the inverse of infinite matrices with small eigenvalues. They succeeded in constructing periodic solutions of the one-dimensional semi-linear wave equations with periodic boundary conditions. Bourgain [5, 6, 7, 8, 9] further developed the Craig–Wayne's method and proved the existence of quasi-periodic solutions for Hamiltonian partial differential equations in higher dimensional spaces with Dirichlet boundary conditions or periodic boundary conditions. In a similar way, Bourgain and Wang [10] constructed time quasi-periodic solutions to the nonlinear random Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}q = (\varepsilon\Delta + V)q + \delta|q|^{2p}q \quad (p > 0)$$

on $\mathbb{Z}^d \times [0, +\infty)$, which is considered as a perturbation of (1.3), with ε, δ sufficiently small. We point out that the Craig-Wayne-Bourgain's method allows one to avoid explicitly using the Hamiltonian structure of the systems. We will not introduce their approaches in detail. The reader is referred to Craig–Wayne [14], Bourgain [5, 6, 7, 8, 9], and Bourgain-Wang[10].

Comparing with Craig-Wayne-Bourgain's approach, the KAM approach has its own advantages. Besides obtaining the existent results it allows one to construct a local normal form in a neighborhood of the obtained solutions, and this is useful for better understanding of the dynamics. For example, one can obtain the linear stability and zero Lyapunov exponents. The KAM method was successfully applied by Kuksin[31] and Wayne[39] (see also [32, 34, 36, 37]) to, as typical examples, one-dimensional semi-linear Schrödinger equations

$$\mathrm{i}u_t - u_{xx} + mu = f(u),$$

and wave equations

$$u_{tt} - u_{xx} + mu = f(u),$$

with Dirichlet boundary conditions. Geng–You [25, 26] proved that the higher dimensional nonlinear beam equations and nonlocal Schrödinger equations admit small–amplitude linearly–stable quasi–periodic solutions. The breakthrough of constructing quasi-periodic solutions for more interesting higher dimensional Schrödinger equation by modified KAM method was made recently by Eliasson–Kuksin [17]. They proved that the higher dimensional nonlinear Schrödinger equations admit small–amplitude linearly–stable quasi–periodic solutions. Very recently, quasi–periodic solutions of two dimensional cubic Schrödinger equation

$$\mathrm{i}u_t - \triangle u + |u|^2 u = 0, \qquad x \in \mathbb{T}^2,\ t \in \mathbb{R},$$

with periodic boundary conditions are obtained by Geng–Xu–You [23]. By carefully choosing tangential sites $\{i_1, \cdots, i_b\} \in \mathbb{Z}^2$, the authors proved that the above nonlinear Schrödinger equation admits a family of small-amplitude quasi-periodic solutions.

However, all the above mentioned KAM results fail in dealing with the cases of random Hamiltonian systems as Craig-Wayne-Bourgain's method. In this paper, we try to attack the case of random lattice Hamiltonian PDEs. Concretely, we consider the equation (1.1) as a model, note that $\{v_n\}_{n\in\mathbb{Z}}$ is dense on the interval $[0, 1]$, thus all the above mentioned KAM results fail for this case. In this paper we give an abstract KAM theorem which can be applied to (1.1). We use the theorem to construct the quasi-periodic solutions and, different from the Craig-Wayne-Bourgain's method, prove their linear stability for the equation (1.1). To establish the KAM theorem, we have to impose further restrictions both on the unperturbed part and on the perturbation besides smallness. In the existent infinite dimensional KAM theorems, e.g., Kuksin [31], Pöschel [37], Wayne [39], Eliasson–Kuksin [17], Geng–Viveros–Yi [29], Geng–Xu–You [23], some assumptions on the regularity of the frequencies and the perturbation are required (See (**A1**) – (**A5**) in Section 2). In addition, we also assume that the perturbation has a special form defined in (**A6**) in Section(2), which is called gauge invariance. Our proof benefits a lot from such speciality of the perturbation. With the speciality of the form of the perturbation, we can prove that the normal form part of the Hamiltonian remains simple during the iteration. Compared with the proof of the previous KAM theorems, an additional job done in this paper is to prove that the perturbation always has the special form along the KAM iteration.

Now we are going to state our main result.

Let $b > 1$ be an integer and $\mathcal{J} = \{n_1, \cdots, n_b\} \subset \mathbb{Z}$, $\mathbb{Z}_1 = \mathbb{Z} \setminus \mathcal{J}$. We consider the case with frequencies $\tilde{\omega} = (\tilde{\omega}_1, \cdots, \tilde{\omega}_b)$ parametrized by $\omega = (\omega_1, \cdots, \omega_b)$, which is treated as parameters in a closed region $\mathcal{O}$ in $\mathbb{R}^b_+$ satisfying $|\mathcal{O}| > 0$. (Hereafter, for simplicity, we use the symbol $|\cdot|$ to denote the Lebesgue measure of a subset of $\mathbb{R}^b$). Given $\rho > 0$, let $\ell^1_\rho(\mathbb{Z})$ to be the Banach space of summable complex valued sequences $q = \{q_n\}_{n\in\mathbb{Z}}$, with the norm

$$\|q\|_\rho = \sum_{n\in\mathbb{Z}} |q_n| e^{|n|\rho} < \infty.$$

Our main result can be stated as follows.

Theorem 1 *Consider the lattice equations*

$$\mathrm{i}\dot{q}_n + v_n q_n + \delta |q_n|^2 q_n + \varepsilon_n (q_{n+1} + q_{n-1}) = 0, \quad n \in \mathbb{Z}$$

where $\{v_n\}_{n\in\mathbb{Z}}$ is a family of i.i.d. random variables with common distribution g satisfying (1.2), and and $|\varepsilon_n| \le \varepsilon e^{-\varrho|n|}$ with $\varepsilon, \varrho > 0$. Let b, $\tilde{\omega}$, $\mathcal{O}$, $\mathcal{J}$ and $\mathbb{Z}_1$ be defined as above. There exists a sufficiently small positive number $\tilde{\varepsilon}_0$ such that the following holds for $0 < \varepsilon, \delta < \tilde{\varepsilon}_0$.

There exists $X_{\varepsilon,\delta} \subset [0,1]^{\mathbb{Z}_1}$ with

$$\mathrm{prob}(X_{\varepsilon,\delta}) > e^{-\varepsilon^\sigma}$$

for some $0 < \sigma < 1$ such that if we fix $\{v_n\}_{n\in\mathbb{Z}_1} \in X_{\varepsilon,\delta}$, there exists a family of Cantor sets $\mathcal{O}_{\varepsilon,\delta} \subset \mathcal{O}$ for $0 < \varepsilon, |\delta| \ll 1$ with $|\mathcal{O} \setminus \mathcal{O}_{\varepsilon,\delta}| \to 0$ as $\varepsilon, \delta \to 0$ and C^1_W (i.e., C^1 in the sense of Whitney) maps $\omega_{\varepsilon,\delta} : \mathcal{O}_{\varepsilon,\delta} \to \mathbb{R}^b_+$, such that for every $\omega \in \mathcal{O}_{\varepsilon,\delta}$, the Hamiltonian associated with ω admits a small amplitude, linearly stable, quasi-periodic solution $q(t) = \{q_n(t)\}$ of b-frequency $\omega_{\varepsilon,\delta} = \omega_{\varepsilon,\delta}(\omega)$ that is slightly deformed from ω. Moreover, for each t, $q(t) = \{q_n(t)\} \in \ell^1_\rho(\mathbb{Z})$ for some $\rho > 0$.

The rest of this paper is organized as follows. In Section 2, we define the weighted norms, the decay property and gauge invariance, and present the abstract KAM theorem, which can be applied to the equation (1.1). In Section 3, we give the details for one step of the KAM iteration. The proof of the theorem is completed in Section 4 and 5 by an iteration lemma, giving a convergence result, and finally conducting the measure estimates of the remaining parameters. Some technical lemmas are proved in Section 6, which is regarded as an appendix of this paper.

2 An abstract KAM theorem

2.1 Function space norms

We start with some necessary notations. Fix $b > 1$ an integer. For given b vectors in $\mathbb{Z}$, say $n_1, \cdots, n_b$, we denote $\mathbb{Z}_1 = \mathbb{Z} \setminus \{n_1, \cdots, n_b\}$. Let $q = (\cdots, q_n, \cdots)_{n\in\mathbb{Z}_1}$, and its complex conjugate $\bar{q} = (\cdots, \bar{q}_n, \cdots)_{n\in\mathbb{Z}_1}$, with the norm

$$\|q\|_\rho = \sum_{n\in\mathbb{Z}_1} |q_n| e^{|n|\rho} < \infty.$$

Given real numbers $r, s > 0$, we let $D_\rho(r,s)$ be the complex b-dimensional neighborhood of $\mathbb{T}^b \times \{0\} \times \{0\}$ in $\mathbb{T}^b \times \mathbb{R}^b \times \ell^1_\rho(\mathbb{Z}_1)$, i.e.,

$$D_\rho(r,s) = \{(\theta, I, q) : |\mathrm{Im}\theta| = |\mathrm{Im}(\theta_1, \cdots, \theta_b)| < r, |I| < s^2, \|q\|_\rho < s\},$$

where $|\cdot|$ is the sup-norm of complex vectors.

Let $F(\theta, I, q, \bar{q})$ be a real analytic function on $D_\rho(r,s)$ which depends C^1_W-smoothly on a parameter $\omega \in \mathcal{O}$. In the rest of the paper, all dependencies on ω are assumed of class C^1_W, thus all derivatives with respective to the parameter $\omega \in \mathcal{O}$ will be interpreted in this sense. We expand F into the Taylor-Fourier series with respect to $\theta, I, q, \bar{q}$:

$$F(\theta, I, q, \bar{q}) = \sum_{\alpha,\beta} F_{\alpha\beta} q^\alpha \bar{q}^\beta, \tag{2.1}$$

where, for multi-indices $\alpha := (\cdots, \alpha_n, \cdots)$, $\beta := (\cdots, \beta_n, \cdots)$, $\alpha_n, \beta_n \in \mathbb{N}$ with finitely many non-vanishing components,

$$F_{\alpha\beta} = \sum_{k\in\mathbb{Z}^b, l\in\mathbb{N}^b} F_{kl\alpha\beta}(\omega) I^l e^{\mathrm{i}\langle k,\theta\rangle}.$$

The norm of the function F on $D_\rho(r,s)\times\mathcal{O}$ is given by

$$\|F\|_{D_\rho(r,s),\mathcal{O}} := \sup_{\|q\|_\rho < s} \sum_{\alpha,\beta} \|F_{\alpha\beta}\| \, |q^\alpha||\bar{q}^\beta|, \tag{2.2}$$

where $|q^\alpha| = \prod_{\alpha_n \neq 0} |q_n|^{\alpha_n}$, $|\bar{q}^\beta| = \prod_{\beta_n\neq 0} |\bar{q}_n|^{\beta_n}$, and

$$\|F_{\alpha\beta}\| := \sum_{k,l} |F_{kl\alpha\beta}|_{\mathcal{O}} s^{2|l|} e^{|k|r}, \quad |F_{kl\alpha\beta}|_{\mathcal{O}} := \sup_{\omega\in\mathcal{O}} \left(|F_{kl\alpha\beta}| + \left| \frac{\partial F_{kl\alpha\beta}}{\partial\omega} \right| \right).$$

In the case of a vector-valued function $G : D_\rho(r,s)\times\mathcal{O} \to \mathbb{C}^n$ (with $n < \infty$), we define its norm as

$$\|G\|_{D_\rho(r,s),\mathcal{O}} := \sum_{i=1}^n \|G_i\|_{D_\rho(r,s),\mathcal{O}}.$$

For the Hamiltonian vector field

$$X_F = (F_I, -F_\theta, (-\mathrm{i}F_{q_n})_{n\in\mathbb{Z}_1}, (\mathrm{i}F_{\bar{q}_n})_{n\in\mathbb{Z}_1})$$

associated with a function F on $D_\rho(r,s)\times\mathcal{O}$, we define its norm by

$$\begin{aligned}\|X_F\|_{D_\rho(r,s),\mathcal{O}} := \quad & \|\partial_I F\|_{D_\rho(r,s),\mathcal{O}} + \frac{1}{s^2}\|\partial_\theta F\|_{D_\rho(r,s),\mathcal{O}} \\ & + \frac{1}{s}\Big(\sum_{n\in\mathbb{Z}_1} \|\partial_{q_n} F\|_{D_\rho(r,s),\mathcal{O}} e^{|n|\rho} + \sum_{n\in\mathbb{Z}_1} \|\partial_{\bar{q}_n} F\|_{D_\rho(r,s),\mathcal{O}} e^{|n|\rho}\Big).\end{aligned}$$

All vector fields are going to be estimated in this kind of norm as well, which will imply the exponential decay of the vector field components in the index $n \in \mathbb{Z}$. Sometimes, for the sake of notational simplification, we shall not write the subscript $D_\rho(r,s)$ or $\mathcal{O}$ if it is obvious enough.

In what follows in the formulations and proofs of various assertions we shall encounter absolute constants as well as ones depending on the function F, the dimension b, and so on. All such constants will be denoted by c, c_1 ,c_2, $\cdots$, and sometimes even different constants will be denoted by the same symbol.

Let F, G be two real analytic functions on $D_\rho(r,s)$ which depend C^1_W-smoothly on a parameter $\xi \in \mathcal{O}$, and let $\{\cdot,\cdot\}$ denote the Poisson bracket of smooth functions, i.e.,

$$\{F,G\} = \left\langle \frac{\partial F}{\partial I}, \frac{\partial G}{\partial \theta} \right\rangle - \left\langle \frac{\partial F}{\partial \theta}, \frac{\partial G}{\partial I} \right\rangle + \mathrm{i} \sum_{n\in\mathbb{Z}_1} \left(\frac{\partial F}{\partial q_n}\frac{\partial G}{\partial \bar{q}_n} - \frac{\partial F}{\partial \bar{q}_n}\frac{\partial G}{\partial q_n} \right),$$

which is perhaps the most important quantity to be estimated in this norm defined for the vector fields, as it is significant to Hamiltonian mechanics. Some basic estimates about the vector field and the Poisson bracket are given in the appendix.

2.2 Decay property and gauge invariance

As before, we consider the real analytic function F, given in terms of their Fourier–Taylor series expansion. We decompose F into $\breve{F}$, $\acute{F}$ and $\grave{F}$, where $\acute{F} + \grave{F}$ is the projection onto the components which are independent of the tangential variables (I, θ):

$$\acute{F} = \sum_{|\alpha|+|\beta|\le 2} \acute{F}_{\alpha\beta} q^\alpha \bar{q}^\beta, \quad \acute{F}_{\alpha\beta} = F_{00\alpha\beta}(\omega) \quad (|\alpha|+|\beta| \le 2),$$

$$\grave{F} = \sum_{|\alpha|+|\beta|\ge 3} \grave{F}_{\alpha\beta} q^\alpha \bar{q}^\beta, \quad \grave{F}_{\alpha\beta} = F_{00\alpha\beta}(\omega) \quad (|\alpha|+|\beta| \ge 3).$$

Then $\breve{F}$ is the result of the complementary projection, i.e.

$$\breve{F} = \sum_{\alpha,\beta} \breve{F}_{\alpha\beta} q^\alpha \bar{q}^\beta, \quad \breve{F}_{\alpha\beta} = \sum_{(k,l)\ne 0} F_{kl\alpha\beta}(\omega) I^l e^{\mathrm{i}\langle k,\theta\rangle}.$$

For each multi–index $(\alpha,\beta) = (\cdots, \alpha_n, \beta_n, \cdots)$, $n \in \mathbb{Z}_1$, define the quantities

$$\begin{aligned} n^+ &:= n^+(\alpha,\beta) = \max\{n \in \mathbb{Z}_1 : (\alpha_n,\beta_n) \ne 0\}, \\ n^- &:= n^-(\alpha,\beta) = \min\{n \in \mathbb{Z}_1 : (\alpha_n,\beta_n) \ne 0\}, \\ n^* &:= n^*(\alpha,\beta) = \max\{|n^+|, |n^-|\}, \end{aligned}$$

and

$$\mathrm{supp}(\alpha,\beta) = \{n \in \mathbb{Z}_1 : (\alpha_n,\beta_n) \ne 0\}$$

Remark. The above notations are closely related to the notations of support and diameter for the monomials in [11]. The decay properties of functions on phase space in terms of the index n is important to this study. We distinguish the decay behaviors of functions $\grave{F}_{\alpha\beta}$ which are independent of the tangent variable (I,θ) with $|\alpha|+|\beta| \ge 3$, $\acute{F}_{\alpha\beta}$ with $|\alpha|+|\beta| \le 2$ and $\breve{F}_{\alpha\beta}$ which do depend on (I,θ).

