Diffusion Bound and Reducibility for Discrete Schrödinger Equations with Tangent Potential

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Abstract

In this paper, we consider the following lattice Schrödinger equations

$$\mathrm{i}\dot{q}_n(t) = \tan\pi(n\alpha + x)q_n(t) + \varepsilon(q_{n+1}(t) + q_{n-1}(t)) + \delta v_n(t)|q_n(t)|^{2\tau-2}q_n(t),$$

with α satisfying a certain Diophantine condition, $x \in \mathbb{R}/\mathbb{Z}$ and $\tau = 1$ or 2, where $v_n(t)$ is a spatial localized real bounded potential (i.e. $|v_n(t)| \leq Ce^{-\rho|n|}$). We prove that the growth of H^1 norm of the solution $\{q_n(t)\}_{n\in\mathbb{Z}}$ is at most logarithmic if the initial data $\{q_n(0)\}_{n\in\mathbb{Z}} \in H^1$ for ε sufficiently small and a.e. x fixed.

Furthermore, suppose the linear equation has a time quasi-periodic potential, i.e.,

$$\mathrm{i}\dot{q}_n(t) = \tan\pi(n\alpha + x)q_n(t) + \varepsilon(q_{n+1}(t) + q_{n-1}(t)) + \delta v_n(\theta^0 + t\omega)q_n(t),$$

then the linear equation can be reduced to an autonomous equation for a.e. x and most values of the frequency vectors ω if ε and δ are sufficiently small.

Keywords:Tangent potential. Birkhoff normal form. Reducibility.

1 Introduction and statement of main results

The diffusion for a class of lattice Schrödinger equations with time-dependent linear or nonlinear perturbation has been studied for several years. This problem falls within the same general category of bounds on the higher Sobolev norms (H^1 or beyond) for the continuum nonlinear Hamiltonian PDE in a compact domain, e.g., a circle or a finite interval with Dirichlet boundary conditions, see e.g., [B1]. (Recall that typically the $L^2(\ell^2)$ norm is conserved, so the H^1 norm is the first non-trivial norm to consider.)

In previous papers [BW1, BW2, WZ], the following random Schrödinger equations under perturbation were considered,

$$\mathrm{i}\dot{q}_n = \mu_n q_n + \varepsilon(q_{n+1} + q_{n-1}) + \delta_n \mathcal{W}(t)q_n, \tag{1.1}$$

$$\mathrm{i}\dot{q}_n = \mu_n q_n + \varepsilon(q_{n+1} + q_{n-1}) + \delta_n |q_n|^2 q_n, \tag{1.2}$$

where $\{q_n(t)\}_{n\in\mathbb{Z}}$ are complex variables in $\ell^2(\mathbb{Z})$ for each $t \in [0, +\infty)$, the dot over q_n denotes the partial derivative with respect to t, μ_n are independent randomly chosen variables in $[0, 1]$ (uniform distribution), δ_n decay with n fast and $\mathcal{W}(t)$ is quasi-periodic in t with a Diophantine [1] frequency. Spectral properties of the Schrödinger operator and the

[1] If

$$|\langle k, \omega\rangle|_1 \geq \frac{\gamma}{|k|^\sigma}, \quad \forall 0 \neq k \in \mathbb{Z}^d, \tag{1.3}$$

dynamical localized phenomenons of $q_n(t)$ were studied in those papers.

In this paper, we focus on a different model, where μ_n are chosen to be the tangent potential. Indeed, we consider the one-dimensional discrete equations

$$\mathrm{i}\dot{q}_n(t) = \tan\pi(n\alpha + x)q_n(t) + \varepsilon(q_{n+1}(t) + q_{n-1}(t)) + \delta v_n(t)q_n(t), \tag{1.4}$$
$$\mathrm{i}\dot{q}_n(t) = \tan\pi(n\alpha + x)q_n(t) + \varepsilon(q_{n+1}(t) + q_{n-1}(t)) + \delta v_n(t)|q_n(t)|^2 q_n(t), \tag{1.5}$$

where $n \in \mathbb{Z}$, $x \in \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and α satisfies Diophantine condition. Moreover, the potential $\{v_n(t)\}_{n\in\mathbb{Z}}$ satisfies

$$|v_n(t)| \leq Ce^{-\rho|n|}, \quad \rho > 0. \tag{1.6}$$

We are mainly interested in two topics about equations (1.4) and (1.5). We will discuss them separately in the following two subsections.

1.1 Growth of Soblev Norm

First, we are going to study the time evolution of the above equations, more precisely, to bound the norm

$$\Big(\sum_{n\in\mathbb{Z}}(1 + |n|^2)|q_n(t)|^2\Big)^{\frac{1}{2}}$$

in terms of t as $t \to +\infty$ for the initial condition

$$\sum_{n\in\mathbb{Z}}(1 + |n|^2)|q_n(0)|^2 = 1 \tag{1.7}$$

The expression in (1.8) is sometimes called the diffusion norm. The ℓ^2(i.e., H^0) norm $\sum_{n\in\mathbb{Z}} |q_n|^2$ is a conserved quantity for (1.4) and (1.5)(see (5.2) in the Appendix). The initial condition (1.7) shows the concentration on the lower modes q_n at $t = 0$, with $|n|$ not too large. The diffusion norm (1.8) measures the propagation into higher ones, $q_n, |n| \gg 1$. If one interprets n as an index of Fourier series, then (1.8) is the equivalent of H^1 norm for continuous nonlinear Schrödinger equations (NLS) on a circle. So in fact one could also pursue higher moments:

$$\|q\|_{H^p} := \Big\{\sum_{n\in\mathbb{Z}}(1 + |n|^{2p})|q_n(t)|^2\Big\}^{\frac{1}{2}}, \tag{1.8}$$

for $p > 1$, which correspond to H^p norms.

We have the following bound on the diffusion norm (1.8) for equations (1.4) and (1.5).

Theorem 1 *Let δ be a fixed positive number, α be a fixed Diophantine numbe and $x \in \mathbb{T}$ satisfy $n\alpha+x \neq \frac{1}{2}+k$ for any $n, k \in \mathbb{Z}$. Suppose that $v_n(t)$ satisfies the decay condition (1.6) with $\rho > 0$ fixed. There exists $\varepsilon_0 = \varepsilon_0(\alpha, x, \rho)$ sufficiently small such that if $0 < |\varepsilon| < \varepsilon_0$,*

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with some $\gamma > 0$ and $\sigma > d$,then the vector $\omega \in \mathbb{R}^d$ is said to be a Diophantine vector of type (γ, σ). Here $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathbb{R}^d$, $|\cdot|_1$ is the distance on $\mathbb{T}^1$ and $|k| = |k_1| + \cdots + |k_d|$. If $\omega = (\alpha, 1) \in \mathbb{R}^2$ is Diophantine, we say α is Diophantine.

then for any $p \geq 1$ and $\beta > 0$, the solutions $\{q_n(t)\}_{n\in\mathbb{Z}}$ for (1.4) and (1.5) with initial condition

$$\|q(0)\|_{H^{p+1+\beta}} = \Big(\sum_{n\in\mathbb{Z}}(1+|n|^{2(p+1+\beta)})|q_n(0)|^2\Big)^{\frac{1}{2}} = 1$$

will satisfy the following estimate:

$$\sum_{n\in\mathbb{Z}} |n|^{2p}|q_n(t)|^2 \leq C + C(\log t)^{(2p+1+\beta)}, \tag{1.9}$$

where the positive constant $C = C(\alpha, x, \delta, \varepsilon, \rho, p, \beta)$ is independent of t .