Definition 2.1 *A real analytic function*

$$F = F(\theta, I, q, \bar{q}) = \sum_{\alpha,\beta} F_{\alpha\beta} q^\alpha \bar{q}^\beta$$

on $D_\rho(r,s)$ is said to satisfy the **decay property** *if*

$$\begin{aligned} \|\breve{F}_{\alpha\beta}\| &\le c e^{-\varrho n^*}, \quad |\alpha|+|\beta| \ge 1, \\ \|\acute{F}_{\alpha\beta}\| &\le c e^{-\varrho n^*}, \quad 1 \le |\alpha|+|\beta| \le 2, \\ \|\grave{F}_{\alpha\beta}\| &\le c e^{-\varrho(n^+ - n^-)}, \quad |\alpha|+|\beta| \ge 3 \end{aligned}$$

with some c, $\varrho > 0$.

It is important that this decay property can be preserved by the procedure of making KAM iterations. It allow us to consider a finite dimensional small divisor problems at each iteration step. This property is not preserved by products or sums of coefficients, but it is preserved by the Poisson bracket.

Lemma 2.1 *Consider two real analytic functions defined on $D_\rho(r, s)$*

$$
\begin{aligned}
G(\theta, I, q, \bar{q}) &= \sum_{\hat{\alpha},\hat{\beta}} \breve{G}_{\hat{\alpha}\hat{\beta}} q^{\hat{\alpha}} \bar{q}^{\hat{\beta}} + \sum_{|\hat{\alpha}|+|\hat{\beta}|\le 2} \acute{G}_{\hat{\alpha}\hat{\beta}} q^{\hat{\alpha}} \bar{q}^{\hat{\beta}} + \sum_{|\hat{\alpha}|+|\hat{\beta}|\ge 3} \grave{G}_{\hat{\alpha}\hat{\beta}} q^{\hat{\alpha}} \bar{q}^{\hat{\beta}}, \\
F(\theta, I, q, \bar{q}) &= \sum_{\substack{\tilde{\alpha},\tilde{\beta} \\ \tilde{n}^* \le M}} \breve{F}_{\tilde{\alpha}\tilde{\beta}} q^{\tilde{\alpha}} \bar{q}^{\tilde{\beta}} + \sum_{\substack{|\tilde{\alpha}|+|\tilde{\beta}|\le 2 \\ \tilde{n}^* \le M}} \acute{F}_{\tilde{\alpha}\tilde{\beta}} q^{\tilde{\alpha}} \bar{q}^{\tilde{\beta}},
\end{aligned}
$$

with

$$
\begin{aligned}
&\|\breve{G}_{\hat{\alpha}\hat{\beta}}\| \le c_G e^{-\varrho \hat{n}^*}, \quad |\hat{\alpha}| + |\hat{\beta}| \ge 1, \\
&\|\acute{G}_{\hat{\alpha}\hat{\beta}}\| \le c_G e^{-\varrho \hat{n}^*}, \quad 1 \le |\hat{\alpha}| + |\hat{\beta}| \le 2, \\
&\|\grave{G}_{\hat{\alpha}\hat{\beta}}\| \le c_G e^{-\varrho (\hat{n}^+ - \hat{n}^-)}, \quad |\hat{\alpha}| + |\hat{\beta}| \ge 3, \\
&\|\breve{F}_{\tilde{\alpha}\tilde{\beta}}\| \le c_F e^{-\varrho \tilde{n}^*}, \quad |\tilde{\alpha}| + |\tilde{\beta}| \ge 1, \\
&\|\acute{F}_{\tilde{\alpha}\tilde{\beta}}\| \le c_F e^{-\varrho \tilde{n}^*}, \quad 1 \le |\tilde{\alpha}| + |\tilde{\beta}| \le 2,
\end{aligned}
$$

for some positive c_G, c_F and ϱ, where

$$
\begin{aligned}
&\hat{n}^+ = \hat{n}^+(\hat{\alpha}, \hat{\beta}) = \max\{n : (\hat{\alpha}_n, \hat{\beta}_n) \ne 0\}, \\
&\hat{n}^- = \hat{n}^-(\hat{\alpha}, \hat{\beta}) = \min\{n : (\hat{\alpha}_n, \hat{\beta}_n) \ne 0\}, \\
&\hat{n}^* = \hat{n}^*(\hat{\alpha}, \hat{\beta}) = \max\{|\hat{n}^+|, |\hat{n}^-|\}, \\
&\tilde{n}^+ = \tilde{n}^+(\tilde{\alpha}, \tilde{\beta}) = \max\{n : (\tilde{\alpha}_n, \tilde{\beta}_n) \ne 0\}, \\
&\tilde{n}^- = \tilde{n}^-(\tilde{\alpha}, \tilde{\beta}) = \min\{n : (\tilde{\alpha}_n, \tilde{\beta}_n) \ne 0\}, \\
&\tilde{n}^* = \tilde{n}^*(\tilde{\alpha}, \tilde{\beta}) = \max\{|\tilde{n}^+|, |\tilde{n}^-|\},
\end{aligned}
$$

then on $D_\rho(r - \sigma, \frac{s}{2})$,

$$
K = \{G, F\} = \sum_{\alpha,\beta} K_{\alpha\beta} q^\alpha \bar{q}^\beta
$$

satisfies

$$
\|K_{\alpha\beta}\| \le c_K e^{-\varrho n^*}, \quad |\alpha| + |\beta| \ge 1,
$$

for some positive c_K, where

$$
n^* = n^*(\alpha, \beta) = \max\{|n^+|, |n^-|\},
$$

and

$$
\begin{aligned}
n^+ &= n^+(\alpha, \beta) = \max\{n : (\alpha_n, \beta_n) \ne 0\}, \\
n^- &= n^-(\alpha, \beta) = \min\{n : (\alpha_n, \beta_n) \ne 0\}.
\end{aligned}
$$

Proof: A straightforward calculation yields that

$$
\begin{aligned}
&\{G,F\}\\
=\ & \sum_{\substack{\hat\alpha,\hat\beta,\tilde\alpha,\tilde\beta\\ \tilde n^*\le M}} \left\langle \frac{\partial \breve G_{\alpha_1\hat\beta}}{\partial I}, \frac{\partial \breve F_{\tilde\alpha\tilde\beta}}{\partial \theta} \right\rangle q^{\hat\alpha+\tilde\alpha}\bar q^{\hat\beta+\tilde\beta} && (2.3)\\
& - \sum_{\substack{\hat\alpha,\hat\beta,\tilde\alpha,\tilde\beta\\ \tilde n^*\le M}} \left\langle \frac{\partial \breve G_{\hat\alpha\hat\beta}}{\partial \theta}, \frac{\partial \breve F_{\tilde\alpha\tilde\beta}}{\partial I} \right\rangle q^{\hat\alpha+\tilde\alpha}\bar q^{\hat\beta+\tilde\beta} && (2.4)\\
& + \mathrm{i} \sum_{\substack{\tilde n^*\le M\\ \tilde n^-\le n\le \tilde n^+}} \sum_{\substack{\hat\alpha,\tilde\beta\\ \tilde\alpha,\tilde\beta}} \breve G_{\hat\alpha\hat\beta} \breve F_{\tilde\alpha\tilde\beta} q^{\hat\alpha-e_n}\bar q^{\hat\beta} q^{\tilde\alpha}\bar q^{\tilde\beta-e_n} && (2.5)\\
& + \mathrm{i} \sum_{\substack{\tilde n^*\le M\\ \tilde n^-\le n\le \tilde n^+}} \sum_{\substack{\hat\alpha,\hat\beta\\ \tilde\alpha,\tilde\beta}} \breve G_{\hat\alpha\hat\beta} \acute F_{\tilde\alpha\tilde\beta} q^{\hat\alpha-e_n}\bar q^{\hat\beta} q^{\tilde\alpha}\bar q^{\tilde\beta-e_n} && (2.6)\\
& + \mathrm{i} \sum_{\substack{\tilde n^*\le M\\ \tilde n^-\le n\le \tilde n^+}} \sum_{\substack{\hat\alpha,\hat\beta\\ \tilde\alpha,\tilde\beta}} \acute G_{\hat\alpha\hat\beta} \breve F_{\tilde\alpha\tilde\beta} q^{\hat\alpha-e_n}\bar q^{\hat\beta} q^{\tilde\alpha}\bar q^{\tilde\beta-e_n} && (2.7)\\
& + \mathrm{i} \sum_{\substack{\tilde n^*\le M\\ \tilde n^-\le n\le \tilde n^+}} \sum_{\substack{\hat\alpha,\hat\beta\\ \tilde\alpha,\tilde\beta}} \acute G_{\hat\alpha\hat\beta} \acute F_{\tilde\alpha\tilde\beta} q^{\hat\alpha-e_n}\bar q^{\hat\beta} q^{\tilde\alpha}\bar q^{\tilde\beta-e_n} && (2.8)\\
& + \mathrm{i} \sum_{\substack{\tilde n^*\le M\\ \tilde n^-\le n\le \tilde n^+}} \sum_{\substack{\hat\alpha,\hat\beta\\ \tilde\alpha,\tilde\beta}} \grave G_{\hat\alpha\hat\beta} \breve F_{\tilde\alpha\tilde\beta} q^{\hat\alpha-e_n}\bar q^{\hat\beta} q^{\tilde\alpha}\bar q^{\tilde\beta-e_n} && (2.9)\\
& + \mathrm{i} \sum_{\substack{\tilde n^*\le M\\ \tilde n^-\le n\le \tilde n^+}} \sum_{\substack{\hat\alpha,\hat\beta\\ \tilde\alpha,\tilde\beta}} \grave G_{\hat\alpha\hat\beta} \acute F_{\tilde\alpha\tilde\beta} q^{\hat\alpha-e_n}\bar q^{\hat\beta} q^{\tilde\alpha}\bar q^{\tilde\beta-e_n} && (2.10)\\
& - \mathrm{i} \sum_{\substack{\tilde n^*\le M\\ \tilde n^-\le n\le \tilde n^+}} \sum_{\substack{\hat\alpha,\tilde\beta\\ \tilde\alpha,\tilde\beta}} \breve G_{\hat\alpha\hat\beta} \breve F_{\tilde\alpha\tilde\beta} q^{\hat\alpha}\bar q^{\hat\beta-e_n} q^{\tilde\alpha-e_n}\bar q^{\tilde\beta} && (2.11)\\
& - \mathrm{i} \sum_{\substack{\tilde n^*\le M\\ \tilde n^-\le n\le \tilde n^+}} \sum_{\substack{\hat\alpha,\hat\beta\\ \tilde\alpha,\tilde\beta}} \breve G_{\hat\alpha\hat\beta} \acute F_{\tilde\alpha\tilde\beta} q^{\hat\alpha}\bar q^{\hat\beta-e_n} q^{\tilde\alpha-e_n}\bar q^{\tilde\beta} && (2.12)\\
& - \mathrm{i} \sum_{\substack{\tilde n^*\le M\\ \tilde n^-\le n\le \tilde n^+}} \sum_{\substack{\hat\alpha,\hat\beta\\ \tilde\alpha,\tilde\beta}} \acute G_{\hat\alpha\hat\beta} \breve F_{\tilde\alpha\tilde\beta} q^{\hat\alpha}\bar q^{\hat\beta-e_n} q^{\tilde\alpha-e_n}\bar q^{\tilde\beta} && (2.13)\\
& - \mathrm{i} \sum_{\substack{\tilde n^*\le M\\ \tilde n^-\le n\le \tilde n^+}} \sum_{\substack{\hat\alpha,\hat\beta\\ \tilde\alpha,\tilde\beta}} \acute G_{\hat\alpha\hat\beta} \acute F_{\tilde\alpha\tilde\beta} q^{\hat\alpha}\bar q^{\hat\beta-e_n} q^{\tilde\alpha-e_n}\bar q^{\tilde\beta} && (2.14)\\
& - \mathrm{i} \sum_{\substack{\tilde n^*\le M\\ \tilde n^-\le n\le \tilde n^+}} \sum_{\substack{\hat\alpha,\hat\beta\\ \tilde\alpha,\tilde\beta}} \grave G_{\hat\alpha\hat\beta} \breve F_{\tilde\alpha\tilde\beta} q^{\hat\alpha}\bar q^{\hat\beta-e_n} q^{\tilde\alpha-e_n}\bar q^{\tilde\beta} && (2.15)\\
& - \mathrm{i} \sum_{\substack{\tilde n^*\le M\\ \tilde n^-\le n\le \tilde n^+}} \sum_{\substack{\hat\alpha,\hat\beta\\ \tilde\alpha,\tilde\beta}} \grave G_{\hat\alpha\hat\beta} \acute F_{\tilde\alpha\tilde\beta} q^{\hat\alpha}\bar q^{\hat\beta-e_n} q^{\tilde\alpha-e_n}\bar q^{\tilde\beta} && (2.16)
\end{aligned}
$$

where e_n is the multi-index whose n^{th} component is 1 and other components are all 0. In

(2.3), (2.4), $n^* = \max\{\hat{n}^*, \tilde{n}^*\}$, then according to Lemma 6.2, on $D_\rho(r-\sigma, \frac{s}{2})$, we have

$$\left\| \left\langle \frac{\partial \breve{G}_{\hat{\alpha}\hat{\beta}}}{\partial I}, \frac{\partial \breve{F}_{\tilde{\alpha}\tilde{\beta}}}{\partial \theta} \right\rangle \right\| \le \frac{4c_G c_F M}{\sigma s^2} e^{-\varrho n^*},$$

$$\left\| \left\langle \frac{\partial \breve{G}_{\hat{\alpha}\hat{\beta}}}{\partial \theta}, \frac{\partial \breve{F}_{\tilde{\alpha}\tilde{\beta}}}{\partial I} \right\rangle \right\| \le \frac{4c_G c_F M}{\sigma s^2} e^{-\varrho n^*};$$

in (2.5), (2.6), (2.7), (2.8), (2.11), (2.12), (2.13), (2.14), $n^* = \max\{\hat{n}^*, \tilde{n}^*\}$, then $\hat{n}^* + \tilde{n}^* \ge n^*$, hence

$$\| \sum_{\substack{\tilde{n}^* \le M \\ \tilde{n}^- \le n \le \tilde{n}^+}} \sum_{\substack{\hat{\alpha},\hat{\beta} \\ \tilde{\alpha},\tilde{\beta}}} \breve{G}_{\hat{\alpha}\hat{\beta}} \breve{F}_{\tilde{\alpha}\tilde{\beta}} \| \le c_G c_F M e^{-\varrho n^*},$$

$$\| \sum_{\substack{\tilde{n}^* \le M \\ \tilde{n}^- \le n \le \tilde{n}^+}} \sum_{\substack{\hat{\alpha},\hat{\beta} \\ \tilde{\alpha},\tilde{\beta}}} \breve{G}_{\hat{\alpha}\hat{\beta}} \acute{F}_{\tilde{\alpha}\tilde{\beta}} \| \le c_G c_F M e^{-\varrho n^*},$$

$$\| \sum_{\substack{\tilde{n}^* \le M \\ \tilde{n}^- \le n \le \tilde{n}^+}} \sum_{\substack{\hat{\alpha},\hat{\beta} \\ \tilde{\alpha},\tilde{\beta}}} \acute{G}_{\hat{\alpha}\hat{\beta}} \breve{F}_{\tilde{\alpha}\tilde{\beta}} \| \le c_G c_F M e^{-\varrho n^*},$$

$$\| \sum_{\substack{\tilde{n}^* \le M \\ \tilde{n}^- \le n \le \tilde{n}^+}} \sum_{\substack{\hat{\alpha},\hat{\beta} \\ \tilde{\alpha},\tilde{\beta}}} \acute{G}_{\hat{\alpha}\hat{\beta}} \acute{F}_{\tilde{\alpha}\tilde{\beta}} \| \le c_G c_F M e^{-\varrho n^*};$$

in (2.9), (2.10), (2.15), (2.16), $n^* = \max\{\hat{n}^*, \tilde{n}^*\}$, note $\hat{n}^- \le \tilde{n}^+$, and $\tilde{n}^- \le \hat{n}^+$, then $\hat{n}^+ - \hat{n}^- + \tilde{n}^* \ge n^*$, hence

$$\| \sum_{\substack{\tilde{n}^* \le M \\ \tilde{n}^- \le n \le \tilde{n}^+}} \sum_{\substack{\hat{\alpha},\hat{\beta} \\ \tilde{\alpha},\tilde{\beta}}} \grave{G}_{\hat{\alpha}\hat{\beta}} \breve{F}_{\tilde{\alpha}\tilde{\beta}} \| \le c_G c_F M e^{-\varrho n^*},$$

$$\| \sum_{\substack{\tilde{n}^* \le M \\ \tilde{n}^- \le n \le \tilde{n}^+}} \sum_{\substack{\hat{\alpha},\hat{\beta} \\ \tilde{\alpha},\tilde{\beta}}} \grave{G}_{\hat{\alpha}\hat{\beta}} \acute{F}_{\tilde{\alpha}\tilde{\beta}} \| \le c_G c_F M e^{-\varrho n^*}.$$

Thus Lemma 2.1 is shown to hold. ■

During the KAM steps, we often apply the following formula

$$G \circ \Psi_F^1 = G + \{G, F\} + \frac{1}{2!}\{\{G, F\}, F\} + \cdots + \frac{1}{n!}\{\cdots\{G, \underbrace{F\}\cdots, F\}}_{n} + \cdots.$$

Note that $n^* \le \frac{1}{2}(n^+ - n^-)$, then we have

Corollary 1 *If G and F satisfy the assumption of Lemma 2.1, then on $D_\rho(r-\sigma, \frac{s}{2})$, $\tilde{G} := G \circ \Psi_F^1$ satisfies that*

$$\begin{aligned} &\|\breve{\tilde{G}}_{\hat{\alpha}\hat{\beta}}\| \le c_{\tilde{G}} e^{-\varrho n^*}, \quad |\alpha| + |\beta| \ge 1, \\ &\|\acute{\tilde{G}}_{\hat{\alpha}\hat{\beta}}\| \le c_{\tilde{G}} e^{-\varrho n^*}, \quad 1 \le |\alpha| + |\beta| \le 2, \\ &\|\grave{\tilde{G}}_{\hat{\alpha}\hat{\beta}}\| \le c_{\tilde{G}} e^{-\frac{\varrho}{2}(n^+ - n^-)}, \quad |\alpha| + |\beta| \ge 3, \end{aligned}$$

for some positive $c_{\tilde G}$*, where*

$$\begin{aligned} n^+ &= \max\{n : (\alpha_n, \beta_n) \neq 0\}, \\ n^- &= \min\{n : (\alpha_n, \beta_n) \neq 0\}, \\ n^* &= \max\{|n^+|, |n^-|\}. \end{aligned}$$

Besides the decay property, the gauge invariance, which concerns the relation between k, α, β appearing in the Taylor-Fourier series, can be kept during the KAM iteration. The precise definition of the gauge invariance is given below.