The linear equation

$$\mathrm{i}\dot{q}_n(t) = \tan\pi(n\alpha + x)q_n(t) + \varepsilon(q_{n+1}(t) + q_{n-1}(t)), n \in \mathbb{Z} \tag{1.10}$$

is called the Maryland model(see Appendix for more details about the Maryland model), which corresponds to $\delta = 0$ in (1.4) or (1.5). The equation (1.10) corresponds to the 1-body approximation to the many body problem in the study of electron conduction. In the many body cases, one needs to take into account the interactions among electrons, which is a hard problem. In Maryland model, interaction is approximated by the tangent potential and diffusion is obstructed by proving A. L., i.e., the existence of a complete set of ℓ^2 eigenfunctions which are well localized (actually uniformly localized) with respect to the canonical basis of $\mathbb{Z}$.

In this paper, we consider (1.10) perturbed by bounded, localized (in space), time-dependent terms $\delta v_n(t)q_n$(linear case) or $\delta v_n(t)|q_n|^2 q_n$(nonlinear case). Respectively, (1.4) and (1.5). In these perturbated cases, spectral theory for linear operator is no longer available. However, we are still interested in the persistence of A.L. with perturbations.

Considering the nonlinear case first, Geng and Zhao [GZ] have constructed time quasi-periodic solutions to the standard nonlinear Schrödinger equation on $\mathbb{Z}^1$:

$$\mathrm{i}\dot{q}_n(t) = \tan\pi(n\alpha + x)q_n(t) + \varepsilon(q_{n+1}(t) + q_{n-1}(t)) + \delta|q_n|^2 q_n, \tag{1.11}$$

which corresponds to $v_n(t) \equiv 1$ for all n in (1.5). The desired soloution survived on a set of (x, α) with positive measure and for a corresponding appropriate set of small initial data with compact support. Clearly such initial data are a subset of q_n satisfying (1.7). The present theorem is an attempt to address the growth of Sobolev norm for more generic initial data.

However, we have no idea for the Soblev bound for the standard equation (1.11) without any decay condition. Decay condition (1.6) is crucial for the proof of Theorem 1. A similar artificial decay condition appears in [BW2] when bounding the Soblev norm for nonlinear random Schrödinger equation (1.2). Bourgain and Wang proved in [BW2] that $\|q\|_{H^1}$ grows more slowly than the polynomial of t with any positive degree under the assumption $\delta_n \leq \varepsilon/|n|^A$ for ε sufficiently small and A sufficiently large, compared with the logarithmic growth (1.9) under the exponentially-decay condition (1.6) in (1.5). We need to point out that the same strategy used in Theorem 1 is sufficient to prove polynomial growth in (1.5) under the weaker decay condition $|v_n(t)| \leq 1/|n|^A$.

The soblev norm growth of the solution to the standard nonlinear Schrödinger equation with either random or general quasi-periodic potential remains largely open. [WZ] proved the so-called long time Anderson localization for the nonlinear random Schrödinger equation for arbitrary ℓ^2 initial data. The term 'long time' indicates that the diffusion remains small only between long enough but limited time interval. Few results were known about the turely long time behavior of the solution.

In the present paper, we recast the Schrödinger equation as infinite-dimensional Hamiltonian equations of motion and uses Birkhoff normal form type transformations. However, the proof of Theorem 1 is unlike [BW2] nor [GZ]. We do not need a series of symplectic changes to get a better normal form but only one change. Even the multiplier δ need not to be small in comparison with [BW2, GZ]. In [BW2], symplectic transforms were make to construct energy barriers centered at some $n_0 \in \mathbb{Z}, n_0 > 1$ of width $\log n_0$, where the terms responsible for mode propagation are small. In [GZ], the KAM scheme for infinite Hamiltonian PDE was developed to construct finite dimensional invariant torus. Both of them need to deal with small divisor problem and remove the resonant frequencies.

The proof of the present theorem relies strongly on the localization result for the Maryland model due to Bellissard, Lima and Scoppola [BLS]. According to [BLS](see Theorem 4 in Appendix), under the assumption of Theorem 1, we make a symplectic transform from $H^{p+\frac{1+\beta}{2}}$ to H^p(for any $p \geq 1$ and $\beta > 0$) to render the Hamiltonian (2.4) into a normal form amenable to the proof. The main feature of this normal form, to be spelt out completely in (2.7) is that the mode propagation terms decay fast along the diagonal direction. In the following two sections, we shall show the structure of the transformed Hamiltonian and then calculate the growth of Sobolev norm(1.9) directly from the new Hamiltonian(2.7). Then (1.9) will be a consequence of the bound of the unitary transformation. We note that in those two sections, linear case (1.4) and nonlinear case (1.5) will be treated together, no essential difference is involved.

1.2 Reducibility for Linear Equation

For linear equation (1.4), we want to proceed further. Indeed, we can consider a more general linear equation under time-dependent perturbations, i.e.,

$$\mathrm{i}\dot{q}_n = \tan \pi(n\alpha + x)q_n + \varepsilon(q_{n+1} + q_{n-1}) + \delta(\mathcal{V}(t)q)_n, \quad n \in \mathbb{Z}, \tag{1.12}$$

where $\mathcal{V}(t) : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is a bounded operator for any $t \in [0, +\infty)$, defined by

$$(\mathcal{V}(t)q)_n := \sum_{m\in\mathbb{Z}} v_{mn}(t)q_m, \ n \in \mathbb{Z}, \tag{1.13}$$

for each $q = (q_n)_{n\in\mathbb{Z}} \in \ell^2$ with the coefficients $v_{mn}(t)$ satisfying

$$|v_{mn}(t)| \leq Ce^{-\rho\cdot(|m|+|n|)}, \quad \rho > 0. \tag{1.14}$$

In Section 2, we will see that the Hamiltonian associated to (1.12) can be treated in the same way as the Hamiltonian associated to (1.4). Consequently, the diffusion bound (1.9) remains correct for (1.12).

A more special case is

$$v_{mn}(t) = \begin{cases} v(t), & m = n, \\ 0, & m \neq n, \end{cases} \quad \forall m, n \in \mathbb{Z}, \tag{1.15}$$

which corresponds to equation

$$\mathrm{i}\dot{q}_n = \tan \pi(n\alpha + x)q_n + \varepsilon(q_{n+1} + q_{n-1}) + \delta v(t)q_n, \quad n \in \mathbb{Z}. \tag{1.16}$$

This equation can also be transformed to an explicitly solvable one

$$\mathrm{i}\dot{Q}_n(t) = (\Omega_n + \delta v(t))Q_n(t),\ n \in \mathbb{Z}. \tag{1.17}$$

The corresponding Hamiltonian is completely diagonal and there is no interaction between different modes(i.e., $Q_n, Q_m, n \neq m$). Thus,

$$\|Q(t)\|_{H^1} = \|Q(0)\|_{H^1}$$

and

$$\|q(t)\|_{H^1} \leq C\|q(0)\|_{H^1}, \quad \forall t > 0. \tag{1.18}$$

The reason we are interested in the linear case (1.4) is not only the above generalization but also the following reducibility result: if the time-dependence on t of the potential $v_{mn}(t)$ is quasi-periodic, then the linear Schrödinger equation with such potential can be reduced to an autonomous equation for most values of the frequency vector.

To be precise, we study the following time quasi-periodic Schrödinger equation:

$$\mathrm{i}\dot{q}_n = \tan \pi(n\alpha + x)q_n + \varepsilon(q_{n+1} + q_{n-1}) + \delta \sum_{m\in\mathbb{Z}} v_{mn}(\omega t)q_m,\ n \in \mathbb{Z}, \tag{1.19}$$

where $\omega = (\omega_1, \cdots, \omega_d) \in \mathbb{T}^d$.

To proceed further, we assume $v_{mn}(\theta_1, \cdots, \theta_d) = v_{mn}(\theta)$ are functions on the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, and the frequency vector ω is regarded as a parameter chosen from $\mathcal{O} = \{y \in \mathbb{R}^d : |y| \leq 1\} \subset \mathbb{R}^d$. The function $v_{mn}(\theta; \omega)$ is C^1-smooth in (θ, ω) and is analytic in θ for any $m, n \in \mathbb{Z}$. For some $R > 0$, $v_{mn}(\theta)$ can be extended analytically in θ to the domain

$$\mathbb{T}^d_R = \{z = (z_1, \cdots, z_d) \in \mathbb{C}^d/\mathbb{Z}^d : |\Im z_i| < R,\ i = 1, \cdots, d\}.$$

We have the following reducibility result for (1.19).