Let $|\alpha| := \sum_n \alpha_n$ for any multi-index $\alpha = (\cdots, \alpha_n, \cdots)_{n\in\mathbb{Z}_1}$, $\alpha_n \in \mathbb{N}$, with finitely many non-vanishing components.

Definition 2.2 *The function* $F(\theta, I, q, \bar q)$ *is called to have* **gauge invariance***, if*

$$F_{kl\alpha\beta}(\xi) = 0, \quad \text{when } k_1 + k_2 + \cdots + k_b + |\alpha| - |\beta| \neq 0.$$

Let $\mathcal{A}$ denote the collection of the functions which has gauge invariance.

Lemma 2.2 *If* $G(\theta, I, q, \bar q), F(\theta, I, q, \bar q) \in \mathcal{A}$*, then* $K(\theta, I, q, \bar q) = \{G, F\} \in \mathcal{A}$*.*

Proof: Let

$$G = \sum_{k,\alpha,\beta} G_{k\alpha\beta}(I) e^{\mathrm{i}\langle k,\theta\rangle} q^\alpha \bar q^\beta,$$

$$F = \sum_{\tilde k,\tilde\alpha,\tilde\beta} F_{\tilde k\tilde\alpha\tilde\beta}(I) e^{\mathrm{i}\langle \tilde k,\theta\rangle} q^{\tilde\alpha} \bar q^{\tilde\beta},$$

where the summations are taken over

$$\{(k, \alpha, \beta) : \sum_{j=1}^b k_j + |\alpha| - |\beta| = 0\}, \tag{2.17}$$

and

$$\{(\tilde k, \tilde\alpha, \tilde\beta) : \sum_{j=1}^b \tilde k_j + |\tilde\alpha| - |\tilde\beta| = 0\} \tag{2.18}$$

respectively. Since

$$\begin{aligned} \{G, F\} &= \mathrm{i}\sum_{A_1}\sum_{A_2}\langle \frac{\partial G_{k\alpha\beta}(I)}{\partial I}, \tilde k\rangle F_{\tilde k\tilde\alpha\tilde\beta}(I) e^{\mathrm{i}\langle k,\theta\rangle} q^\alpha \bar q^\beta e^{\mathrm{i}\langle \tilde k,\theta\rangle} q^{\tilde\alpha} \bar q^{\tilde\beta} \\ &\quad - \mathrm{i}\sum_{A_1}\sum_{A_2}\langle k, \frac{\partial F_{\tilde k\tilde\alpha\tilde\beta}(I)}{\partial I}\rangle G_{k\alpha\beta}(I) e^{\mathrm{i}\langle k,\theta\rangle} q^\alpha \bar q^\beta e^{\mathrm{i}\langle \tilde k,\theta\rangle} q^{\tilde\alpha} \bar q^{\tilde\beta} \\ &\quad + \mathrm{i}\sum_{m\in\mathbb{Z}_1}\sum_{A_3} G_{k\alpha\beta}(I) F_{\tilde k\tilde\alpha\tilde\beta}(I) e^{\mathrm{i}\langle k,\theta\rangle} e^{\mathrm{i}\langle \tilde k,\theta\rangle} q^{\alpha-e_m} \bar q^\beta q^{\tilde\alpha} \bar q^{\tilde\beta-e_m} \\ &\quad - \mathrm{i}\sum_{m\in\mathbb{Z}_1}\sum_{A_4} G_{k\alpha\beta}(I) F_{\tilde k\tilde\alpha\tilde\beta}(I) e^{\mathrm{i}\langle k,\theta\rangle} e^{\mathrm{i}\langle \tilde k,\theta\rangle} q^\alpha \bar q^{\beta-e_m} q^{\tilde\alpha-e_m} \bar q^{\tilde\beta} \\ &= \sum_{A_5} K_{(k+\tilde k)(\alpha+\tilde\alpha)(\beta+\tilde\beta)}(I) e^{\mathrm{i}\langle k+\tilde k,\theta\rangle} q^{\alpha+\tilde\alpha} \bar q^{\beta+\tilde\beta} \\ &\quad + \sum_{A_6} K_{(k+\tilde k)(\alpha+\tilde\alpha-e_m)(\beta+\tilde\beta-e_m)}(I) e^{\mathrm{i}\langle k+\tilde k,\theta\rangle} q^{\alpha+\tilde\alpha-e_m} \bar q^{\beta+\tilde\beta-e_m}, \end{aligned}$$

where e_m denotes the vector with the m^{th} component being 1 and the other components being zero; A_1 denotes

$$\sum_{j=1}^{b} k_j + |\alpha| - |\beta| = 0;$$

A_2 denotes

$$\sum_{j=1}^{b} \tilde{k}_j + |\tilde{\alpha}| - |\tilde{\beta}| = 0;$$

A_3 denotes

$$\sum_{j=1}^{b} k_j + |\alpha - e_m| - |\beta| = -1,$$

and

$$\sum_{j=1}^{b} \tilde{k}_j + |\tilde{\alpha}| - |\tilde{\beta} - e_m| = 1;$$

A_4 denotes

$$\sum_{j=1}^{b} k_j + |\alpha| - |\beta - e_m| = 1,$$

and

$$\sum_{j=1}^{b} \tilde{k}_j + |\tilde{\alpha} - e_m| - |\tilde{\beta}| = -1;$$

A_5 denotes

$$\sum_{j=1}^{b} (k_j + \tilde{k}_j) + |\alpha + \tilde{\alpha}| - |\beta + \tilde{\beta}| = 0;$$

A_6 denotes

$$\sum_{j=1}^{b} (k_j + \tilde{k}_j) + |\alpha + \tilde{\alpha} - e_m| - |\beta + \tilde{\beta} - e_m| = 0.$$

Thus Lemma 2.2 is obtained. ■

We also have

Corollary 2 *If $G(\theta, I, q, \bar{q}), F(\theta, I, q, \bar{q}) \in \mathcal{A}$, then $G \circ \Psi_F^1 \in \mathcal{A}$.*

2.3 Statement of the abstract KAM theorem

The starting point will be a family of integrable Hamiltonians of the form

$$N = e + \langle \omega, I \rangle + \sum_{n \in \mathbb{Z}_1} \Omega_n q_n \bar{q}_n, \tag{2.19}$$

where $\omega \in \mathcal{O}$ is a parameter, $\{\Omega_n\}_{n \in \mathbb{Z}_1} \in \mathbb{R}^{\mathbb{Z}_1}$ is a family of i.i.d. bounded random variables with common distribution $g(\Omega_n) = d\Omega_n$ equipped with the product measure

$$\prod_{n \in \mathbb{Z}_1} g(\Omega_n) = \prod_{n \in \mathbb{Z}_1} d\Omega_n,$$

and independent of ω. The phase space is endowed with the symplectic structure $dI \wedge d\theta + \mathrm{i} \sum_{n \in \mathbb{Z}_1} dq_n \wedge d\bar{q}_n$.

For each $\omega \in \mathcal{O}$, the Hamiltonian equations of motion for N, i.e.,

$$\frac{d\theta}{dt} = \omega, \quad \frac{dI}{dt} = 0, \quad \frac{dq_n}{dt} = -\mathrm{i}\Omega_n q_n, \quad \frac{d\bar{q}_n}{dt} = \mathrm{i}\Omega_n \bar{q}_n, \quad n \in \mathbb{Z}_1, \tag{2.20}$$

admit special solutions $(\theta, 0, 0, 0) \to (\theta + \omega t, 0, 0, 0)$ that corresponds to an invariant torus in the phase space.

Consider the new perturbed Hamiltonian

$$H = N + P = e + \langle \omega, I \rangle + \sum_{n \in \mathbb{Z}_1} \Omega_n q_n \bar{q}_n + P(\theta, I, q, \bar{q}; \omega). \tag{2.21}$$

Our goal is to prove that, for most of $\{\Omega_n\}_{n \in \mathbb{Z}_1} \in \mathbb{R}^{\mathbb{Z}_1}$(in product measure sense), the Hamiltonians $H = N + P$ still admit invariant tori for most of the parameter $\omega \in \mathcal{O}$ (in Lebesgue measure sense), provided that $\|X_P\|_{D_\rho(r,s),\mathcal{O}}$ is sufficiently small.

To this end, we need to impose some conditions on $\{\Omega_n\}_{n \in \mathbb{Z}_1}$ and the perturbation P.

(**A1**) *Regularity of normal frequencies:* For each $n \in \mathbb{Z}_1$, Ω_n is independent of the parameter ω.

(**A2**) *Gap condition of normal frequencies:* There exist $\gamma > 0$, $\tau > b$ such that for $n, m \in \mathbb{Z}_1$, $n \neq m$, and $0 \leq |m|, |n| \leq K_0 \sim \ln \frac{1}{\gamma}$,

$$|\Omega_m - \Omega_n| \geq \frac{\gamma}{|n - m|^\tau}. \tag{2.22}$$

Remark. We shall use X_0 to denote the subset of $\mathbb{R}^{Z_1}$ such that if $\{\Omega_n\}_{n \in \mathbb{Z}_1} \in X_0$ then (**A2**) holds.

(**A3**) *Melnikov's nondegeneracy:* There exist $\gamma > 0$, $\tau > b$ such that for any $k \neq 0$, and $0 \leq |m|, |n| \leq K_0$,

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{|k|^\tau}, \tag{2.23}$$

$$|\langle k, \omega \rangle + \Omega_n| \geq \frac{\gamma}{|k|^\tau}, \tag{2.24}$$

$$|\langle k, \omega \rangle + \Omega_n + \Omega_m| \geq \frac{\gamma}{|k|^\tau}, \tag{2.25}$$

$$|\langle k, \omega \rangle + \Omega_m - \Omega_n| \geq \frac{\gamma}{|k|^\tau}. \tag{2.26}$$

(**A4**) *Regularity of the perturbation:* The perturbation P is real analytic in I, θ, q, $\bar{q}$ and Whitney smoothly parametrized by $\omega \in \mathcal{O}$; in addition $\|X_P\|_{D_\rho(r,s),\mathcal{O}} < \varepsilon_0$ for some sufficiently small ε_0.

$(\mathbf{A5})$*Decay property of the perturbation:* If we write that $P=\breve{P}+\acute{P}+\grave{P}$, where

$$\breve{P}=\breve{P}(\theta,I,q,\bar{q};\omega)=\sum_{\alpha,\beta}\breve{P}_{\alpha\beta}q^{\alpha}\bar{q}^{\beta}=\sum_{\substack{(k,l)\neq 0\\ \alpha,\beta}}P_{kl\alpha\beta}q^{\alpha}\bar{q}^{\beta}e^{\mathrm{i}\langle k,\theta\rangle}I^{l}, \tag{2.27}$$

$$\acute{P}=\acute{P}(q,\bar{q};\omega)=\sum_{|\alpha|+|\beta|\leq 2}\acute{P}_{\alpha\beta}q^{\alpha}\bar{q}^{\beta}=\sum_{|\alpha|+|\beta|\leq 2}P_{00\alpha\beta}q^{\alpha}\bar{q}^{\beta}, \tag{2.28}$$

$$\grave{P}=\grave{P}(q,\bar{q};\omega)=\sum_{|\alpha|+|\beta|\geq 3}\grave{P}_{\alpha\beta}q^{\alpha}\bar{q}^{\beta}=\sum_{|\alpha|+|\beta|\geq 3}P_{00\alpha\beta}q^{\alpha}\bar{q}^{\beta}, \tag{2.29}$$

then the coefficients satisfy

$$\|\breve{P}_{\alpha\beta}\|\leq ce^{-\varrho n^{*}},\quad |\alpha|+|\beta|\geq 1, \tag{2.30}$$

$$\|\acute{P}_{\alpha\beta}\|\leq ce^{-\varrho n^{*}},\quad 1\leq|\alpha|+|\beta|\leq 2, \tag{2.31}$$

$$\|\grave{P}_{\alpha\beta}\|\leq ce^{-\varrho(n^{+}-n^{-})},\quad |\alpha|+|\beta|\geq 3 \tag{2.32}$$

for some positive constant c and ϱ, where

$$\begin{aligned}
n^{+}&=n^{+}(\alpha,\beta)=\max\{n\in\mathbb{Z}_1:(\alpha_n,\beta_n)\neq 0\},\\
n^{-}&=n^{-}(\alpha,\beta)=\min\{n\in\mathbb{Z}_1:(\alpha_n,\beta_n)\neq 0\},\\
n^{*}&=n^{*}(\alpha,\beta)=\max\{|n^{+}|,|n^{-}|\}.
\end{aligned}$$

$(\mathbf{A6})$ *Gauge invariance of the perturbation:* We expand the perturbation P into the Taylor-Fourier series with respect to θ, I, q, $\bar{q}$:

$$P=\sum_{\substack{k\in\mathbb{Z}^b,l\in\mathbb{N}^b\\ \alpha,\beta}}P_{kl\alpha\beta}(\omega)I^{l}e^{\mathrm{i}\langle k,\theta\rangle}q^{\alpha}\bar{q}^{\beta},$$

then the coefficients $P_{kl\alpha\beta}(\omega)\equiv 0$ if $\sum_{j=1}^{b}k_j+|\alpha|-|\beta|\neq 0$.

Our abstract KAM theorem states as follows.

Theorem 2 *Assume that the unperturbed Hamiltonian N in (2.19) satisfies* $(\mathbf{A1})-(\mathbf{A3})$*, and P satisfies* $(\mathbf{A4})-(\mathbf{A6})$*. Let $\gamma>0$ small enough, there is a positive constant $\varepsilon_0=\varepsilon_0(\mathcal{O},K_0,\gamma,r,s)\sim\gamma^8$ and $X_{\varepsilon_0}\subset\mathbb{R}^{\mathbb{Z}_1}$ with*

$$\mathrm{prob}(X_\gamma)>e^{-\gamma^{\sigma}}$$

with some $0<\sigma<1$ such that if $\|X_P\|_{D_\rho(r,s),\mathcal{O}}<\varepsilon_0$ and $\{\Omega_n\}_{n\in\mathbb{Z}_1}\in X_\gamma$ is fixed, then the following holds.

There exist a Cantor set $\mathcal{O}_\gamma\subset\mathcal{O}$ with $|\mathcal{O}\setminus\mathcal{O}_\gamma|=O(\gamma)$ and maps

$$\Psi:\mathbb{T}^b\times\mathcal{O}_\gamma\to D_\rho(r,s),\quad \tilde{\omega}:\mathcal{O}_\gamma\to\mathbb{R}^b,$$

which are real-analytic in θ and C^1_W-smooth in ω with $\|\Psi-\Psi_0\|_{D_0(\frac{r}{2},0),\mathcal{O}_\gamma}\to 0$ and $|\tilde{\omega}(\omega)-\omega|\to 0$ as $\gamma\to 0$, where Ψ_0 is the trivial embedding: $\mathbb{T}^b\times\mathcal{O}\to\mathbb{T}^b\times\{0,0\}$, such that each $\omega\in\mathcal{O}_\gamma$ and $\theta\in\mathbb{T}^b$ correspond to a linear stable, b-frequency quasi-periodic solution $\Psi(\theta,\omega)=(\theta+\tilde{\omega}t,q_n(t),\bar{q}_n(t))$ of equations of motion associated with the Hamiltonian (2.21).

2.4 Proof of Theorem 1

In Theorem 1, the Hamiltonian function associated with the lattice equation is

$$H = \Lambda + G \tag{2.33}$$

with

$$\Lambda := \sum_{n\in\mathbb{Z}} v_n q_n \bar{q}_n,$$

and

$$G := \frac{1}{2}\delta \sum_{n\in\mathbb{Z}} |q_n|^4 + \sum_{n\in\mathbb{Z}} \varepsilon_n \bar{q}_n (q_{n+1} + q_{n-1}),$$

where $\{v_n\}_{n\in\mathbb{Z}} \in [0,1]^{\mathbb{Z}}$ is a family of i.i.d. random variables with common distribution g satisfying (1.2), and and $|\varepsilon_n| \leq \varepsilon e^{-\varrho|n|}$ with $\varepsilon, \varrho > 0$. The symplectic structure is $\mathrm{i}\sum_{n\in\mathbb{Z}} dq_n \wedge d\bar{q}_n$.

Moreover, the perturbation G in (2.33) has the following regularity property.

Lemma 2.3 *For any fixed $0 < \rho < \varrho$, the gradient $G_{\bar{q}}$ is real analytic as a map in a neighborhood of the origin in $\ell^1_\rho(\mathbb{Z})$ into $\ell^1_\rho(\mathbb{Z})$ with*

$$\|G_{\bar{q}}\|_\rho \leq c \max\{\varepsilon, \delta\} \|q\|_\rho.$$

Proof: Since $G = \frac{1}{2}\delta \sum_{n\in\mathbb{Z}} |q_n|^4 + \sum_{n\in\mathbb{Z}} \varepsilon_n \bar{q}_n (q_{n+1} + q_{n-1})$, we have that

$$\|G_{\bar{q}}\|_\rho = \sum_{n\in\mathbb{Z}} \left|\frac{\partial G}{\partial \bar{q}_n}\right| e^{|n|\rho} \leq \delta \sum_{n\in\mathbb{Z}} |q_n^2 \bar{q}_n| e^{|n|\rho} + \sum_{n\in\mathbb{Z}} \varepsilon_n (|q_{n+1}| + |q_{n-1}|) e^{|n|\rho} \leq c \max\{\varepsilon, \delta\} \|q\|_\rho,$$

where

$$\delta \sum_{n\in\mathbb{Z}} |q_n^2 \bar{q}_n| e^{|n|\rho} \leq c\delta \|q\|^3_\rho,$$

and

$$\sum_{n\in\mathbb{Z}} \varepsilon_n (|q_{n+1}| + |q_{n-1}|) e^{|n|\rho} \leq c\varepsilon \|q\|_\rho.$$

Then the regularity of $G_{\bar{q}}$ is proved. ■

Next, fix $\mathcal{J} = \{n_1, \cdots, n_b\}$, and $\mathbb{Z}_1 = \mathbb{Z} \setminus \mathcal{J}$. We introduce action-angle variables and parameters to the Hamiltonian function (2.33). Fix $\xi = (\xi_{n_1}, \cdots, \xi_{n_b})$ with $0 < \xi_{n_i} < \varepsilon$, $i = 1, \cdots, b$ and $(I, \theta) = (I_{n_1}, \cdots, I_{n_b}, \theta_{n_1}, \cdots, \theta_{n_b})$ be the standard action-angle variables in the $(q_n, \bar{q}_n)_{n\in\mathcal{J}}$-space around ξ. Then

$$q_{n_1} = \sqrt{I_{n_1} + \xi_{n_1}} e^{\mathrm{i}\theta_{n_1}}, \cdots, q_{n_b} = \sqrt{I_{n_b} + \xi_{n_b}} e^{\mathrm{i}\theta_{n_b}},$$

$$\bar{q}_{n_1} = \sqrt{I_{n_1} + \xi_{n_1}} e^{-\mathrm{i}\theta_{n_1}}, \cdots, \bar{q}_{n_b} = \sqrt{I_{n_b} + \xi_{n_b}} e^{-\mathrm{i}\theta_{n_b}},$$

denote the remaining normal coordinates by $(q, \bar{q})$, and the Hamiltonian (2.33) becomes

$$H = e + \langle \omega, I \rangle + \sum_{n\in\mathbb{Z}_1} \Omega_n |q_n|^2 + P(\theta, I, q, \bar{q}; \omega),$$

where

$$
\begin{aligned}
e &= \sum_{n\in\mathcal{J}}(v_n\xi_n+\frac{1}{2}\delta\xi_n^2),\\
\omega &= (v_{n_1}+\xi_{n_1},\cdots,v_{n_b}+\xi_{n_b}),\\
\Omega_n &= v_n,\quad n\in\mathbb{Z}_1,\\
P &= \frac{1}{2}\delta\sum_{n\in\mathbb{Z}_1}|q_n|^4+\sum_{\substack{n\notin\mathcal{J}\\ n+1\notin\mathcal{J}}}\varepsilon_n\bar{q}_nq_{n+1}+\sum_{\substack{n\notin\mathcal{J}\\ n-1\notin\mathcal{J}}}\varepsilon_n\bar{q}_nq_{n-1} \qquad (2.34)\\
&+\sum_{\substack{n\in\mathcal{J}\\ n+1\notin\mathcal{J}}}\varepsilon_n\sqrt{I_n+\xi_n}e^{-\mathrm{i}\theta_n}q_{n+1}+\sum_{\substack{n\in\mathcal{J}\\ n-1\notin\mathcal{J}}}\varepsilon_n\sqrt{I_n+\xi_n}e^{-\mathrm{i}\theta_n}q_{n-1} \qquad (2.35)\\
&+\sum_{\substack{n\notin\mathcal{J}\\ n+1\in\mathcal{J}}}\varepsilon_n\sqrt{I_{n+1}+\xi_{n+1}}e^{\mathrm{i}\theta_{n+1}}\bar{q}_n+\sum_{\substack{n\notin\mathcal{J}\\ n-1\in\mathcal{J}}}\varepsilon_n\sqrt{I_{n-1}+\xi_{n-1}}e^{\mathrm{i}\theta_{n-1}}\bar{q}_n \qquad (2.36)\\
&+\sum_{\substack{n\in\mathcal{J}\\ n+1\in\mathcal{J}}}\varepsilon_n\sqrt{I_n+\xi_n}\sqrt{I_{n+1}+\xi_{n+1}}e^{-\mathrm{i}(\theta_n-\theta_{n+1})} \qquad (2.37)\\
&+\sum_{\substack{n\in\mathcal{J}\\ n-1\in\mathcal{J}}}\varepsilon_n\sqrt{I_n+\xi_n}\sqrt{I_{n-1}+\xi_{n-1}}e^{-\mathrm{i}(\theta_n-\theta_{n-1})}+\frac{1}{2}\delta\sum_{n\in\mathcal{J}}I_n^2 \qquad (2.38)\\
&=: \sum_{\alpha,\beta}\check{P}_{\alpha\beta}q^\alpha\bar{q}^\beta+\sum_{|\alpha|+|\beta|\le 2}\acute{P}_{\alpha\beta}q^\alpha\bar{q}^\beta+\sum_{|\alpha|+|\beta|\ge 3}\acute{P}_{\alpha\beta}q^\alpha\bar{q}^\beta.
\end{aligned}
$$

Now we show that this Hamiltonian satisfies the assumptions (**A1**)$-$(**A6**) of the KAM theorem.
Verification of (**A1**): Since $\{v_n\}_{n\in\mathbb{Z}}$ is a family of i.i.d. random variables, for each $n\in\mathbb{Z}_1$,

$\Omega_n=v_n$ is independent of $\omega=(v_{n_1}+\xi_{n_1},\cdots,v_{n_b}+\xi_{n_b})$.

Verification of (**A2**): First, we order the integers such that $n\in\mathbb{Z}_1$ and $|n|\le K_0$ as

$$j_1<j_2<\cdots<j_N,$$

where $N\le 2K_0+1$ denotes the number of such integers. Then we choose any value $v_{j_1}\in[0,1]$ for Ω_{j_1}. With $\Omega_{j_1}=v_{j_1}$ fixed, we have that

$$\mathrm{mes}\left\{v_{j_2}:|v_{j_2}-\Omega_{j_1}|<\frac{\gamma}{|j_2-j_1|^\tau}\right\}<\frac{2\gamma}{|j_2-j_1|^\tau}.$$

Excluding the set of such values for v_{j_2}, we can choose any value left for Ω_{j_2}. Now we proceed inductively. With $\Omega_{j_1}=v_{j_1},\cdots,\Omega_{j_i}=v_{j_i}$, $1<i\le N-1$ fixed, we choose $\Omega_{j_{i+1}}=v_{j_{i+1}}$ such that $v_{j_{i+1}}$ does not belong to the set

$$\left\{v_{j_{i+1}}:|v_{j_{i+1}}-\Omega_j|<\frac{\gamma}{|j_{i+1}-j|^\tau},\quad j=j_1,\cdots,j_i\right\},$$

whose measure is less than $c\gamma$. Thus (2.22) holds for any $n\neq m$ and $|n|,|m|\le K_0=c\ln\frac{1}{\gamma}$. The product measure of the set of remaining values for the variables $\{v_n\}_{|n|\le K_0}$ is not less than

$$(1-c\gamma)^{cK_0}\ge e^{-\gamma^{\frac{1}{2}}},$$

if γ is small enough.

Verification of (**A3**): We check (2.26), which is the most complicated case. For any $k \neq 0$ and $|n|, |m| \leq K_0$ fixed,

$$\left|\frac{\partial(\langle k, \omega\rangle + \Omega_m - \Omega_n)}{\partial\omega}\right| \geq \frac{1}{2}|k|$$

Therefore, by excluding some parameter set with measure $O(\gamma)$, we have that

$$|\langle k, \omega\rangle + \Omega_m - \Omega_n| \geq \frac{\gamma}{|k|^\tau}.$$

We can show that (2.23)–(2.25) hold similarly, so (**A3**) is verified.

Verification of (**A4**): By Lemma 2.3, together with Lemma 6.2 and Lemma 6.3, we obtain that

Lemma 2.4 *For any $\varepsilon > 0$ sufficiently small and $s \leq \varepsilon$, if $|I| < s^2$ and $\|q\|_\rho < s$, then*

$$\|X_P\|_{D_\rho(r,s),\mathcal{O}} \leq \varepsilon.$$

Verification of (**A5**): We focus on the expression of P. The (I, θ)-dependent terms of P are (2.35) – (2.38), whose coefficients corresponding to q_n, $\bar{q}_n$(or q_{n+1}, $\bar{q}_{n+1}$, q_{n-1}, $\bar{q}_{n-1}$) are not more than $c\varepsilon_n \leq ce^{-\varrho|n|}$. This means (2.30) holds. Since

$$\acute{P} = \sum_{\substack{n\notin\mathcal{J}\\ n+1\notin\mathcal{J}}} \varepsilon_n\bar{q}_n q_{n+1} + \sum_{\substack{n\notin\mathcal{J}\\ n-1\notin\mathcal{J}}} \varepsilon_n\bar{q}_n q_{n-1},$$

and

$$\grave{P} = \frac{1}{2}\delta \sum_{n\in\mathbb{Z}_1} |q_n|^4,$$

(2.31) and (2.32) is obviously verified.

Verification of (**A6**): It is obvious that the initial perturbation $\frac{1}{2}\sum_{n\in\mathbb{Z}} \delta|q_n|^4 + \sum_{n\in\mathbb{Z}} \varepsilon_n\bar{q}_n(q_{n+1} + q_{n-1})$ has gauge invariance. After introducing the action-angle variables, any term $e^{\mathrm{i}\langle k,\theta\rangle}$ originates from $\prod_{\substack{(\alpha_n,\beta_n)\neq 0\\ n\in\mathcal{J}}} q_n^{\alpha_n}\bar{q}_n^{\beta_n}$, and we have that

$$\sum_{j=1}^{b} k_j = \sum_{n\in\mathcal{J}} \alpha_n - \sum_{n\in\mathcal{J}} \beta_n.$$

Then $\sum_{j=1}^b k_j + |\alpha| - |\beta|$ remains zero if its initial value $\sum_{n\in\mathbb{Z}} \alpha_n - \sum_{n\in\mathbb{Z}} \beta_n$ is zero. Thus (**A6**) is verified.

Thus Theorem 1 can be viewed as a corollary of Theorem 2.

3 KAM step

In this section we present the KAM iteration scheme applied to (2.33). This is a succession of infinitely many steps whose purpose is to eliminate lower-order θ-dependent terms in P. At each KAM step the perturbation is made smaller at the cost of excluding a small-measure set of parameters. It will be shown that the KAM iterations converge and that, in the end, the total measure of the set of parameters that has been excluded is small.

At the ν^{th} step of the KAM iteration, we consider a Hamiltonian vector field with

$$\begin{aligned} H_\nu &= N_\nu + P_\nu \\ &= e_\nu + \langle \tilde{\omega}_\nu(\omega), I\rangle + \sum_{n\in\mathbb{Z}_1} \Omega_n^\nu(\omega) q_n \bar{q}_n + P_\nu(\theta, I, q, \bar{q}; \omega), \end{aligned}$$

where N_ν is an "integrable normal form", $P_\nu \in \mathcal{A}$ with decay property is defined in $D_{\rho_\nu}(r_\nu, s_\nu) \times \mathcal{O}_\nu$.

Assume that at the ν^{th} step, $\nu \geq 1$, the frequencies have the following properties. The tangential frequencies

$$\tilde{\omega}_\nu(\omega) = \omega + \hat{\omega}_\nu(\omega), \quad \omega \in \mathcal{O}_\nu, \tag{3.1}$$

where $\hat{\omega}_\nu(\omega)$ is a C_W^1 function of ω with C_W^1-norm bounded by ε_0. $\{\Omega_n^\nu(\omega)\}_{n\in\mathbb{Z}_1}$ satisfies

$$\Omega_n^\nu(\omega) = \begin{cases} \Omega_n^0 + \hat{\Omega}_n^\nu(\omega), & |n| \leq \ln \frac{1}{\varepsilon_\nu}, \\ \Omega_n^0, & |n| > \ln \frac{1}{\varepsilon_\nu}. \end{cases} \tag{3.2}$$

with $\{\Omega_n^0\}_{n\in\mathbb{Z}_1} \in X_\nu$ being the initial normal frequencies and $\hat{\Omega}_n^\nu(\omega)$'s are C_W^1 functions of ω with C_W^1-norm bounded by ε_0.

We then construct a map

$$\Phi_\nu : D_{\rho_{\nu+1}}(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_{\nu+1} \to D_{\rho_\nu}(r_\nu, s_\nu) \times \mathcal{O}_\nu$$

so that the vector field $X_{H_\nu\circ\Phi_\nu}$ defined on $D_{\rho_{\nu+1}}(r_{\nu+1}, s_{\nu+1})$ satisfies

$$\|X_{P_{\nu+1}}\|_{D_{\rho_{\nu+1}}(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}} = \|X_{H_\nu\circ\Phi_\nu} - X_{N_{\nu+1}}\|_{D_{\rho_{\nu+1}}(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}} \leq \varepsilon_\nu^\kappa, \quad \kappa > 1$$

with some new normal form $N_{\nu+1}$, which has properties similar to that of N_ν. Moreover, the new perturbation $P_{\nu+1}$ still has the gauge invariance and the corresponding decay property. Here, the quantities r_ν and ρ_ν satisfies that, $\frac{1}{2}r < \rho_\nu \leq r_{\nu+1}$.

To simplify notations, in what follows, the quantities without subscripts refer to quantities at the ν^{th} step, while the quantities with subscripts $+$ denote the corresponding quantities at the $(\nu+1)^{\text{th}}$ step. We now let $0 < r_+ < r$ and define

$$s_+ = \frac{1}{4} s \varepsilon^{\frac{1}{3}}, \quad \varepsilon_+ = c\gamma^{-2}(r - r_+)^{-c} \varepsilon^{\frac{6}{5}}. \tag{3.3}$$

Here and later, the letter c denotes suitable (possibly different) constants that do not depend on the iteration steps.