Theorem 2 *Suppose that v_{mn} in (1.19) satisfies*

$$|v_{mn}(\omega t)| \leq e^{-\rho(|m|+|n|)}, \quad \rho > 0.$$

Let α be a fixed Diophantine number, $x \in \mathbb{T}$ satisfies $n\alpha + x \neq \frac{1}{2} + k$ for any $n, k \in \mathbb{Z}$. For $\varepsilon = \varepsilon(\alpha, x, \rho)$ sufficiently small, there exists $\delta_0 = \delta_0(\varepsilon, d, R) > 0$, such that the non-autonomous equation (1.19) can be analytically reduced to an autonomous equation.

More precisely, for any $0 < \delta < \delta_0(\varepsilon, d, R)$, there exists a Cantor set $\mathcal{O}_\delta \subset \mathbb{T}^d$, such that for any $\omega \in \mathcal{O}_\delta$, $\theta \in \mathbb{T}^d$, there exists a complex linear isomorphism $\Phi(\theta) = \Phi(\omega t; \delta)$ in the space $\ell^2(\mathbb{Z})$ with the following properties:

(T1) $\Phi(\theta) = \Phi(\omega t; \delta)$ *is analytic in* $\theta \in \mathbb{T}^d_R$ *and depends smoothly on* ω*;*
(T2) $\Phi(\theta)$ *transforms equation (1.19) into an autonomous system of the form*

$$\mathrm{i}\dot{Q}_n(t) = (\mathcal{K}Q)_n(t) = \Omega_n Q_n(t) + \delta \sum_{m\in\mathbb{Z}} A_{mn} Q_n(t), \quad Q(t) = \Phi(\omega t) q(t), \tag{1.20}$$

here $\mathcal{K}$ *is a Hermitian operator depending smoothly on* ω *but independent of* t*, and* $\{\Omega_n\}_{n\in\mathbb{Z}}$ *are of the form* $\Omega(n\alpha + x)$ [2]*;*
(T3) $\Phi, \Omega, A, \mathcal{O}_\delta$ *meet the estimates*

$$\begin{aligned}
&\|\Phi(\theta) - Id\|_{\ell^2} \le \beta\delta, \\
&\|\Omega(n\alpha + x) - \tan\pi(n\alpha + x)\|_{\sup(\mathbb{T}^1)} \le C\varepsilon, \\
&|A_{mn}| \le Ce^{-\frac{1}{10}\rho\cdot max\{|m|,|n|\}}, \\
&\mathrm{mes}(\mathbb{T}^d - \mathcal{O}_\delta) \le C\delta^\kappa,
\end{aligned}$$

where $C > 0, \kappa > 0$ *depend only on* d, R *and* β *depends on* d, R, ω*.*

Remark 1.1 *The linear operators in the right hand side of linear Hamiltonian equations (1.19) and (1.20) are complex linear Hermitian transformations. So the flow-maps of these equations are complex linear, symplectic and unitary. The conjugating transformations* $U(\theta)$ *are complex linear. It can be shown that they also are symplectic. Hence, they are unitary. So the conjugations respect all structures, preserved by Eqs. (1.19) and (1.20).*

The motivation for studying (1.19) comes from questions of Anderson localization for non-linear Schrödinger equations (see e.g., [DS, FSW] for random model), which in turn is a approximation to the many body problem mentioned earlier.

On the other hand, we remark that linear Schrödinger equations with time-dependent potential in the form

$$\mathrm{i}\frac{\partial}{\partial t}u = \Delta u + V(t,x)u, \;\; u(t,x) \in L^2_x(\mathbb{T}^d) \tag{1.21}$$

is considered more extensively in the continuum medium. For details, see e.g. [B1, B2, W1, W2, EK, N]. The problem of growth of solutions for the linear Schrödinger equation with time quasi-periodic and with smooth bounded potentials was considered by J. Bourgain in [B1, B2], respectively. In [B1], it was shown that for a Diophantine frequency vector ω Sobolev norms of any solution for (1.21) grow with t at most logarithmically, while results of [B2] imply that for any ω each Sobolev norm grows slower than any positive degree of t. Wang has strengthened the result of [B2] on the circle in [W1]. In contrast, it was proven by Nersesyanin in [N] that with the additional assumption that the time dependence is random, the Sobolev norms are unbounded with probability 1.

Eliasson and Kuksin have specified these results for 'typical' vectors ω in any dimension. In [EK], it was shown that the Sobolev norms of solutions for Eq. (1.21) remain bounded in time, provided that the frequency vector ω is 'typical'. Wang [W2] proved the norms of the solutions may stay bounded also in the opposite case when ω is 'completely

[2] $\Omega(z)$ is a period 1 meromorphic function on $\mathcal{D}_R = \{z \in \mathbb{C} : |\Im z| < R\}$. See details in the Appendix.

resonant' for Eq. (1.21) where $d = 1$ and $\omega = 1$.

However, in the discrete case, there are very few results on the bound of Soblev norms. At present, it seems difficult to completely understand the long time behavior of the solutions for lattice PDEs with generic intial value. Only the nonlinear random model was treated in few articles. Bourgain and Wang [BW1] have studied the discrete case with random potential and time quasi-periodic perturbation(i.e.,(1.1)). They proved that the displacement of the wave front is uniform in t, i.e., for any $\gamma > 0$, there exists N such that

$$\sup_t \sum_{|n|>N} |q_n(t)|^2 \leq \gamma \tag{1.22}$$

compared with the standard nonlinear case in [WZ] where the displacement of the wave front N can not be uniformly bounded in t, which is believed to be the true behavior of the nonlinear equation. Bourgain and Wang got the desired dynamical behavior of the solution from the spectral property of the so-called quasi energy operator

$$K = -\mathrm{i}\frac{\partial}{\partial t} + \varepsilon\Delta + V + \mathcal{W}(t)$$

acting on $\ell^2(\mathbb{Z}^\nu) \times L^2(\mathbb{T}^d)$. See more related work in [H1, SW, YK] about the quasi energy operator and the quantum stability.

We need to point out that the growth of Sobolev norm in (1.1) can not be derived from (1.22) directly, since the relation between γ and N in (1.22) was not specified from the spectral theory in [BW1]. While dealing with the tangent model (1.4) in this paper, though in Theorem 1 we get $\log t$ bound of the Sobolev norm with no more specific condition on the time dependence, we have no idea of the spectral property of the quasi energy operator nor such diffusion tail estimate as in (1.22).

Theorem 1 and 2 of this paper is an attemp to address more dynamical results(long time behavior and stability) for general discrete quasi-periodic Schrödinger equation. We shall focus on the model with tangent potential. Our results extend automatically to a class of quasiperiodic functions $\mathcal{P}$ constructed by Bellissard, Lima and Scoppola (see [BLS] for precise definition of class $\mathcal{P}$). The class $\mathcal{P}$ has singularities, containing the tangent model and the spectrum of the corresponding Schrödinger operator has been studied extensively. Our aim is to generalize the results to general quasi-periodic Schrödinger operator(e.g., the almost Mathieu operator with cosine potential). The ideas and techniques needed for general quasi-periodic potential might be more involved and distinct from the random model. The linear operator theory for quasi-periodic case is more difficult and the theory is far less developed than for the random case since, among other reasons, quasiperiodicity does not allow for nice perturbations. When we deal with the discrete nonlinear quasi-periodic Schrödinger equations, the small-divisor problem does not only come from the linear part but also from the nonlinear interaction. See [J] for more comparison between random Schrödinger operator and quasi-periodic one. The literature [J] also contains the most extensive results about linear random and quasi-periodic Schrödinger operators. As we have mentioned above, there are many open problems related to these two models(as well as their nonlinear version), there is still a long way to go to truly understand any of the problems.