Let us then consider the Hamiltonian

$$H = N + P \equiv e + \langle \tilde{\omega}, I\rangle + \sum_{n\in\mathbb{Z}_1} \Omega_n(\omega) q_n \bar{q}_n + P(\theta, I, q, \bar{q}; \omega) \tag{3.4}$$

defined in $D_\rho(r,s)\times\mathcal{O}$. We assume that for each $k\neq 0$, $\tilde\omega$ and $\{\Omega_n(\omega)\}_{|n|\le \ln\frac{1}{\varepsilon}}$ satisfies

$$|\Omega_n(\omega)-\Omega_m(\omega)|\ge \frac{\gamma}{|n-m|^\tau},\quad n\neq m, \tag{3.5}$$

$$|\langle k,\tilde\omega\rangle|\ge \frac{\gamma}{|k|^\tau}, \tag{3.6}$$

$$|\langle k,\tilde\omega\rangle+\Omega_n(\omega)|\ge \frac{\gamma}{|k|^\tau}, \tag{3.7}$$

$$|\langle k,\tilde\omega\rangle+\Omega_n(\omega)+\Omega_m(\omega)|\ge \frac{\gamma}{|k|^\tau}, \tag{3.8}$$

$$|\langle k,\tilde\omega\rangle+\Omega_n(\omega)-\Omega_m(\omega)|\ge \frac{\gamma}{|k|^\tau}, \tag{3.9}$$

with $\gamma>0$, $\tau>b$, while $\{\Omega_n\}_{|n|>\ln\frac{1}{\varepsilon}}$ is independent of ω. As for P, we have that

$$\|X_P\|_{D_\rho(r,s),\mathcal{O}}\le\varepsilon, \tag{3.10}$$

and $P=\sum_{k,l,\alpha,\beta}P_{kl\alpha\beta}I^l e^{\mathrm{i}\langle k,\theta\rangle}q^\alpha\bar q^\beta$ has the gauge invariance. Moreover, if we write that $P=\breve P+\acute P+\grave P$, where

$$\begin{aligned}
\breve P&=\breve P(\theta,I,q,\bar q;\omega)=\sum_{\alpha,\beta}\breve P_{\alpha\beta}q^\alpha\bar q^\beta=\sum_{\substack{(k,l)\neq 0\\ \alpha,\beta}}P_{kl\alpha\beta}q^\alpha\bar q^\beta e^{\mathrm{i}\langle k,\theta\rangle}I^l,\\
\acute P&=\acute P(q,\bar q;\omega)=\sum_{|\alpha|+|\beta|\le 2}\acute P_{\alpha\beta}q^\alpha\bar q^\beta=\sum_{|\alpha|+|\beta|\le 2}P_{00\alpha\beta}q^\alpha\bar q^\beta,\\
\grave P&=\grave P(q,\bar q;\omega)=\sum_{|\alpha|+|\beta|\ge 3}\grave P_{\alpha\beta}q^\alpha\bar q^\beta=\sum_{|\alpha|+|\beta|\ge 3}P_{00\alpha\beta}q^\alpha\bar q^\beta,
\end{aligned}$$

then P has decay property, i.e.

$$\begin{aligned}
&\|\breve P_{\alpha\beta}\|\le ce^{-\varrho n^*},\quad |\alpha|+|\beta|\ge 1,\\
&\|\acute P_{\alpha\beta}\|\le ce^{-\varrho n^*},\quad 1\le|\alpha|+|\beta|\le 2,\\
&\|\grave P_{\alpha\beta}\|\le ce^{-\varrho(n^+-n^-)},\quad |\alpha|+|\beta|\ge 3,
\end{aligned}$$

where

$$\begin{aligned}
n^+&=n^+(\alpha,\beta)=\max\{n\in\mathbb{Z}:(\alpha_n,\beta_n)\neq 0\},\\
n^-&=n^-(\alpha,\beta)=\min\{n\in\mathbb{Z}:(\alpha_n,\beta_n)\neq 0\},\\
n^*&=n^*(\alpha,\beta)=\max\{|n^+|,|n^-|\}.
\end{aligned}$$

We now describe how to construct a set $\mathcal{O}_+\subset\mathcal{O}$ and a change of variables $\Phi: D_+\times\mathcal{O}_+=D_{\rho_+}(r_+,s_+)\times\mathcal{O}_+\to D_\rho(r,s)\times\mathcal{O}$ such that the transformed Hamiltonian $H_+=N_++P_+:=H\circ\Phi$ satisfies all the above iterative assumptions with new parameters s_+, ε_+, r_+, γ_+ and with $\omega_+\in\mathcal{O}_+$.

3.1 Solving the linearized equations

Expand P into the Fourier-Taylor series

$$P = \sum_{k,l,\alpha,\beta} P_{kl\alpha\beta} e^{\mathrm{i}\langle k,\theta\rangle} \; I^l q^\alpha \bar{q}^\beta$$

where $k \in \mathbb{Z}^b, l \in \mathbb{N}^b$ and the multi–indices α and β run over the set of all infinite dimensional vectors $\alpha \equiv (\cdots, \alpha_n, \cdots)_{n\in\mathbb{Z}_1}$, $\beta \equiv (\cdots, \beta_n, \cdots)_{n\in\mathbb{Z}_1}$, $\alpha_n, \beta_n \in \mathbb{N}$ with finitely many nonzero components of positive integers.

Let R be the truncation of P given by

$$\begin{aligned} R(\theta, I, q, \bar{q}) &= \breve{R} + \acute{R} \\ &= \sum_{\substack{n^* \le \ln \frac{1}{\varepsilon} \\ 2|l|+|\alpha|+|\beta| \le 2}} \sum_{(k,l)\neq 0} P_{kl\alpha\beta} q^\alpha \bar{q}^\beta + \sum_{\substack{n^* \le \ln \frac{1}{\varepsilon} \\ |\alpha|+|\beta| \le 2}} P_{00\alpha\beta} q^\alpha \bar{q}^\beta. \end{aligned} \tag{3.11}$$

Hence

$$P - R = \sum_{\substack{n^* > \ln \frac{1}{\varepsilon} \\ 2|l|+|\alpha+\beta| \le 2}} \sum_{(k,l)\neq 0} P_{kl\alpha\beta} q^\alpha \bar{q}^\beta + \sum_{2|l|+|\alpha+\beta|\ge 3} \sum_{(k,l)\neq 0} P_{kl\alpha\beta} q^\alpha \bar{q}^\beta + \grave{P}.$$

Since $P \in \mathcal{A}$, we can rewrite $\breve{R}$ and $\acute{R}$ as

$$\begin{aligned} \breve{R} &= \sum_{\substack{(k,l)\neq 0 \\ |l|\le 1}} P_{kl00} e^{\mathrm{i}\langle k,\theta\rangle} I^l + \sum_{\substack{k\neq 0 \\ |n| \le \ln \frac{1}{\varepsilon}}} (P_n^{k10} q_n + P_n^{k01} \bar{q}_n) \\ &\quad + \sum_{\substack{k\neq 0 \\ |n|,|m| \le \ln \frac{1}{\varepsilon}}} (P_{nm}^{k20} q_n q_m + P_{nm}^{k02} \bar{q}_n \bar{q}_m) + \sum_{\substack{k \\ |n|,|m| \le \ln \frac{1}{\varepsilon}}} P_{nm}^{k11} q_n \bar{q}_m, \\ \acute{R} &= \sum_{|n|,|m| \le \ln \frac{1}{\varepsilon}} P_{nm}^{011} q_n \bar{q}_m + P_{0000}, \end{aligned}$$

where $P_n^{k10} = P_{kl\alpha\beta}$ with $\alpha = e_n, \beta = 0$, here e_n denotes the vector with the n^{th} component being 1 and the other components being zero; $P_n^{k01} = P_{kl\alpha\beta}$ with $\alpha = 0, \beta = e_n$; $P_{nm}^{k20} = P_{kl\alpha\beta}$ with $\alpha = e_n + e_m, \beta = 0$; $P_{nm}^{k11} = P_{kl\alpha\beta}$ with $\alpha = e_n, \beta = e_m$; $P_{nm}^{k02} = P_{kl\alpha\beta}$ with $\alpha = 0, \beta = e_n + e_m$. Due to the assumption $(A4)$, $P \in \mathcal{A}$ implies that

$$P_{kl00} = 0, \qquad \text{if } \sum_{j=1}^{b} k_j \neq 0$$

$$P_n^{k10} = 0, \qquad \text{if } \sum_{j=1}^{b} k_j + 1 \neq 0$$

$$P_n^{k01} = 0, \qquad \text{if } \sum_{j=1}^{b} k_j - 1 \neq 0$$

$$P_{nm}^{k20} = 0, \qquad \text{if } \sum_{j=1}^{b} k_j + 2 \neq 0$$

$$P_{nm}^{k11}=0, \qquad \text{if } \sum_{j=1}^{b}k_j\neq 0$$

$$P_{nm}^{k02}=0, \qquad \text{if } \sum_{j=1}^{b}k_j-2\neq 0$$

Rewrite H as $H=N+R+(P-R)$. By the choice of s_+ in (3.3) and the definition of the norms, it follows immediately that

$$\|X_R\|_{D_\rho(r,s),\mathcal{O}}\leq \|X_P\|_{D_\rho(r,s),\mathcal{O}}\leq \varepsilon. \tag{3.12}$$

Moreover, we take $s_+\ll s$ such that in a domain $D_\rho(r,s_+)$,

$$\|X_{(P-R)}\|_{D_\rho(r,s_+)}\leq c\varepsilon_+. \tag{3.13}$$

In the following, we will look for an F in the class $\mathcal{A}$, defined in a domain $D_+=D_{\rho_+}(r_+,s_+)$, such that the time one map Φ_F^1 of the Hamiltonian vector field X_F defines a map from D_+ to D and transforms H into H_+. More precisely, by second order Taylor formula, we have

$$\begin{aligned} H\circ\Phi_F^1 &= (N+R)\circ\Phi_F^1+(P-R)\circ\Phi_F^1\\ &= N+\{N,F\}+R\\ &\quad +\int_0^1(1-t)\{\{N,F\},F\}\circ\Phi_F^t dt+\int_0^1\{R,F\}\circ\Phi_F^t dt+(P-R)\circ\Phi_F^1\\ &= N_++P_++\{N,F\}+R-P_{0000}-\langle\omega',I\rangle-\sum_{|n|\leq\ln\frac{1}{\varepsilon}}P_{nn}^{011}q_n\bar{q}_n, \end{aligned} \tag{3.14}$$

where

$$\omega'=\int\frac{\partial P}{\partial I}d\theta|_{q=\bar{q}=0,I=0},$$

$$N_+=N+P_{0000}+\langle\omega',I\rangle+\sum_{|n|\leq\ln\frac{1}{\varepsilon}}P_{nn}^{011}q_n\bar{q}_n,$$

$$P_+=\int_0^1(1-t)\{\{N,F\},F\}\circ\Phi_F^t dt+\int_0^1\{R,F\}\circ\Phi_F^t dt+(P-R)\circ\Phi_F^1.$$

We shall find a function $F\in\mathcal{A}$ of the form

$$\begin{aligned} F(\theta,I,q,\bar{q}) &= \breve{F}+\acute{F} \\ &= \sum_{\substack{k\neq 0\\ n^*\leq\ln\frac{1}{\varepsilon}\\ 2|l|+|\alpha|+|\beta|\leq 2}}F_{kl\alpha\beta}q^\alpha\bar{q}^\beta+\sum_{\substack{n^*\leq\ln\frac{1}{\varepsilon}\\ |\alpha|+|\beta|\leq 2\\ \alpha\neq\beta}}F_{00\alpha\beta}q^\alpha\bar{q}^\beta \end{aligned} \tag{3.15}$$

satisfying the equation

$$\{N,F\}+R-P_{0000}-\langle\omega',I\rangle-\sum_{|n|\leq\ln\frac{1}{\varepsilon}}P_{nn}^{011}q_n\bar{q}_n=0. \tag{3.16}$$

Similarly, we rewrite $\breve{F}$ and $\acute{F}$ as

$$
\begin{aligned}
\breve{F} &= \sum_{\substack{k\neq 0\\|l|\leq 1}} F_{kl00}e^{\mathrm{i}\langle k,\theta\rangle}I^l + \sum_{\substack{k\neq 0\\|n|\leq \ln\frac{1}{\varepsilon}}} (F_n^{k10}q_n + F_n^{k01}\bar{q}_n)\\
&\quad + \sum_{\substack{k\neq 0\\|n|,|m|\leq \ln\frac{1}{\varepsilon}}} (F_{nm}^{k20}q_nq_m + F_{nm}^{k11}q_n\bar{q}_m + F_{nm}^{k02}\bar{q}_n\bar{q}_m),\\
\acute{F} &= \sum_{\substack{|n|,|m|\leq \ln\frac{1}{\varepsilon}\\ n\neq m}} F_{nm}^{011}q_n\bar{q}_m + P_{0000}.
\end{aligned}
$$

Lemma 3.1 *Equation (3.16) is equivalent to the following system*

$$
\begin{aligned}
-(\Omega_n-\Omega_m)F_{nm}^{011} &= \mathrm{i}P_{nm}^{011}, \quad |n|,|m|\leq \ln\frac{1}{\varepsilon}, \quad n\neq m,\\
\langle k,\tilde{\omega}\rangle F_{kl00} &= \mathrm{i}P_{kl00}, \quad k\neq 0, \quad |l|\leq 1,\\
(\langle k,\tilde{\omega}\rangle-\Omega_n)F_n^{k10} &= \mathrm{i}P_n^{k10}, \quad k\neq 0, \quad |n|\leq \ln\frac{1}{\varepsilon},\\
(\langle k,\tilde{\omega}\rangle+\Omega_n)F_n^{k01} &= \mathrm{i}P_n^{k01}, \quad k\neq 0, \quad |n|\leq \ln\frac{1}{\varepsilon},\\
(\langle k,\tilde{\omega}\rangle-\Omega_n-\Omega_m)F_{nm}^{k20} &= \mathrm{i}P_{nm}^{k20}, \quad k\neq 0, \quad |n|,|m|\leq \ln\frac{1}{\varepsilon},\\
(\langle k,\tilde{\omega}\rangle-\Omega_n+\Omega_m)F_{nm}^{k11} &= \mathrm{i}P_{nm}^{k11}, \quad k\neq 0, \quad |n|,|m|\leq \ln\frac{1}{\varepsilon},\\
(\langle k,\tilde{\omega}\rangle+\Omega_n+\Omega_m)F_{nm}^{k02} &= \mathrm{i}P_{nm}^{k02}, \quad k\neq 0, \quad |n|,|m|\leq \ln\frac{1}{\varepsilon},
\end{aligned}
$$

where $\Omega = (\cdots,\Omega_n,\cdots)_{n\in\mathbb{Z}_1}$.