The rest of this paper is organized as follows. We transform (1.4) and (1.5) into new forms via the change of coordinates in [BLS] in Section 2. Then we shall complete the proof of Theorem 1 in Section 3 based on the special structure of the transformed system. In section 4 we derive Theorem 2 from an abstract KAM theorem in [GZ] and will discuss more details. Section 5 is regarded as an appendix, in which we prove the ℓ^2 conservation law and present Localization results for the Maryland model in [BLS].

2 Structure of the Transformed Hamiltonians and Analysis of the Symplectic Transformation

In this section, we use the transformation in [BLS] to render equations (1.4) and (1.5) into a new form amenable to the proof of Theorem 1. Recall from Section 1, the Schrödinger equation:

$$\mathrm{i}\dot{q}_n = \tan \pi(n\alpha + x)q_n + \varepsilon(q_{n+1} + q_{n-1}) + \delta v_n(t)|q_n|^{2\tau-2}q_n \tag{2.1}$$

where $\tau = 1$ in the linear case (1.4) and $\tau = 2$ in the nonlinear case (1.5). Here α is a fixed Diophantine number, x satisfies that $n\alpha + x \neq \frac{1}{2} + k$ for any $n, k \in \mathbb{Z}$ and $0 < \varepsilon \ll 1$. The potential $v_n(t)$ satisfies the condition

$$|v_n(t)| \leq e^{-\rho|n|} \tag{2.2}$$

with $\rho > 0$ fixed. As mentioned in Section 1, (2.2) (previously (1.6)) is crucial for the transformation below.

Equation (2.1) can be recast as infinite-dimensional Hamiltonian equations of motion

$$\mathrm{i}\dot{q}_n = \frac{\partial H}{\partial \bar{q}_n}, \quad n \in \mathbb{Z}, \tag{2.3}$$

with canonical variables $(q, \bar{q})$ and the Hamiltonian

$$H(q,\bar{q}) = \sum_{n\in\mathbb{Z}} \tan \pi(n\alpha + x)|q_n|^2 + \varepsilon \sum_{n\in\mathbb{Z}} (q_{n+1} + q_{n-1})\bar{q}_n + \delta \sum_{n\in\mathbb{Z}} v_n(t)|q_n|^{2\tau}, \quad \tau = 1, 2. \tag{2.4}$$

As mentioned in Section 1, we use the complete set of ℓ^2 eigenfunctions for Maryland model to obstruct energy transfer from low to high modes. This obstruction is achieved by controlling the truncated sum of higher modes

$$\sum_{|n|>n_0} |q_n|^2, \quad n_0 \gg 1, \tag{2.5}$$

which in turn enables us to control the sum

$$\sum_{n\in\mathbb{Z}} |n|^2|q_n|^2. \tag{2.6}$$

We construct a unitary transformation $U : \ell^2 \to \ell^2$ by the set of eigenfunctions and do coordinates change $q = UQ$. Then $\Gamma = (U, \bar{U})$ will be sympletic.

Since the eigenfunctions are uniformly localized, the decay property (2.2) is preserved in some sense and the transformation U are bounded on H^1. Through Γ, H in (2.4) is transformed into $H' = H \circ \Gamma$ of the form:

$$H'(Q,\bar{Q}) = \sum_{n\in\mathbb{Z}} \Omega_n |Q_n|^2 + \delta P'(Q,\bar{Q}), \tag{2.7}$$

where $\{\Omega_n\}_{n\in\mathbb{Z}}$ are the eigenvalues of the linear operator according to the Maryland model, and

$$P'(Q,\bar{Q}) = \sum_{m,n\in\mathbb{Z}} P'_{mn}(t) Q_m \bar{Q}_n \quad \text{when } \tau = 1, \tag{2.8}$$

$$P'(Q,\bar{Q}) = \sum_{m,n,j,k\in\mathbb{Z}} P'_{mnjk}(t) Q_m \bar{Q}_n Q_j \bar{Q}_k \quad \text{when } \tau = 2, \tag{2.9}$$

satisfying

$$|P'_{mn}(t)| \le 5e^{-\frac{1}{4}\rho\cdot max\{|n|,|m|\}}, \tag{2.10}$$

$$|P'_{mnjk}(t)| \le 5e^{-\frac{1}{4}\rho\cdot max\{|n|,|m|,|j|,|k|\}}. \tag{2.11}$$

We state the above assertions about Hamiltonian (2.4) as the following Lemma:

Lemma 2.1 *Let α be a fixed Diophantine number, $x \in \mathbb{T}$ satisfy that $n\alpha + x \neq \frac{1}{2} + k$ for any $n, k \in \mathbb{Z}$ and δ, ρ be fixed positive numbers. Suppose that $v_n(t)$ in (2.4) satisfies the decay condition (2.2). Then for $\varepsilon = \varepsilon(\alpha, x, \rho)$ sufficiently small, there exist a unitary operator U and a corresponding symplectic transformation $\Gamma = (U, \bar{U})$ such that the Hamiltonian H in (2.4) is transformed into $H' = H \circ \Gamma$ in (2.7), satisfying decay properties (2.10) when $\tau = 1$ and (2.11) when $\tau = 2$.*

U, U^* preserve ℓ^2 norm. Moreover, for any $p \ge 1$ and $\beta > 0$, U, U^* are bounded operators from $H^{p+\frac{2+\beta}{2}}$ to H^p, i.e.,

$$\|UQ\|_{\ell^2} = \|Q\|_{\ell^2}, \quad \|U^*q\|_{\ell^2} = \|q\|_{\ell^2},$$

$$\|UQ\|_{H^p} \le C\|Q\|_{H^{p+\frac{2+\beta}{2}}}, \quad \|U^*q\|_{H^p} \le C\|q\|_{H^{p+\frac{2+\beta}{2}}}, \tag{2.12}$$

where the constant C depends on α, x, ρ but is independent of t.

Proof: Let

$$H_0 = \sum_{n\in\mathbb{Z}} \tan\pi(n\alpha + x)|q_n|^2 + \varepsilon \sum_{n\in\mathbb{Z}} (q_{n+1} + q_{n-1})\bar{q}_n, \tag{2.13}$$

where α is a fixed Diophantine number, and $x \in \mathbb{T}$ satisfies that $n\alpha + x \neq \frac{1}{2} + k$ for any $n, k \in \mathbb{Z}$. According to Theorem 4 in the Appendix, given any positive ρ_u, for sufficiently small ε, there exists a unitary operator $U = U_{\alpha,x,\rho_u} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$(independent of t), such that under the coordinate change

$$q_n(t) = \sum_{j\in\mathbb{Z}} U_{nj} Q_j(t), \quad n \in \mathbb{Z}, \tag{2.14}$$

H_0 is transformed into

$$H'_0(Q,\bar{Q}) = \sum_{n\in\mathbb{Z}} \Omega_n |Q_n(t)|^2, \tag{2.15}$$

where $\{Q_j(t)\}_{j\in\mathbb{Z}} \in \ell^2$ is the new coordinate and U_{nj} is the (n,j) matrix element of U satisfying

$$|U_{nj}| \leq e^{-\rho_u|n-j|}. \tag{2.16}$$

Applying the coordinate change (2.14) to $H = H_0 + \delta P$ in (2.4), where

$$P = \sum_{n\in\mathbb{Z}} v_n(t)|q_n|^{2\tau} \quad \tau = 1, 2, \tag{2.17}$$

one gets immediately that

$$H'(Q, \bar{Q}) = \sum_{n\in\mathbb{Z}} \Omega_n |Q_n|^2 + \delta P'(Q, \bar{Q}). \tag{2.18}$$

When $\tau = 1$ in (2.17),

$$P = \sum_{n\in\mathbb{Z}} v_n(t) q_n \bar{q}_n,$$

then

$$\begin{aligned} P' = P \circ \Gamma = P(UQ, \overline{UQ}) &= \sum_{n\in\mathbb{Z}} v_n(t)\Big(\sum_{j\in\mathbb{Z}} U_{nj} Q_j\Big)\Big(\sum_{k\in\mathbb{Z}} \bar{U}_{nk} \bar{Q}_k\Big) \\ &= \sum_{j,k\in\mathbb{Z}} \Big(\sum_{n\in\mathbb{Z}} v_n(t) U_{nj} \bar{U}_{nk}\Big) Q_j \bar{Q}_k \\ &:= \sum_{j,k\in\mathbb{Z}} P'_{jk} Q_j \bar{Q}_k, \end{aligned} \tag{2.19}$$

where

$$P'_{jk} = \sum_{n\in\mathbb{Z}} v_n(t) U_{nj} \bar{U}_{nk}. \tag{2.20}$$

Considering the decay condition (2.2), (2.16) for $v_n(t)$ and U, and assuming $|j| \geq |k| > 0$, we have

$$\begin{aligned} |P'_{jk}| &\leq \sum_{n\in\mathbb{Z}} e^{-\rho|n|} e^{-\rho_u|n-j|} e^{-\rho_u|n-k|} \\ &\leq \sum_{n\in\mathbb{Z}} e^{-\rho|n|} e^{-\rho_u|n-j|} \\ &\leq \sum_{|n-j|>\frac{|j|}{2}} e^{-\rho|n|} e^{-\rho_u|n-j|} + \sum_{|n-j|\leq\frac{|j|}{2}} e^{-\rho|n|} e^{-\rho_u|n-j|} \\ &\leq e^{-\rho_u\frac{|j|}{2}} \sum_{|n-j|>\frac{|j|}{2}} e^{-\rho|n|} + e^{-\rho\frac{|j|}{2}} \sum_{|n-j|\leq\frac{|j|}{2}} e^{-\delta|n-j|} \\ &\leq 5e^{-\rho_0|j|}, \end{aligned}$$

where $\rho_0 = \frac{1}{2}min\{\rho_u, \rho\} > 0$. Take $\rho_u = \rho$, then $\rho_0 \geq \rho/4$.

The same computation can be applied to the nonlinear case. Indeed, when $\tau = 2$ in (2.17),

$$P = \sum_{n\in\mathbb{Z}} v_n(t) q_n \bar{q}_n q_n \bar{q}_n,$$

then

$$\begin{aligned}
& P'(Q,\bar{Q}) \\
= & \sum_{j,k,l,m\in\mathbb{Z}} \Big(\sum_{n\in\mathbb{Z}} v_n(t)U_{nj}\bar{U}_{nk}U_{nl}\bar{U}_{nm}\Big)Q_j\bar{Q}_kQ_l\bar{Q}_m \\
:= & \sum_{j,k,l,m\in\mathbb{Z}} P'_{jklm}Q_j\bar{Q}_kQ_l\bar{Q}_m.
\end{aligned}$$

Suppose that $|j|\geq|k|\geq|l|\geq|m|$, then

$$\begin{aligned}
|P'_{jklm}| &\leq \sum_{n\in\mathbb{Z}} |v_n(t)U_{nj}\bar{U}_{nk}U_{nl}\bar{U}_{nm}| \\
&\leq \sum_{n\in\mathbb{Z}} e^{-\rho|n|}e^{-\rho_u|n-j|}e^{-\rho_u|n-k|}e^{-\rho_u|n-l|}e^{-\rho_u|n-m|} \\
&\leq \sum_{n\in\mathbb{Z}} e^{-\rho|n|}e^{-\rho_u|n-j|} \\
&\leq 5e^{-\frac{1}{4}\rho|j|}.
\end{aligned}$$

Since U and U^* are unitary, the ℓ^2 conservation is obvious. The H^1 boundness (2.12) follows immediately from (2.16):

$$\begin{aligned}
\Big(\sum_{n\in\mathbb{Z}}|n|^{2s}|(UQ)_n|^2\Big)^{\frac{1}{2}} &= \Big(\sum_{n\in\mathbb{Z}}|n|^{2p}|\sum_{j\in\mathbb{Z}}U_{nj}Q_j|^2\Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{n\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}|n|^pe^{-\rho_u|n-j|}|Q_j|\Big)^2\Big)^{\frac{1}{2}} \\
&\leq \sum_{j\in\mathbb{Z}}\Big(\sum_{n\in\mathbb{Z}}|n|^{2p}e^{-2\rho_u|n-j|}|Q_j|^2\Big)^{\frac{1}{2}} \\
&\leq C_{\rho_u}\sum_{j\in\mathbb{Z}}|Q_j||j|^s \\
&\leq C\Big(\sum_{j\in\mathbb{Z}}|j|^{2p+1+\beta}|Q_j|^2\Big)^{\frac{1}{2}}\Big(\sum_{j\in\mathbb{Z}}\frac{1}{|j|^{1+\beta}}\Big)^{\frac{1}{2}} \\
&\leq C\Big(\sum_{j\in\mathbb{Z}}|j|^{2p+1+\beta}|Q_j|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

$$\|UQ\|_{H^p}\leq\|UQ\|_{\ell^2}+\Big(\sum_{n\in\mathbb{Z}}|n|^{2p}|(UQ)_n|^2\Big)^{\frac{1}{2}}\leq\|Q\|_{\ell^2}+C\|Q\|_{H^{p+\frac{2+\beta}{2}}}\leq 2C\|Q\|_{H^{p+\frac{2+\beta}{2}}}.$$

Thus Lemma 2.1 is proved. ■

3 Growth of Soblev Norm and Proof of Theorem 1

In the previous section, the Hamiltonian H in (2.4) is transformed into H' in (2.7) with decay properties (2.10) and (2.11) when $\tau=1$ and $\tau=2$ respectively. Now we are ready to calculate the growth of H^1 norm of the solution $\{Q_n(t)\}_{n\in\mathbb{Z}}$ associated to Hamiltonian equations of motion

$$\mathrm{i}\dot{Q}_n = \frac{\partial H'}{\partial \bar{Q}_n}, \quad n \in \mathbb{Z}. \tag{3.1}$$

We only deal with the case P' in (2.8) with decay condition (2.10), i.e.,

$$H'(Q,\bar{Q}) = \sum_{n\in\mathbb{Z}} \Omega_n |Q_n|^2 + \delta \sum_{m,n\in\mathbb{Z}} P'_{mn}(t) Q_m \bar{Q}_n, \tag{3.2}$$

with

$$|P'_{mn}(t)| \leq e^{-\frac{1}{4}\rho\cdot max\{|n|,|m|\}}, \quad \rho > 0. \tag{3.3}$$

The nonlinear case (2.9) can be proved in a similar way and will be discussed later.

For convenience, we omit the prime in (3.1)-(3.3) when we state the following result.

Lemma 3.1 *Suppose that $P_{mn}(t)$ in (3.2) satisfies (3.3), then the solution $\{Q_n(t)\}_{n\in\mathbb{Z}}$ of (3.1) satisfies that*

$$\sum_{n\in\mathbb{Z}} |n|^{2p} |Q_n(t)|^2 \leq C_p \Big(1 + (\frac{32}{\rho}\log t)^{2p}\Big), \tag{3.4}$$

provided that the initial value $Q(0) = \{Q_n(0)\}_{n\in\mathbb{Z}} \in H^p$ with $\|Q(0)\|_{H^p} \leq C$. The constant C_p is independent of t.