Proof: Inserting F defined in (3.15), into (3.16) one sees that (3.16) is equivalent to the following system of equations

$$\{N,\breve{F}\} + \breve{R} = \langle \omega', I\rangle, \tag{3.17}$$

$$\{N,\acute{F}\} + \acute{R} = P_{0000} + \sum_{|n|\leq \ln\frac{1}{\varepsilon}} P_{nn}^{011}q_n\bar{q}_n. \tag{3.18}$$

We note that

$$\{N,\acute{F}\} = \mathrm{i}\sum_{|n|,|m|\leq \ln\frac{1}{\varepsilon}} (\Omega_m-\Omega_n)F_{nm}^{011}q_n\bar{q}_m.$$

It follows that F_{nm}^{011}, are determined by the linear algebraic system

$$\mathrm{i}(\Omega_m-\Omega_n)F_{nm}^{011}q_n\bar{q}_m + P_{nm}^{011} = 0, \quad |n|,|m|\leq \ln\frac{1}{\varepsilon}, \quad n\neq m.$$

Similarly, from

$$\{N,\breve{F}\} = \mathrm{i}\sum_{\substack{k\neq 0\\|l|\leq 1}} \langle k,\tilde{\omega}\rangle F_{kl00}e^{\mathrm{i}\langle k,\theta\rangle}I^l$$

$$+\ \mathrm{i} \sum_{\substack{k\neq 0\\ |n|\leq \ln\frac{1}{\varepsilon}}} [(\langle k,\tilde\omega\rangle-\Omega_n)F_n^{k10}q_n+(\langle k,\omega\rangle+\Omega_n)F_n^{k01}\bar q_n]$$
$$+\ \mathrm{i} \sum_{\substack{k\neq 0\\ |n|,|m|\leq \ln\frac{1}{\varepsilon}}} [(\langle k,\tilde\omega\rangle-\Omega_n-\Omega_m)F_{nm}^{k20}q_nq_m+(\langle k,\tilde\omega\rangle-\Omega_n+\Omega_m)F_{nm}^{k11}q_n\bar q_m$$
$$+(\langle k,\tilde\omega\rangle+\Omega_n+\Omega_m)F_{nm}^{k02}\bar q_n\bar q_m],$$

it follows that F_n^{k10},F_n^{k01},F_{nm}^{k20}, F_{nm}^{k11} and F_{nm}^{k02} are determined by the following linear algebraic systems

$$\begin{aligned}
\mathrm{i}\langle k,\tilde\omega\rangle F_{kl00}+P_{kl00} &= 0, \quad k\neq 0, \quad |l|\leq 1,\\
\mathrm{i}(\langle k,\tilde\omega\rangle-\Omega_n)F_n^{k10}+P_n^{k10} &= 0, \quad k\neq 0, \quad |n|\leq \ln\frac{1}{\varepsilon},\\
\mathrm{i}(\langle k,\tilde\omega\rangle+\Omega_n)F_n^{k01}+P_n^{k01} &= 0, \quad k\neq 0, \quad |n|\leq \ln\frac{1}{\varepsilon},\\
\mathrm{i}(\langle k,\tilde\omega\rangle-\Omega_n-\Omega_m)F_{nm}^{k20}+P_{nm}^{k20} &= 0, \quad k\neq 0, \quad |n|,|m|\leq \ln\frac{1}{\varepsilon},\\
\mathrm{i}(\langle k,\tilde\omega\rangle-\Omega_n+\Omega_m)F_{nm}^{k11}+P_{nm}^{k11} &= 0, \quad k\neq 0, \quad |n|,|m|\leq \ln\frac{1}{\varepsilon},\\
\mathrm{i}(\langle k,\tilde\omega\rangle+\Omega_n+\Omega_m)F_{nm}^{k02}+P_{nm}^{k02} &= 0, \quad k\neq 0, \quad |n|,|m|\leq \ln\frac{1}{\varepsilon}.
\end{aligned}$$

Thus Lemma 3.1 is obtained. ∎

Remark. $P\in\mathcal{A}$ implies $F\in\mathcal{A}$.

3.2 Estimation on the coordinate transformation

We proceed to estimate X_F and Φ_F^1. We start with the following

Lemma 3.2 *Let* $D_i=D_{\rho_+}(r_++\frac{i}{4}(r-r_+),\frac{i}{4}s)$, $0<i\leq 4$. *If* $\varepsilon\ll(\frac{1}{2}\gamma^2(r-r_+)^c)^{\frac{3}{2}}$, *then*

$$\|X_F\|_{D_3,\mathcal{O}}\leq c\gamma^{-2}\varepsilon^{\frac{9}{10}}. \tag{3.19}$$

Proof: By the definition of $\mathcal{O}$, Lemma 3.1, and , we have that

$$|F_{nm}^{011}|_{\mathcal{O}}\leq|(\Omega_n-\Omega_m)^{-1}P_{nm}^{011}|_{\mathcal{O}}<c\gamma^{-2}|n-m|^{2\tau+1}|P_{nm}^{011}|_{\mathcal{O}}, \quad |n|,|m|\leq\ln\frac{1}{\varepsilon}, \quad n\neq m$$

and

$$\begin{aligned}
|F_{kl00}|_{\mathcal{O}} &\leq |\langle k,\tilde\omega\rangle^{-1}P_{kl00}|_{\mathcal{O}}<c\gamma^{-2}|k|^{2\tau+1}|P_{kl00}|_{\mathcal{O}}, \quad k\neq 0, \quad |l|\leq 1,\\
|F_n^{k10}|_{\mathcal{O}} &\leq c\gamma^{-2}|k|^{2\tau+1}|P_n^{k10}|_{\mathcal{O}}, \quad k\neq 0, \quad |n|\leq\ln\frac{1}{\varepsilon},\\
|F_n^{k01}|_{\mathcal{O}} &\leq c\gamma^{-2}|k|^{2\tau+1}|P_n^{k01}|_{\mathcal{O}}, \quad k\neq 0, \quad |n|\leq\ln\frac{1}{\varepsilon},\\
|F_{nm}^{k20}|_{\mathcal{O}} &\leq c\gamma^{-2}|k|^{2\tau+1}|P_{nm}^{k20}|_{\mathcal{O}}, \quad k\neq 0, \quad |n|,|m|\leq\ln\frac{1}{\varepsilon},\\
|F_{nm}^{k11}|_{\mathcal{O}} &\leq c\gamma^{-2}|k|^{2\tau+1}|P_{nm}^{k11}|_{\mathcal{O}}, \quad k\neq 0, \quad |n|,|m|\leq\ln\frac{1}{\varepsilon},\\
|F_{nm}^{k02}|_{\mathcal{O}} &\leq c\gamma^{-2}|k|^{2\tau+1}|P_{nm}^{k02}|_{\mathcal{O}}, \quad k\neq 0, \quad |n|,|m|\leq\ln\frac{1}{\varepsilon}.
\end{aligned}$$

It follows that

$$
\begin{aligned}
& \frac{1}{s^2}\|\breve{F}_\theta\|_{D_3,\mathcal{O}} \\
\le\ & \frac{1}{s^2}\sum_{\substack{k\neq 0\\ |l|\le 1}} |F_{kl00}|_{\mathcal{O}} s^{2|l|}|k|e^{|k|(r-\frac{1}{4}(r-r_+))} \\
& + \sum_{\substack{k\neq 0\\ |n|\le \ln\frac{1}{\varepsilon}}} (|F_n^{k10}|_{\mathcal{O}}|q_n| + |F_n^{k01}|_{\mathcal{O}}|\bar{q}_n|)|k|e^{|k|(r-\frac{1}{4}(r-r_+))} \\
& + \sum_{\substack{k\neq 0\\ |n|,|m|\le \ln\frac{1}{\varepsilon}}} (|F_{nm}^{k20}|_{\mathcal{O}}|q_n||q_m| + |F_{nm}^{k11}|_{\mathcal{O}}|q_n||\bar{q}_m| + |F_{nm}^{k02}|_{\mathcal{O}}|\bar{q}_n||\bar{q}_m|)|k|e^{|k|(r-\frac{1}{4}(r-r_+))} \\
\le\ & c\gamma^{-2}(r-r_+)^{-c}\|X_R\| \\
\le\ & c\gamma^{-2}(r-r_+)^{-c}\varepsilon.
\end{aligned}
$$

Similarly,

$$
\|\breve{F}_I\|_{D_3,\mathcal{O}} = \sum_{\substack{k\neq 0\\ |l|=1}} |F_{kl00}|e^{|k|(r-\frac{1}{4}(r-r_+))} \le c\gamma^{-2}(r-r_+)^{-c}\varepsilon.
$$

From

$$
\begin{aligned}
\|\breve{F}_{q_n}\|_{D_3,\mathcal{O}} &= \|\sum_{\substack{k\neq 0\\ |m|\le \ln\frac{1}{\varepsilon}}} F_{nm}^{k11}e^{\mathrm{i}\langle k,\theta\rangle}\bar{q}_m\|_{D_3,\mathcal{O}} + \|\sum_{\substack{k\neq 0\\ |m|\le \ln\frac{1}{\varepsilon}}} F_{nm}^{k20}e^{\mathrm{i}\langle k,\theta\rangle}q_m\|_{D_3,\mathcal{O}} \\
&\quad + \|\sum_{k\neq 0} F_n^{k10}e^{\mathrm{i}\langle k,\theta\rangle}\|_{D_3,\mathcal{O}} \\
&\le \sum_{\substack{k\neq 0\\ |m|\le \ln\frac{1}{\varepsilon}}} |F_{nm}^{k11}|_{\mathcal{O}}e^{|k|(r-\frac{1}{4}(r-r_+))}|\bar{q}_m| + \sum_{\substack{k\neq 0\\ |m|\le \ln\frac{1}{\varepsilon}}} |F_{nm}^{k20}|_{\mathcal{O}}e^{|k|(r-\frac{1}{4}(r-r_+))}|q_m| \\
&\quad + \sum_{k\neq 0} |F_n^{k10}|_{\mathcal{O}}e^{|k|(r-\frac{1}{4}(r-r_+))},
\end{aligned}
$$

and

$$
\begin{aligned}
\|\breve{F}_{\bar{q}_n}\|_{D_3,\mathcal{O}} &= \|\sum_{\substack{k\neq 0\\ |m|\le \ln\frac{1}{\varepsilon}}} F_{mn}^{k11}e^{\mathrm{i}\langle k,\theta\rangle}q_m\|_{D_3,\mathcal{O}} + \|\sum_{\substack{k\neq 0\\ |m|\le \ln\frac{1}{\varepsilon}}} F_{nm}^{k02}e^{\mathrm{i}\langle k,\theta\rangle}\bar{q}_m\|_{D_3,\mathcal{O}} \\
&\quad + \|\sum_{k\neq 0} F_n^{k01}e^{\mathrm{i}\langle k,\theta\rangle}\|_{D_3,\mathcal{O}} \\
&\le \sum_{\substack{k\neq 0\\ |m|\le \ln\frac{1}{\varepsilon}}} |F_{mn}^{k11}|_{\mathcal{O}}e^{|k|(r-\frac{1}{4}(r-r_+))}|q_m| + \sum_{\substack{k\neq 0\\ |m|\le \ln\frac{1}{\varepsilon}}} |F_{nm}^{k02}|_{\mathcal{O}}e^{|k|(r-\frac{1}{4}(r-r_+))}|\bar{q}_m| \\
&\quad + \sum_{k\neq 0} |F_n^{k01}|_{\mathcal{O}}e^{|k|(r-\frac{1}{4}(r-r_+))},
\end{aligned}
$$

we have that

$$
\|X_{\breve{F}}\|_{D_3,\mathcal{O}} = \|\breve{F}_I\|_{D_3,\mathcal{O}} + \frac{1}{s^2}\|\breve{F}_\theta\|_{D_3,\mathcal{O}} + \frac{1}{s}(\sum_{n\in\mathbb{Z}_1}\|\breve{F}_{q_n}\|_{D_3,\mathcal{O}}e^{|n|\rho_+} + \sum_{n\in\mathbb{Z}_1}\|\breve{F}_{\bar{q}_n}\|_{D_3,\mathcal{O}}e^{|n|\rho_+})
$$

$$
\begin{aligned}
&\leq c\gamma^{-2}(r-r_+)^{-c}\|X_R\| \\
&\leq c\gamma^{-2}(r-r_+)^{-c}\varepsilon.
\end{aligned}
$$

Since

$$
\begin{aligned}
\|\acute{F}_{q_n}\|_{D_3,\mathcal{O}} &= \|\sum_{\substack{n\neq m\\ |n|,|m|\leq \ln\frac{1}{\varepsilon}}} F^{011}_{nm}\bar{q}_m\|_{D_3,\mathcal{O}} \\
&\leq c\gamma^{-2}\sum_{\substack{n\neq m\\ |n|,|m|\leq \ln\frac{1}{\varepsilon}}} |n-m|^{2\tau+1}|P^{011}_{nm}|_{\mathcal{O}}|\bar{q}_m|,
\end{aligned}
$$

and

$$
\begin{aligned}
\|\acute{F}_{\bar{q}_n}\|_{D_3,\mathcal{O}} &= \|\sum_{\substack{n\neq m\\ |n|,|m|\leq \ln\frac{1}{\varepsilon}}} F^{011}_{mn}q_m\|_{D_3,\mathcal{O}} \\
&\leq c\gamma^{-2}\sum_{\substack{n\neq m\\ |n|,|m|\leq \ln\frac{1}{\varepsilon}}} |n-m|^{2\tau+1}|P^{011}_{mn}|_{\mathcal{O}}|q_m|,
\end{aligned}
$$

we have that

$$
\begin{aligned}
\|X_{\acute{F}}\|_{D_3,\mathcal{O}} &\leq \frac{1}{s}(\sum_n \|\acute{F}_{q_n}\|_{D_3,\mathcal{O}}e^{|n|\rho_+} + \sum_n \|\acute{F}_{\bar{q}_n}\|_{D_3,\mathcal{O}}e^{|n|\rho_+}) \\
&\leq c\gamma^{-2}\left(\ln\frac{1}{\varepsilon}\right)^{2\tau+1}\|X_R\| \\
&\leq c\gamma^{-2}\left(\ln\frac{1}{\varepsilon}\right)^{2\tau+1}\varepsilon.
\end{aligned}
$$

Under the assumption that $\varepsilon \ll (\frac{1}{2}\gamma^2(r-r_+)^c)^{\frac{3}{2}}$,

$$
\max\{\left(\ln\frac{1}{\varepsilon}\right)^{2\tau+1}, (r-r_+)^{-c}\} < \varepsilon^{-\frac{1}{10}}.
$$

Then the conclusion of the lemma follows from the estimates above. ∎

In the next lemma, we give some estimates for Φ^t_F. The formula (3.20) will be used to prove our coordinate transformation is well defined. Inequality (3.21) will be used to check the convergence of the iteration.

Lemma 3.3 *Let* $\eta = \varepsilon^{\frac{1}{3}}, D_{i\eta} = D_{\rho_+}(r_+ + \frac{i}{4}(r-r_+), \frac{i}{4}\eta s), 0 < i \leq 4$. *If* $\varepsilon \ll (\frac{1}{2}\gamma^2(r-r_+)^c)^{\frac{3}{2}}$, *we then have*

$$
\Phi^t_F : D_{2\eta} \to D_{3\eta}, \quad -1 \leq t \leq 1, \tag{3.20}
$$

Moreover,

$$
\|D\Phi^t_F - Id\|_{D_{1\eta}} < c\gamma^{-2}\varepsilon^{\frac{9}{10}}. \tag{3.21}
$$

Proof: Let

$$
\|D^m F\|_{D,\mathcal{O}} = \max\left\{\left\|\frac{\partial^{|i|+|l|+|\alpha|+|\beta|}}{\partial\theta^i\partial I^l\partial q^\alpha_{\vec{n}}\partial\bar{q}^\beta_{\vec{n}}}F\right\|_{D,\mathcal{O}}, |i|+|l|+|\alpha|+|\beta| = m \geq 2\right\}.
$$

Notice that F is a polynomial of degree 1 in I and degree 2 in q, $\bar{q}$. From Lemma 3.2 and the Cauchy inequality, it follows that

$$\|D^m F\|_{D_2,\mathcal{O}} < c\gamma^{-2}\varepsilon^{\frac{9}{10}}, \tag{3.22}$$

for any $m \geq 2$.

To get the estimates for Φ_F^t, we start from the integral equation,

$$\Phi_F^t = id + \int_0^t X_F \circ \Phi_F^s \, ds$$

so that $\Phi_F^t : D_{2\eta} \to D_{3\eta}, \quad -1 \leq t \leq 1$, which follows directly from (3.22). Since

$$D\Phi_F^t = Id + \int_0^t (DX_F) D\Phi_F^s \, ds = Id + \int_0^t J(D^2F) D\Phi_F^s \, ds,$$

where J denotes the standard symplectic matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, it follows that

$$\|D\Phi_F^t - Id\| \leq 2\|D^2 F\| < c\gamma^{-2}\varepsilon^{\frac{9}{10}}. \tag{3.23}$$

Consequently Lemma 3.3 follows. ■

3.3 Estimation for the new normal form

The map Φ_F^1 defined above transforms H into $H_+ = N_+ + P_+$(see (3.14) and (3.16)). Here the new normal form N_+ is

$$\begin{aligned} N_+ &= N + P_{0000} + \langle \omega', I \rangle + \sum_n P_{nn}^{011} q_n \bar{q}_n \\ &= e_+ + \langle \tilde{\omega}_+, I \rangle + \sum_n \Omega_n^+ q_n \bar{q}_n, \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} e_+ &= e + P_{0000}, \\ \tilde{\omega}_+ &= \tilde{\omega} + P_{0l00}(|l| = 1), \\ \Omega_n^+ &= \begin{cases} \Omega_n + P_{nn}^{011}, & |n| \leq \ln \frac{1}{\varepsilon}, \\ \Omega_n, & |n| > \ln \frac{1}{\varepsilon}. \end{cases} \end{aligned}$$

Note that the new normal frequencies Ω_n^+ do not change for $|n| > \ln \frac{1}{\varepsilon}$, thus they remain the initial random variables.

Now we show that N_+ has properties similar to those of N. By the regularity of P, we have that

$$|\tilde{\omega}_+ - \tilde{\omega}|_{\mathcal{O}} < \varepsilon, \quad |P_{nn}^{011}|_{\mathcal{O}} < \varepsilon. \tag{3.25}$$

It follows that

$$|\Omega_n^+ - \Omega_m^+| \geq \frac{\gamma}{|n-m|^\tau} - 2\varepsilon \geq \frac{\gamma_+}{|n-m|^\tau}, \quad n \neq m, \quad |n|, |m| \leq \ln \frac{1}{\varepsilon},$$

$$|\langle k,\tilde{\omega}+P_{0l00}\rangle| \geq |\langle k,\tilde{\omega}\rangle| - |\langle k,P_{0l00}\rangle| \geq \frac{\gamma}{|k|^{\tau}} - \varepsilon|k| \geq \frac{\gamma_+}{|k|^{\tau}}, \quad k\neq 0,$$

and similarly

$$\begin{aligned} |\langle k,\tilde{\omega}+P_{0l00}\rangle+\Omega_n^+| &\geq \frac{\gamma_+}{|k|^{\tau}}, \quad k\neq 0, \quad |n|\leq \ln\frac{1}{\varepsilon} \\ |\langle k,\tilde{\omega}+P_{0l00}\rangle+\Omega_n^+ + \Omega_m^+| &\geq \frac{\gamma_+}{|k|^{\tau}}, \quad k\neq 0, \quad |n|,|m|\leq \ln\frac{1}{\varepsilon} \\ |\langle k,\tilde{\omega}+P_{0l00}\rangle+\Omega_n^+ - \Omega_m^+| &\geq \frac{\gamma_+}{|k|^{\tau}}, \quad k\neq 0, \quad |n|,|m|\leq \ln\frac{1}{\varepsilon} \end{aligned}$$

provided that $\varepsilon|k|^{\tau+1} \leq c(\gamma-\gamma_+)$. This means that in the succeeding KAM step, small divisor conditions are automatically satisfied for $|n|,|m|\leq \ln\frac{1}{\varepsilon}$ and $|k|\leq K$, where $\varepsilon K^{\tau+1}\leq c(\gamma-\gamma_+)$.

As for the condition associated with $|k|>K$ or $\ln\frac{1}{\varepsilon}\leq |n|,|m|\leq \ln\frac{1}{\varepsilon_+}$, which is necessary for the next KAM step, we shall verify them by measure-estimating in Section 6. Note that the bounds in (3.25) will be used for the measure estimates.

3.4 Estimation for the new perturbation

Since

$$\begin{aligned} P_+ &= \int_0^1 (1-t)\{\{N,F\},F\}\circ\Phi_F^t dt + \int_0^1 \{R,F\}\circ\Phi_F^t dt + (P-R)\circ\Phi_F^1 \\ &= \int_0^1 \{R(t),F\}\circ\Phi_F^t dt + (P-R)\circ\Phi_F^1, \end{aligned}$$

where $R(t)=(1-t)(N_+-N)+tR$. Hence

$$X_{P_+} = \int_0^1 (\Phi_F^t)^* X_{\{R(t),F\}} dt + (\Phi_F^1)^* X_{(P-R)}.$$

According to Lemma 3.3,

$$\|D\Phi_F^t - Id\|_{D_{1\eta}} < c\gamma^{-2}\varepsilon^{\frac{9}{10}}, \quad -1\leq t\leq 1,$$

thus

$$\|D\Phi_F^t\|_{D_{1\eta}} \leq 1 + \|D\Phi_F^t - Id\|_{D_{1\eta}} \leq 2, \quad -1\leq t\leq 1.$$

Due to Lemma 6.3,

$$\|X_{\{R(t),F\}}\|_{D_{2\eta}} \leq c\gamma^{-2}\eta^{-2}\varepsilon^{\frac{19}{10}},$$

and

$$\|X_{(P-R)}\|_{D_{2\eta}} \leq c\eta\varepsilon,$$

we have

$$\|X_{P_+}\|_{D_\rho(r_+,s_+)} \leq c\eta\varepsilon + c\gamma^{-2}\eta^{-2}\varepsilon^{\frac{19}{10}} \leq c\varepsilon_+.$$

3.5 Verification of the assumptions after one KAM step

To continue the iteration we must show that the new Hamiltonian H_+ satisfies the assumptions similar to $(\mathbf{A1})-(\mathbf{A6})$. We have obtained the regularity of $\tilde{\omega}_+$ and $\{\Omega_n^+\}_{n\in\mathbb{Z}_1}$ in the form of (3.1) and (3.2) in view of (3.25). For the next step, we shall prove the Melnikov's nondegeneracy for $\tilde{\omega}_+$ and the gap condition for $\{\Omega_n^+\}_{n\in\mathbb{Z}_1}$ in Section 5 via measure estimates. Since the regularity of P_+, together with its smallness, has been verified in the last subsection, we only need to check the decay property and gauge invariance here.

In Taylor series, P_+ is expressed in terms of the iterated Poisson bracket

$$\begin{aligned}P_+ &= P - R + \{P, F\} + \frac{1}{2!}\{\{N, F\}, F\} + \frac{1}{2!}\{\{P, F\}, F\} \\ &\quad + \cdots + \frac{1}{n!}\{\cdots\{N, \underbrace{F\}\cdots, F}_{n}\} + \frac{1}{n!}\{\cdots\{P, \underbrace{F\}\cdots, F}_{n}\} + \cdots.\end{aligned}$$

The support of any term $F_{kl\alpha\beta}$ is finite, with $n^* \leq \ln\frac{1}{\varepsilon}$, therefore, applying Corollary 1 and 2 with $G = P$ and

$$G = \{N, F\} = P_{0000} + \langle\omega', I\rangle + \sum_{|n|\leq \ln\frac{1}{\varepsilon}} P^{011}_{nn} q_n \bar{q}_n - R,$$

we obtain that the decay property and gauge invariance are satisfied. Note that the new decay property of $P_+ = \check{P}_+ + \acute{P}_+ + \grave{P}_+$ is expressed as

$$\begin{aligned}&\|\check{P}^+_{\alpha\beta}\| \leq ce^{-\varrho n^*} \leq ce^{-\varrho_+ n^*}, \quad \text{for } |\alpha| + |\beta| \geq 1, \\ &\|\acute{P}^+_{\alpha\beta}\| \leq ce^{-\varrho n^*} \leq ce^{-\varrho_+ n^*}, \quad \text{for } 1 \leq |\alpha| + |\beta| \leq 2, \\ &\|\grave{P}^+_{\alpha\beta}\| \leq ce^{-\varrho_+(n^+ - n^-)}, \quad \text{for } |\alpha| + |\beta| \geq 3,\end{aligned}$$

with $\varrho_+ = \frac{1}{2}\varrho$ in view of Corollary 1.

4 Iteration lemma and convergence

For any given r, ε_0, s, ρ, ϱ, γ and for all $\nu \geq 1$, we define the following sequences

$$\begin{aligned}&r_\nu = r(1 - \sum_{i=2}^{\nu+1} 2^{-i}), \\ &\varepsilon_\nu = c\gamma^{-2}(r_{\nu-1} - r_\nu)^{-c}\varepsilon_{\nu-1}^{\frac{6}{5}}, \\ &s_\nu = \frac{1}{4}\eta_{\nu-1}s_{\nu-1} = 2^{-2\nu}(\prod_{i=0}^{\nu-1}\varepsilon_i)^{\frac{1}{3}}s, \\ &\rho_\nu = \rho(1 - \sum_{i=2}^{\nu+1} 2^{-i}), \\ &\varrho_\nu = 2^{-\nu}\varrho \\ &\gamma_\nu = \varepsilon_\nu^{\frac{1}{8}} \\ &\eta_\nu = \varepsilon_\nu^{\frac{1}{3}}, \\ &D_\nu = D_{\rho_\nu}(r_\nu, s_\nu),\end{aligned}$$

where c is a constant. Note that

$$\Psi(r)=\prod_{i=1}^{\infty}[(r_{i-1}-r_i)^{-c}]^{(\frac{5}{6})^i}$$

is a well–defined function of r.

4.1 Iteration lemma

The preceding analysis can be summarized as follows.

Lemma 4.1 *Let ε is small enough and $\nu\geq 0$. Suppose that*
(1). $N_\nu=e_\nu+\langle\tilde{\omega}(\omega)_\nu,I\rangle+\sum_{n\in\mathbb{Z}_1}\Omega_n^\nu(\omega)q_n\bar{q}_n$ is a normal form with the tangential frequencies

$$\tilde{\omega}_\nu(\omega)=\omega+\hat{\omega}_\nu(\omega),\quad \omega\in\mathcal{O}_\nu,$$

where $\mathcal{O}_\nu$ is a closed set in $\mathbb{R}^b$, $\hat{\omega}_\nu(\omega)$ is a C_W^1 function of $\omega\in\mathcal{O}_\nu$ with C_W^1-norm bounded by ε_0, and $\{\Omega_n^\nu(\omega)\}_{n\in\mathbb{Z}_1}$ satisfies

$$\Omega_n^\nu(\omega)=\begin{cases}\Omega_n^0+\hat{\Omega}_n^\nu(\omega), & |n|\leq\ln\frac{1}{\varepsilon_\nu},\\ \Omega_n^0, & |n|>\ln\frac{1}{\varepsilon_\nu}.\end{cases}$$

with $\{\Omega_n^0\}_{n\in\mathbb{Z}_1}\in X_\nu$ being the initial normal frequencies and $\hat{\Omega}_n^\nu(\omega)$'s are C_W^1 functions of ω with C_W^1-norm bounded by ε_0. Moreover,

$$|\tilde{\omega}_\nu-\tilde{\omega}_{\nu-1}|_{\mathcal{O}_\nu}\leq\varepsilon_{\nu-1},\quad |\Omega_n^\nu-\Omega_n^{\nu-1}|_{\mathcal{O}_\nu}\leq\varepsilon_{\nu-1};$$

(2). For fixed $\{\Omega_n^0\}_{n\in\mathbb{Z}_1}\in X_\nu$, the parameters $\omega\in\mathcal{O}_\nu$ satisfying

$$\begin{aligned}&|\Omega_n^\nu-\Omega_m^\nu|\geq\frac{\gamma_\nu}{|n-m|^\tau},\quad n\neq m,\\ &|\langle k,\tilde{\omega}_\nu\rangle|\geq\frac{\gamma_\nu}{|k|^\tau},\\ &|\langle k,\tilde{\omega}_\nu\rangle+\Omega_n^\nu|\geq\frac{\gamma_\nu}{|k|^\tau},\\ &|\langle k,\tilde{\omega}_\nu\rangle+\Omega_n^\nu+\Omega_m^\nu|\geq\frac{\gamma_\nu}{|k|^\tau},\\ &|\langle k,\tilde{\omega}_\nu\rangle+\Omega_n^\nu-\Omega_m^\nu|\geq\frac{\gamma_\nu}{|k|^\tau},\end{aligned}$$

for all $k\neq 0$ and $|n|,|m|\leq\ln\frac{1}{\varepsilon_\nu}$;
*(3). P_ν has the gauge invariance defined in (**A6**) and*

$$\|X_{P_\nu}\|_{D_\nu,\mathcal{O}_\nu}\leq\varepsilon_\nu.$$

Moreover, if we write that $P_\nu=\breve{P}_\nu+\acute{P}_\nu+\grave{P}_\nu$, where

$$\begin{aligned}\breve{P}_\nu&=\breve{P}_\nu(\theta,I,q,\bar{q};\omega)=\sum_{\alpha,\beta}\breve{P}_{\alpha\beta}^\nu q^\alpha\bar{q}^\beta=\sum_{\substack{(k,l)\neq 0\\ \alpha,\beta}}P_{kl\alpha\beta}^\nu q^\alpha\bar{q}^\beta e^{\mathrm{i}\langle k,\theta\rangle}I^l,\\ \acute{P}_\nu&=\acute{P}_\nu(q,\bar{q};\omega)=\sum_{|\alpha|+|\beta|\leq 2}\acute{P}_{\alpha\beta}^\nu q^\alpha\bar{q}^\beta=\sum_{|\alpha|+|\beta|\leq 2}P_{00\alpha\beta}^\nu q^\alpha\bar{q}^\beta,\\ \grave{P}_\nu&=\grave{P}_\nu(q,\bar{q};\omega)=\sum_{|\alpha|+|\beta|\geq 3}\grave{P}_{\alpha\beta}^\nu q^\alpha\bar{q}^\beta=\sum_{|\alpha|+|\beta|\geq 3}P_{00\alpha\beta}^\nu q^\alpha\bar{q}^\beta,\end{aligned}$$

then P has decay property, i.e.

$$\|\breve{P}^{\nu}_{\alpha\beta}\| \leq ce^{-\varrho_\nu n^*}, \quad \text{for } |\alpha|+|\beta|\geq 1,$$
$$\|\acute{P}^{\nu}_{\alpha\beta}\| \leq ce^{-\varrho_\nu n^*}, \quad \text{for } 1\leq |\alpha|+|\beta|\leq 2,$$
$$\|\grave{P}^{\nu}_{\alpha\beta}\| \leq ce^{-\varrho_\nu (n^+-n^-)}, \quad \text{for } |\alpha|+|\beta|\geq 3,$$

where

$$n^+ = n^+(\alpha,\beta) = \max\{n\in\mathbb{Z} : (\alpha_n,\beta_n)\neq 0\},$$
$$n^- = n^-(\alpha,\beta) = \min\{n\in\mathbb{Z} : (\alpha_n,\beta_n)\neq 0\},$$
$$n^* = n^*(\alpha,\beta) = \max\{|n^+|,|n^-|\}.$$

Then there are subsets $\mathcal{O}_{\nu+1}\subset\mathcal{O}_\nu$ and $X_{\nu+1}\subset X_\nu$ such that

$$\mathcal{O}_{\nu+1} = \mathcal{O}_\nu \setminus \left(\bigcup_{\substack{k\neq 0\\ |n|,|m|\leq \ln\frac{1}{\varepsilon_{\nu+1}}}} \left(\mathcal{R}^{\nu+1}_k \bigcup \mathcal{R}^{\nu+1}_{kn} \bigcup \mathcal{R}^{\nu+1}_{knm}\right)\right),$$

where

$$\begin{aligned}
\mathcal{R}^{\nu+1}_k &= \{\omega\in\mathcal{O}_\nu : |\langle k,\tilde\omega_{\nu+1}(\omega)\rangle| < \frac{\gamma_{\nu+1}}{|k|^\tau}\},\\
\mathcal{R}^{\nu+1}_{kn} &= \{\omega\in\mathcal{O}_\nu : |\langle k,\tilde\omega_{\nu+1}(\omega)\rangle + \Omega^{\nu+1}_n| < \frac{\gamma_{\nu+1}}{|k|^\tau}\},\\
\mathcal{R}^{\nu+1}_{knm} &= \{\omega\in\mathcal{O}_\nu : |\langle k,\tilde\omega_{\nu+1}(\omega)\rangle + \Omega^{\nu+1}_n \pm \Omega^{\nu+1}_m| < \frac{\gamma_{\nu+1}}{|k|^\tau}\},
\end{aligned}$$

and $X_{\nu+1}$ is expressed as

$$\left\{\{\Omega^0_n\}_{n\in\mathbb{Z}_1}\in X_\nu : |\Omega^{\nu+1}_n - \Omega^{\nu+1}_m| \geq \frac{\gamma_{\nu+1}}{|n-m|^\tau}, \quad n\neq m, \quad |n|,|m|\leq \ln\frac{1}{\varepsilon_{\nu+1}}\right\}$$

with

$$\tilde\omega_{\nu+1} = \tilde\omega_\nu + P^\nu_{0l00},$$
$$\Omega^{\nu+1}_n = \Omega^\nu_n + P^{011,\nu}_{nn}, \quad |n|\leq \ln\frac{1}{\varepsilon_\nu},$$

and a symplectic transformation of variables

$$\Phi_\nu : D_{\nu+1}\times\mathcal{O}_{\nu+1}\to D_\nu\times\mathcal{O}_\nu,$$

such that on $D_{\rho_{\nu+1}}(r_{\nu+1},s_{\nu+1})\times\mathcal{O}_{\nu+1}$, $H_{\nu+1} = H_\nu\circ\Phi_\nu$ has the form

$$H_{\nu+1} = e_{\nu+1} + \langle\tilde\omega_{\nu+1}, I\rangle + \sum_{n\in\mathbb{Z}_1}\Omega^{\nu+1}_n q_n\bar q_n + P_{\nu+1},$$

with $\{\Omega^0_n\}_{n\in\mathbb{Z}_1}\in X_{\nu+1}$ and

$$|\tilde\omega_{\nu+1}-\tilde\omega_\nu|_{\mathcal{O}_{\nu+1}} \leq \varepsilon_\nu, \quad |\Omega^{\nu+1}_n - \Omega^\nu_n|_{\mathcal{O}_{\nu+1}} \leq \varepsilon_\nu.$$

And also

$$P_{\nu+1} = \breve{P}_{\nu+1} + \acute{P}_{\nu+1} + \grave{P}_{\nu+1}$$

satisfies that

$$\|X_{P_{\nu+1}}\|_{D_{\nu+1},\mathcal{O}_{\nu+1}} \le \varepsilon_{\nu+1},$$

has the gauge invariance defined in (**A6**) *and the decay property, i.e.*

$$\begin{aligned} &\|\breve{P}^{\nu+1}_{\alpha\beta}\| \le ce^{-\varrho_\nu n^*}, \quad \text{for } |\alpha|+|\beta| \ge 1,\\ &\|\acute{P}^{\nu+1}_{\alpha\beta}\| \le ce^{-\varrho_\nu n^*}, \quad \text{for } 1 \le |\alpha|+|\beta| \le 2,\\ &\|\grave{P}^{\nu+1}_{\alpha\beta}\| \le ce^{-\varrho_{\nu+1}(n^+-n^-)}, \quad \text{for } |\alpha|+|\beta| \ge 3. \end{aligned}$$

4.2 Convergence

Let $\Psi^\nu = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu$, $\nu = 1, 2, \cdots$. An induction argument shows that

$$\Psi^\nu : D_\nu \times \mathcal{O}_\nu \to D_0 \times \mathcal{O}$$

and $H_0 \circ \Psi^\nu = H_\nu = N_\nu + P_\nu$ with $\{\Omega^0_n\} \in X_\nu$ for all $\nu = 1, 2, \cdots$.