Proof: For some $s_0(t) \in \mathbb{N}$ large enough($2^{s_0(t)} \sim \log t$, which will be specified later), we do the following partition to P with respect to every $s \in \mathbb{N}$, $s \geq s_0(t)$:

$$P(Q,\bar{Q}) = \sum_{m,n\in\mathbb{Z}} P_{mn}(t) Q_m \bar{Q}_n = \sum_{\substack{m,n\in\mathbb{Z}\\ |m|,|n|\leq 2^s}} P_{mn}(t) Q_m \bar{Q}_n + \sum_{\substack{m,n\in\mathbb{Z}\\ max\{|m|,|n|\}>2^s}} P_{mn}(t) Q_m \bar{Q}_n, \tag{3.5}$$

and use A^s and B^s to denote the two parts respectively,

$$A^s := \sum_{\substack{m,n\in\mathbb{Z}\\ |m|,|n|\leq 2^s}} P_{mn}(t) Q_m \bar{Q}_n \quad \text{and} \quad B^s := \sum_{\substack{m,n\in\mathbb{Z}\\ max\{|m|,|n|\}>2^s}} P_{mn}(t) Q_m \bar{Q}_n.$$

Then we estimate the derivative of the the truncated sum:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \sum_{2^s<|n|\leq 2^{s+1}} |Q_n(t)|^2 &= \sum_{2^s<|n|\leq 2^{s+1}} \dot{Q}_n \bar{Q}_n + Q_n \dot{\bar{Q}}_n \\ &= \frac{1}{\mathrm{i}} \sum_{2^s<|n|\leq 2^{s+1}} \frac{\partial H}{\partial \bar{Q}_n} \bar{Q}_n - \frac{\partial H}{\partial Q_n} Q_n. \end{aligned} \tag{3.6}$$

Notice that indices of terms in the sum A^s are all less than 2^s, which implies

$$\frac{\partial A^s}{\partial \bar{Q}_n} = \frac{\partial A^s}{\partial Q_n} = 0 \tag{3.7}$$

for $|n| > 2^s$ and $s \geq s_0$, so only terms in B^s contribute to (3.6) and according to the decay property (3.3) we have that

$$
\begin{aligned}
& \left| \frac{\mathrm{d}}{\mathrm{dt}} \sum_{2^s<|n|\leq 2^{s+1}} |Q_n(t)|^2 \right| \\
\leq & \left| \sum_{\substack{m,n\in\mathbb{Z} \\ 2^s<|n|\leq 2^{s+1}}} P_{mn}(t) Q_m \bar{Q}_n - \sum_{\substack{m,n\in\mathbb{Z} \\ 2^s<|n|\leq 2^{s+1}}} P_{nm}(t) Q_n \bar{Q}_m \right| \\
\leq & \ 2 \sum_{\substack{m,n\in\mathbb{Z} \\ 2^s<|n|\leq 2^{s+1}}} e^{-\frac{1}{4}\rho\cdot max\{|n|,|m|\}} |Q_m||Q_n| \\
\leq & \ 2 \sum_{\substack{m,n\in\mathbb{Z} \\ |n|>2^s}} e^{-\frac{1}{8}\rho\cdot(|m|+|n|)} |Q_m||Q_n| \\
\leq & \ 2 \sum_{\substack{n\in\mathbb{Z} \\ |n|>2^s}} e^{-\frac{1}{8}\rho\cdot|n|} |Q_n| \sum_{m\in\mathbb{Z}} e^{-\frac{1}{8}\rho\cdot|m|} |Q_m| \\
\leq & \ 4 e^{-\frac{1}{16}\rho\cdot 2^s}.
\end{aligned} \tag{3.8}
$$

Let $\varepsilon_s = 4e^{-\frac{1}{16}\rho\cdot 2^s}$ and integrate (3.8) in t, we obtain

$$
\sum_{2^s<|n|\leq 2^{s+1}} |Q_n(t)|^2 \leq \sum_{2^s<|n|\leq 2^{s+1}} |Q_n(0)|^2 + \varepsilon_s t \tag{3.9}
$$

and

$$
\begin{aligned}
\sum_{|n|>2^{s_0}} |n|^{2p} |Q_n(t)|^2 &= \sum_{s\geq s_0} \sum_{2^s<|n|\leq 2^{s+1}} |n|^{2p} |Q_n(t)|^2 \\
&\leq \sum_{s\geq s_0} \sum_{2^s<|n|\leq 2^{s+1}} (2^{s+1})^{2p} |Q_n(t)|^2 \\
&\leq \sum_{s\geq s_0} (2\cdot 2^s)^{2p} \sum_{2^s<|n|\leq 2^{s+1}} |Q_n(0)|^2 + \sum_{s\geq s_0} (2^{s+1})^{2p} \varepsilon_s t \\
&\leq 4 \sum_{s\geq s_0} \sum_{2^s<|n|\leq 2^{s+1}} |n|^{2p} |Q_n(0)|^2 + \sum_{s\geq s_0} (2^{s+1})^{2p} \varepsilon_s t \qquad (3.10) \\
&\leq C_p (1 + t e^{-\frac{1}{32}\rho\cdot 2^{s_0}}). \qquad (3.11)
\end{aligned}
$$

Note that from (3.10) to (3.11) we use the initial condition $\|Q(0)\|_{H^p} \leq C$ and the super-expotential convergence of ε_s in (3.8). Meanwhile,

$$
\begin{aligned}
\sum_{|n|\leq 2^{s_0}} |n|^{2p} |Q_n(t)|^2 &\leq 4^{ps_0} \sum_{|n|\leq 2^{s_0}} |Q_n(t)|^2 \\
&\leq 4^{ps_0} \sum_{n\in\mathbb{Z}} |Q_n(t)|^2 \qquad (3.12) \\
&= 4^{ps_0} \sum_{n\in\mathbb{Z}} |Q_n(0)|^2 \qquad (3.13) \\
&\leq 4^{ps_0} C. \qquad (3.14)
\end{aligned}
$$

From (3.12) to (3.14), the ℓ^2 conservation law and the initial condition are used again. Now take 2^{s_0} to be about $\frac{32}{\rho}\log t$, one finally gets the bound

$$\sum_{n\in\mathbb{Z}}|n|^{2p}|q_n(t)|^2 \le 4^{ps_0}C + C_p(1+te^{-\frac{1}{32}\rho\cdot 2^{s_0}}) \le C_p\Big(1+(\frac{32}{\rho}\log t)^{2p}\Big). \tag{3.15}$$

■

Remark 3.1 *In the nonlinear case (2.9) where*

$$P'(Q,\bar{Q}) = \sum_{m,n,j,k\in\mathbb{Z}} P'_{mnjk}(t)Q_m\bar{Q}_nQ_j\bar{Q}_k,$$

the only difference lies in the estimates of $\frac{\mathrm{d}}{\mathrm{dt}}\sum_{2^s<|n|\le 2^{s+1}}|Q_n(t)|^2$*. Actually, we have*

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{dt}}\sum_{2^s<|n|\le 2^{s+1}}|Q_n(t)|^2\right| &\le 4\sum_{\substack{m,n,j,k\in\mathbb{Z}\\ max\{|m|,|n|,|j|,|k|\}>2^s}}|P_{mnjk}(t)Q_m\bar{Q}_nQ_j\bar{Q}_k| \\ &\le 4\sum_{\substack{m,n,j,k\in\mathbb{Z}\\ max\{|m|,|n|,|j|,|k|\}>2^s}}e^{-\frac{1}{16}\rho\cdot(|m|+|n|+|j|+|k|)}|Q_m||Q_n||Q_k||Q_j| \\ &\le 16e^{-\frac{1}{32}\rho 2^s}.\end{aligned}$$

Proof of Theorem 1: Consider the lattice equation

$$\mathrm{i}\dot{q}_n = \tan\pi(n\alpha+x)q_n + \varepsilon(q_{n+1}+q_{n-1}) + \delta v_n(t)|q_n|^{\tau-1}q_n, \quad \tau=1,2 \tag{3.16}$$

with initial condition

$$\|q(0)\|_{H^{p+1+\beta}} = \Big(\sum_{n\in\mathbb{Z}}(1+|n|^{2(p+1+\beta)})|q_n(0)|^2\Big)^{\frac{1}{2}} = 1.$$

Fix α, x, ρ as needed in Lemma 2.1, $U = U_{\alpha,x,\rho}$ is the unitary operator given by Lemma 2.1. Let $q = UQ$, then

$$\|Q(0)\|_{H^{p+\frac{1+\beta}{2}}} = \|U^*q(0)\|_{H^{p+\frac{1+\beta}{2}}} \le C\|q(0)\|_{H^{p+1+\beta}} \le C, \quad C = C(\alpha,x,\varepsilon,\rho).$$

and the transformed Hamilton equations of $Q(t)$ satisfies the decay condition in Lemma 3.1. By Lemma 3.1,

$$\sum_{n\in\mathbb{Z}}|n|^{2(p+\frac{1+\beta}{2})}|Q_n(t)|^2 \le C_{p,\beta}\Big(1+(\frac{32}{\rho}\log t)^{(2p+1+\beta)}\Big).$$

Therefore,

$$\sum_{n\in\mathbb{Z}}|n|^{2p}|q_n(t)|^2 \le \|q(t)\|^2_{H^p} = \|UQ(t)\|^2_{H^p} \le C^2\|Q(t)\|^2_{H^{p+\frac{1+\beta}{2}}} \le C_{p,\beta}\Big(1+(\frac{32}{\rho}\log t)^{(2p+1+\beta)}\Big),$$

where $C_{p,\beta}$ is the constant depending on $\alpha, x, \delta, \varepsilon, p, \beta$ only. ■

4 Proof of the reducibility result

In this section, we drive Theorem 2 from a KAM theory in [GZ].(See related KAM scheme in [GVYi, Y].) In order to prove Theorem 2, firstly we re-interpret Eq. (1.19) as an autonomous Hamiltonian system in an extended phase-space:

$$\mathcal{Z}_R = \mathbb{T}_R^d \times \mathbb{C}^d \times \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) = \{(\theta, I, q, \bar{q})\},$$

where

$$\mathbb{T}_R^d := \{\theta = (\theta_1, \cdots, \theta_d) \in \mathbb{C}^d/\mathbb{Z}^d : |\Im\theta_i| < R, \quad i = 1, \cdots, d\}.$$

The symplectic structure on $\mathcal{Z}_R$ corresponds to the Hamiltonian equations of motion:

$$\mathrm{i}\dot{q}_n = \frac{\partial H}{\partial \bar{q}_n}, \quad \dot{\theta} = \frac{\partial H}{\partial I} = \omega, \quad \dot{I} = -\frac{\partial H}{\partial \theta}, \tag{4.1}$$

with the Hamilton function

$$H(\theta, I, q, \bar{q}) = \langle \omega, I\rangle + \sum_{n\in\mathbb{Z}} \tan \pi(x + n\alpha)|q_n|^2 + \varepsilon \sum_{n\in\mathbb{Z}} (q_{n+1} + q_{n-1})\bar{q}_n + \delta \sum_{m,n\in\mathbb{Z}} v_{mn}(\theta) q_m \bar{q}_n, \tag{4.2}$$

which is analytic in $\mathcal{Z}$. Here, the coefficients $v_{mn}(\theta)$ satisfy that

$$|v_{mn}(\theta)| \le Ce^{-\rho\cdot(|m|+|n|)} \tag{4.3}$$

with some $\rho > 0$. The first two equations of (4.1) are independent of I and are equivalent to equation (1.19).

As we have done in Lemma 2.1, given $x \in \mathbb{T}$ satisfying that $n\alpha + x \neq \frac{1}{2} + k$ for any $n, k \in \mathbb{Z}$, let the Diophantine number α and the positive number ρ in (4.3) be fixed and let $\varepsilon = \varepsilon(\alpha, x, \rho)$ be sufficiently small. Let $\Gamma = (U, \bar{U})$ be the symplectic transformation given in Lemma 2.1. Then through $(q, \bar{q}) = (Uq', \bar{U}\bar{q}')$ the Hamiltonian (4.2) can be transformed into H', where

$$H' = \langle \omega, I\rangle + \sum_{n\in\mathbb{Z}} \Omega_n |q_n'|^2 + \delta \sum_{m,n\in\mathbb{Z}} P_{mn}(\theta) q_m' \bar{q}_n', \tag{4.4}$$

with

$$P_{mn}(\theta) = \sum_{j,l\in\mathbb{Z}} v_{jl}(\theta) U_{jm} \bar{U}_{ln}$$

We have known that $v_{mn}(\theta)$ is analytic in $\mathbb{T}_R^d$. Expand $v_{mn}(\theta)$ further into Fourier series with respect to the basis $\{e^{2\pi\mathrm{i}\langle k,\theta\rangle}\}_{k\in\mathbb{Z}^d}$:

$$v_{jl}(\theta) = \sum_{k\in\mathbb{Z}^d} e^{2\pi\mathrm{i}\langle k,\theta\rangle} v_{jlk} \quad \forall j, l \in \mathbb{Z}$$

with

$$|v_{jlk}| \le e^{-R|k|} \|v_{jl}(\theta)\|_{\sup(\mathbb{T}_R^d)}, \quad \forall j, l \in \mathbb{Z}, \quad \forall k \in \mathbb{Z}^d$$

Omit the prime in (4.4) for convenience, then we have

$$H_\omega^\delta(\theta, I, q, \bar{q}) = \langle \omega, I\rangle + \sum_{n\in\mathbb{Z}} \Omega_n |q_n|^2 + \delta \sum_{\substack{k\in\mathbb{Z}^d \\ m,n\in\mathbb{Z}}} P_{mnk} e^{2\pi\mathrm{i}\langle k,\theta\rangle} q_m \bar{q}_n, \tag{4.5}$$

where

$$P_{mnk} = \sum_{j,l\in\mathbb{Z}} v_{jlk} U_{jm} \bar{U}_{ln}.$$

satisfying that

$$|P_{mnk}| \leq e^{-\frac{1}{4}\rho\cdot max\{|n|,|m|\}} e^{-R|k|}, \quad \forall m,n\in\mathbb{Z}, \quad \forall k\in\mathbb{Z}^d.$$

In [GZ], Geng and Zhao have studied nonlinear Hamiltonian perturbations of infinite-dimensional linear systems generated by the nonlinear Schrödinger equation:

$$\mathrm{i}\dot{q}_n = \tan\pi(n\alpha + x)q_n + \varepsilon(q_{n+1} + q_{n-1}) + \delta|q_n|^2 q_n.$$

Results of that work can be applied to the perturbed Hamiltonian H_ω^δ in (4.5). Let $D^a(R,r)$ denote the domain

$$\{(\theta, I, q, \bar{q}) : q\in\ell^2(\mathbb{Z}), \sum_{n\in\mathbb{Z}} e^{a|n|}|q_n|^2 < r, |\Im\theta| \leq R, |I| \leq r^2\}.$$

Results of [GZ] imply the following assertions concerning Hamiltonian H_ω^δ (4.5) and Theorem 2 follows directly.