Let $\tilde{\mathcal{O}} = \cap_{\nu=0}^\infty \mathcal{O}_\nu$ and $\tilde{X} = \cap_{\nu=0}^\infty X_\nu$. Using Lemma 4.1 and standard arguments [36, 37], one can conclude that H_ν, e_ν, N_ν, P_ν, Ψ^ν, $\tilde{\omega}_\nu$ and $\{\Omega^\nu_n\}_{n\in\mathbb{Z}_1}$ converge uniformly on $D_{\frac{\varrho}{2}}(\frac{r}{2}, 0) \times \tilde{\mathcal{O}}$ to, say, H_∞, e_∞, N_∞, P_∞, Ψ^∞, ω_∞ and $\{\Omega^\infty_n\}_{n\in\mathbb{Z}_1}$, respectively, in which case it is clear that

$$N_\infty = e_\infty + \langle \tilde{\omega}_\infty, I\rangle + \sum_{n\in\mathbb{Z}_1} \Omega^\infty_n q_n \bar{q}_n.$$

Since

$$\varepsilon_{\nu+1} = c\gamma_\nu^{-2}(r_\nu - r_{\nu+1})^{-c}\varepsilon_\nu^{\frac{3}{2}} \le (c\gamma^{-2}\Psi(r)\varepsilon)^{(\frac{3}{2})^\nu},$$

we have, by Lemma 4.1 that

$$X_{P_\infty}|_{D_{\frac{\varrho}{2}}(\frac{r}{2},0)\times\tilde{\mathcal{O}}} \equiv 0.$$

Let Φ^t_H denote the flow of any Hamiltonian vector field X_H. Since $H \circ \Psi^\nu = H_\nu$, we have

$$\Phi^t_H \circ \Psi^\nu = \Psi^\nu \circ \Phi^t_{H_\nu}. \tag{4.1}$$

The uniform convergence of Ψ^ν, $D\Psi^\nu$, ω_ν and X_{H_ν} implies that the limits can be taken on both sides of (4.1). Hence, on $D_{\frac{\varrho}{2}}(\frac{r}{2}, 0) \times \tilde{\mathcal{O}}$ we get

$$\Phi^t_H \circ \Psi^\infty = \Psi^\infty \circ \Phi^t_{H_\infty} \tag{4.2}$$

and

$$\Psi^\infty : D_{\frac{\varrho}{2}}(\frac{r}{2}, 0) \times \tilde{\mathcal{O}} \to D_\rho(r, s) \times \mathcal{O}.$$

It follows from (4.2) that

$$\Phi^t_H(\Psi^\infty(\mathbb{T}^b \times \{\omega_\infty\})) = \Psi^\infty\Phi^t_{N_\infty}(\mathbb{T}^b \times \{\omega\}) = \Psi^\infty(\mathbb{T}^b \times \{\omega\})$$

for $\omega \in \tilde{\mathcal{O}}$. This means that $\Psi^\infty(\mathbb{T}^b \times \{\omega\})$ is an embedded torus which is invariant for the original perturbed Hamiltonian system at $\omega_\infty \in \tilde{\mathcal{O}}$. We remark here that the frequencies $\tilde{\omega}_\infty$ associated to $\Psi^\infty(\mathbb{T}^b \times \{\omega\})$ are slightly different from the initial frequencies ω. The normal behavior of the invariant torus is governed by normal frequencies Ω^∞_n.

5 Measure estimate

In the KAM steps, we have assume that the small divisor conditions in the form of (3.5) – (3.9) are satisfied. In this section, we shall estimate the measure of the set of parameters such that these conditions are violated during the iterations.

5.1 Small divisors concerning the tangential frequencies

At the $(\nu+1)^{\text{th}}$ step of the KAM iteration, we have to exclude the following resonant set

$$\mathcal{R}^{\nu+1} = \bigcup_{\substack{k\neq 0 \\ |n|,|m|\leq \ln\frac{1}{\varepsilon_{\nu+1}}}} (\mathcal{R}_k^{\nu+1} \bigcup \mathcal{R}_{kn}^{\nu+1} \bigcup \mathcal{R}_{knm}^{\nu+1}),$$

where

$$\begin{aligned}
\mathcal{R}_k^{\nu+1} &= \left\{\omega \in \mathcal{O}_\nu : |\langle k, \tilde{\omega}_{\nu+1}(\omega)\rangle| < \frac{\gamma_{\nu+1}}{|k|^\tau}\right\}, \\
\mathcal{R}_{kn}^{\nu+1} &= \left\{\omega \in \mathcal{O}_\nu : |\langle k, \tilde{\omega}_{\nu+1}(\omega)\rangle + \Omega_n^{\nu+1}(\omega)| < \frac{\gamma_{\nu+1}}{|k|^\tau}\right\}, \\
\mathcal{R}_{knm}^{\nu+1} &= \left\{\omega \in \mathcal{O}_\nu : |\langle k, \tilde{\omega}_{\nu+1}(\omega)\rangle + \Omega_n^{\nu+1}(\omega) \pm \Omega_m^{\nu+1}(\omega)| < \frac{\gamma_{\nu+1}}{|k|^\tau}\right\}.
\end{aligned}$$

Lemma 5.1 *For any fixed $k \neq 0$, and $|n|, |m| \leq \ln\frac{1}{\varepsilon_{\nu+1}}$,*

$$|\mathcal{R}_k^{\nu+1} \bigcup \mathcal{R}_{kn}^{\nu+1} \bigcup \mathcal{R}_{knm}^{\nu+1}| < c\frac{\gamma_{\nu+1}}{|k|^{\tau+1}}.$$

Proof: Recall that $\tilde{\omega}_{\nu+1}(\omega) = \omega + \sum_{j=0}^{\nu} P_{0l00}^j(\omega)$ with

$$\left|\sum_{j=0}^{\nu} P_{0l00}^j\right|_{\mathcal{O}_\nu} \leq \varepsilon_0, \tag{5.1}$$

and $\Omega_n^{\nu+1}(\omega) = \Omega_n^0 + \sum_{j=0}^{\nu} P_{nn}^{011,j}(\omega)$ with

$$\left|\sum_{j=0}^{\nu} P_{nn}^{011,j}\right|_{\mathcal{O}_\nu} \leq \varepsilon. \tag{5.2}$$

It follows that [1]

$$\left|\frac{\partial(\langle k, \tilde{\omega}_{\nu+1}\rangle \pm \Omega_n^{\nu+1} \pm \Omega_m^{\nu+1})}{\partial\omega}\right| \geq c|k|,$$

then the proof of this lemma is evident, we omit it. ■

[1] Here $|\cdot|$ denotes ℓ^1–norm.

Lemma 5.2 *The total measure we need to exclude along the KAM iteration is*

$$\left|\bigcup_{\nu\geq 0}\mathcal{R}^{\nu+1}\right| = \left|\bigcup_{\nu\geq 0}\bigcup_{\substack{k\neq 0\\ |n|,|m|\leq \ln\frac{1}{\varepsilon_\nu}}}\left(\mathcal{R}_k^\nu\bigcup\mathcal{R}_{kn}^\nu\bigcup\mathcal{R}_{knm}^\nu\right)\right| < c\gamma^\vartheta, \quad \vartheta>0.$$

Proof: By Lemma 5.1,

$$\begin{aligned}\left|\bigcup_{\nu\geq 0}\mathcal{R}^{\nu+1}\right| &\leq \sum_{\nu\geq 0}\sum_{\substack{k\neq 0\\ |n|,|m|\leq \ln\frac{1}{\varepsilon_\nu}}}\frac{\gamma_\nu}{|k|^{\tau+1}}\\ &\leq c\sum_{\nu\geq 0}\sum_{k\neq 0}\left(\ln\frac{1}{\varepsilon_\nu}\right)^2\frac{\gamma_\nu}{|k|^{\tau+1}}\\ &\leq c\sum_{\nu\geq 0}\sum_{k\neq 0}\frac{\gamma_\nu^{\frac{1}{2}}}{|k|^{\tau+1}}\\ &\leq c\sum_{\nu\geq 0}\gamma_\nu^{\frac{1}{2}}\\ &\leq c\gamma^{\frac{1}{2}}.\end{aligned}$$

This completes the measure estimate for the tangential frequencies. ■

5.2 Small divisors concerning the normal frequencies

As we proceed the $\nu+1^{\text{th}}$ KAM step, we need to verify that the inequality

$$|\Omega_n^{\nu+1}-\Omega_m^{\nu+1}| \geq \frac{\gamma_{\nu+1}}{|n-m|^\tau} \tag{5.3}$$

holds for $n,m\in\mathbb{Z}_1$, $n\neq m$ and $|n|,|m|\leq \ln\frac{1}{\varepsilon_{\nu+1}}$, under the assumption that

$$|\Omega_n^\nu-\Omega_m^\nu| \geq \frac{\gamma_\nu}{|n-m|^\tau}$$

for $n,m\in\mathbb{Z}_1$, $n\neq m$ and $|n|,|m|\leq\ln\frac{1}{\varepsilon_\nu}$. Since $|\Omega_n^{\nu+1}-\Omega_n^\nu|_{\mathcal{O}_{\nu+1}}\leq\varepsilon_\nu$, the assumption above implies that (5.3) is automatically satisfied for $|n|,|m|\leq\ln\frac{1}{\varepsilon_\nu}$.

If $|n|\leq\ln\frac{1}{\varepsilon_\nu}$ and $\ln\frac{1}{\varepsilon_\nu}\leq|m|\leq\ln\frac{1}{\varepsilon_{\nu+1}}$, then with $\Omega_n^{\nu+1}$ fixed we can exclude the set of Ω_m^0's

$$\left\{\Omega_m^0 : |\Omega_m^0-\Omega_n^{\nu+1}| < \frac{\gamma_{\nu+1}}{|n-m|^\tau}\right\},$$

whose measure is no more than $\frac{\gamma_{\nu+1}}{|n-m|^\tau}$, recalling that $\Omega_m^{\nu+1}=\Omega_m^0$ for $|m|>\ln\frac{1}{\varepsilon_\nu}$. As for the case that $\ln\frac{1}{\varepsilon_\nu}\leq|n|,|m|\leq\ln\frac{1}{\varepsilon_{\nu+1}}$, with the former normal frequency fixed, we also can estimate the measure of the latter variables such that (5.3) fail, just as in Section 2 where we verify the assumption **(A2)**.

After the procedures above, the remaining values of variables $\{\Omega_n^0\}_{\ln\frac{1}{\varepsilon_\nu}\le|n|\le\ln\frac{1}{\varepsilon_{\nu+1}}}$ form a subset, with its measure more than

$$(1-c\gamma_{\nu+1})^{c\ln\frac{1}{\varepsilon_{\nu+1}}}\ge e^{-\gamma_{\nu+1}^{\frac{1}{2}}}.$$

Thus the total probability of $\{\Omega_n^0\}_{n\in\mathbb{Z}_1}$ we can choose as the normal frequencies is larger than

$$\prod_{\nu=0}^{\infty}e^{-\gamma_\nu^{\frac{1}{2}}}\ge e^{-\gamma^{\frac{1}{3}}}.$$

6 Appendix

Lemma 6.1 *The Banach algebraic property of the norm:*

$$\|FG\|_{D_\rho(r,s),\mathcal{O}}\le\|F\|_{D_\rho(r,s),\mathcal{O}}\|G\|_{D_\rho(r,s),\mathcal{O}}.$$

Proof: Since

$$(FG)_{kl\alpha\beta}=\sum_{k',l',\alpha',\beta'}F_{k-k',l-l',\alpha-\alpha',\beta-\beta'}G_{k'l'\alpha'\beta'},$$

we have that

$$\begin{aligned}\|FG\|_{D_\rho(r,s),\mathcal{O}} &= \sup_{\|q\|_\rho<s}\sum_{k,l,\alpha,\beta}|(FG)_{kl\alpha\beta}|_{\mathcal{O}}|q^\alpha||\bar{q}^\beta|s^{2|l|}e^{|k|r}\\ &\le \sup_{\|q\|_\rho<s}\sum_{k,l,\alpha,\beta}\sum_{k',l',\alpha',\beta'}|F_{k-k',l-l',\alpha-\alpha',\beta-\beta'}G_{k'l'\alpha'\beta'}|_{\mathcal{O}}|q^\alpha||\bar{q}^\beta|s^{2|l|}e^{|k|r}\\ &\le \|F\|_{D_\rho(r,s),\mathcal{O}}\|G\|_{D_\rho(r,s),\mathcal{O}}.\end{aligned}$$

■

Lemma 6.2 (Generalized Cauchy Inequalities) *The various components of the Hamiltonian vector field X_F satisfy the estimates:*

$$\|\partial_\theta F\|_{D_\rho(r-\sigma,s),\mathcal{O}}\le\frac{c}{\sigma}\|F\|_{D_\rho(r,s),\mathcal{O}},$$

$$\|\partial_I F\|_{D_\rho(r,\frac{1}{2}s),\mathcal{O}}\le\frac{c}{s^2}\|F\|_{D_\rho(r,s),\mathcal{O}},$$

and

$$\|\partial_{q_n} F\|_{D_\rho(r,\frac{1}{2}s),\mathcal{O}}\le\frac{c}{s}\|F\|_{D_\rho(r,s),\mathcal{O}}e^{|n|\rho},$$

$$\|\partial_{\bar{q}_n} F\|_{D_\rho(r,\frac{1}{2}s),\mathcal{O}}\le\frac{c}{s}\|F\|_{D_\rho(r,s),\mathcal{O}}e^{|n|\rho}.$$

Proof: The inequalities follow from the standard Cauchy estimate. See [37]. ■

Let $\{\cdot,\cdot\}$ denote the Poisson bracket of smooth functions, i.e.,

$$\{F,G\} = \langle \frac{\partial F}{\partial I}, \frac{\partial G}{\partial \theta}\rangle - \langle \frac{\partial F}{\partial \theta}, \frac{\partial G}{\partial I}\rangle + \mathrm{i}\sum_{n\in\mathbb{Z}_1}(\frac{\partial F}{\partial q_n}\frac{\partial G}{\partial \bar{q}_n} - \frac{\partial F}{\partial \bar{q}_n}\frac{\partial G}{\partial q_n}),$$

which is perhaps the most important quantity to be estimated in this norm defined for the vector fields, as it is significant to Hamiltonian mechanics. Then we have the following lemma:

Lemma 6.3 *If*

$$\|X_F\|_{D_\rho(r,s)} < \varepsilon',\ \|X_G\|_{D_\rho(r,s)} < \varepsilon'',$$

for some $\varepsilon', \varepsilon'' > 0$*, then*

$$\|X_{\{F,G\}}\|_{D_\rho(r-\sigma,\eta s)} < c\sigma^{-1}\eta^{-2}\varepsilon'\varepsilon'',$$

for any $0 < \sigma < r$ *and* $0 < \eta \ll 1$*. In particular, if* $\eta \sim \varepsilon^{\frac{1}{4}}$*,* $\varepsilon' \sim \varepsilon, \varepsilon'' \sim \varepsilon^{\frac{3}{4}}$*, we have that*

$$\|X_{\{F,G\}}\|_{D_\rho(r-\sigma,\eta s)} \sim \varepsilon^{\frac{5}{4}}.$$

For the proof, see [24]. ■

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