Theorem 3 *Consider the Hamiltonian H_ω^δ in (4.5). There is $\delta_0 > 0$ such that for every $0 < \delta \leq \delta_0$, there exists a Borel set $\mathcal{O}_\delta \subset \mathbb{T}^d$, satisfying that*

$$\mathrm{mes}(\mathbb{T}^d - \mathcal{O}_\delta) \leq K\delta^\kappa$$

for some $0 < \kappa < 1$, such that for all $\omega \in \mathcal{O}_\delta$ the following holds:

There exists an analytic symplectic diffeomorphism $\Phi : D^0(R/2, r/2) \to D^0(R, r)$, which is analytic in θ and C_W^1 smooth in ω such that $H_\omega^\delta \circ \Phi$ equals (modulo a constant)

$$\langle\omega, I\rangle + \sum_{n\in\mathbb{Z}} \Omega_n |Q_n|^2 + \delta \sum_{m,n\in\mathbb{Z}} A_{mn} Q_m \bar{Q}_n := \widetilde{H}_\omega^\delta, \tag{4.6}$$

where the coefficients A_{mn} are independent of θ and satisfy

$$|A_{mn}| \leq e^{-\frac{1}{10}\rho\cdot max\{|n|,|m|\}}. \tag{4.7}$$

The transformation $\Phi = (\Phi_q, \Phi_{\bar{q}}, \Phi_\theta, \Phi_I)$ satisfies

$$\|\Phi_q - Id_q\|_{\ell^2} + |\Phi_\theta - Id_\theta| + |\Phi_I - Id_I| \leq \beta\delta$$

for all $(q, \bar{q}, \theta, I) \in D^0(R/2, r/2)$ [3]. The positive constants δ_0, κ, K depend on d, R, r, while β also depends on ω.

[3] $\|\cdot\|_{\ell^2}$ denotes the operator norm on ℓ^2 and $|\cdot|$ denotes the sup norm on $\mathbb{C}^d$

Remark 4.1 *In [EK], the reduced system have a special form where the matrix element* A_{mn} *of the non-diagonal operator* A *satisfies* $A_{mn} = 0$ *if* $|m| \neq |n|$*. As a corollary of the reducibility result, the Sobolev norm of the solution* $q(t)$ *stays bounded.*

In the present model, however, we can not get such uniform bound(w.r.t. time t*) of the the Sobolev norm of the solution. One reason is that the separation property of the normal frequencies of the discrete tangent model is worse than the separation property of the continous model in [EK]. More precisely, the spectrum of Schödinger operator with tangent potential is dense pure point with the gap condition*

$$|\Omega_n - \Omega_m| \geq \frac{\gamma}{|m-n|^\tau},$$

with the lower bound goes to 0 *as* $|m-n| \to \infty$*. While in [EK] the spectrum of the linear operator is* n^2 *and* $|m^2 - n^2| \geq 1$, $m \neq n$*. Therefore, the normal form here in each KAM step can not be diagonalized nor can be as simple as that in [EK].*

Remark 4.2 *The assertions of the theorem follow from Theorem 2 in [GZ] with slight modification. That theorem deals with perturbations of integrable infinite-dimensional Hamiltonian systems of a rather general form. Since the perturbation here is independent of* I*, quadratic in* q *and we use* ω *as interior parameter instead of outer parameter* ξ*, not all conditions* **(A1)–(A6)** *for the KAM theory in [GZ] are needed here, what is more, the vector* ω *stays constant during the transformations.*

Remark 4.3 *Applying Geng and Zhao's KAM scheme directly, exponentially decay weight* ρ *in (4.7) would shrink to zero instead of the positive weight* $\frac{1}{10}\rho$*. However, the perturbation here is independent of* I *and quadratic in* q*, which guarantees the exponential weight* ρ *reduces slower than the general case, e.g.* $\rho/4^\nu$ *at the* ν^{th} *step . Hence the exponential weight in the final state is no less than*

$$\rho - \sum_{\nu \in \mathbb{N}} (\rho/4^\nu) > \frac{1}{10}\rho,$$

which is the needed result.

5 Appendix: ℓ^2 conservation and Localization results for Maryland model

Proposition 1 (ℓ^2 conservation) *Let*

$$\mathrm{i}\dot{q}_n = \Omega_n q_n + \varepsilon(q_{n+1} + q_{n-1}) + f_n(t, |q_n|^2) q_n, \ n \in \mathbb{Z}, \tag{5.1}$$

where ε, Ω_n *are real and* $f_n(t, |q_n|^2)$ *are real functions of* t *and* $|q_n|^2$ *then*

$$\sum_{n \in \mathbb{Z}} |q_n(t)|^2 = \sum_{n \in \mathbb{Z}} |q_n(0)|^2. \tag{5.2}$$

Consequently, let f_n *be* $v_n(t)$ *or* $v_n(t)|q_n|^2$ *as in (1.4) and (1.5), (5.2) hold.*

Proof: After computing the derivative of the ℓ^2 norm directly, one has

$$\begin{aligned}
\mathrm{i}\frac{\mathrm{d}}{\mathrm{dt}}\sum_{n\in\mathbb{Z}}|q_n(t)|^2 &= \mathrm{i}\sum_{n\in\mathbb{Z}}\dot{q}_n\bar{q}_n + q_n\dot{\bar{q}}_n \\
&= \sum_{n\in\mathbb{Z}}(\Omega_n q_n + \varepsilon(q_{n+1}+q_{n-1}) + f_n q_n)\bar{q}_n \\
&\quad -\sum_{n\in\mathbb{Z}} q_n(\Omega_n\bar{q}_n + \varepsilon(\bar{q}_{n+1}+\bar{q}_{n-1}) + f_n\bar{q}_n) \\
&= \varepsilon\sum_{n\in\mathbb{Z}}(q_{n+1}+q_{n-1})\,\bar{q}_n - \varepsilon\sum_{n\in\mathbb{Z}} q_n\,(\bar{q}_{n+1}+\bar{q}_{n-1}) \\
&= 0.
\end{aligned}$$

■

In what follows, we state the conclusion Bellissard, Lima and Scoppola have proved in [BLS] for the Maryland model.

Firstly, let us introduce some necessary notations.

Given $R>0$, for each $x\in\mathbb{R}/\mathbb{Z}$, we are concerned about the spectrum of the linear operator $\mathcal{L}_x:\ell^2(\mathbb{Z}^\nu)\to\ell^2(\mathbb{Z}^\nu)$ by

$$(\mathcal{L}_x\psi)(n) := \tan\pi(x+n\cdot\omega)\psi(n) + \varepsilon(\Delta\psi)(n), \quad n\in\mathbb{Z}^\nu, \tag{5.3}$$

where $(\Delta\psi)(n) := \sum_{|m-n|=1}\psi(m)$ denotes the discrete laplace on $\mathbb{Z}^\nu$.

Let $\mathcal{H}_R$ denote the set of period-one holomorphic bounded functions on the complex region

$$\mathcal{D}_R := \{z\in\mathbb{C} : |\Im z| < R\}$$

equipped with the sup-norm

$$\|f\|_R = \sup_{z\in\mathcal{D}_R}|f(z)|,$$

and let $\mathcal{P}_R$ denote the set of period-one meromorphic functions f on $\mathcal{D}_R$ such that there is a constant $c>0$ with with

$$|f(z)-f(z-a)| \geq c|a|_1, \quad \forall a\in\mathbb{R}, \quad \forall z\in\mathcal{D}_R, \tag{5.4}$$

where $|\cdot|_1$ is defined as in (1.3). Then $|f|_R$ is defined as the biggest possible value of c in (5.4). It is obvious the function $f(z)=\tan\pi z$ belongs to $\mathcal{P}_R$ for any $R>0$, with $|f|_R\geq 1$.

For $R,r>0$ and $\omega\in\mathbb{R}^\nu$ satisfying the Diophantine condition (1.3), we denote by $\mathcal{U}^{\omega}_{R,\varrho}$ the Banach $*$-algebra of kernels $\mathrm{m}=\{\mathrm{m}(z,n)\}_{n\in\mathbb{Z}^\nu, z\in\mathcal{D}_R}$ where for each $n\in\mathbb{Z}$, the map $z\mapsto\mathrm{m}(z,n)$ belongs to $\mathcal{H}_R$(or $\mathcal{P}_R$), and

$$\|\mathrm{m}\|'_{R,\varrho} := \sup_{z\in\mathcal{D}_R}\sum_{n\in\mathbb{Z}^\nu}|\mathrm{m}(z,n)|e^{r|n|}$$

is finite. (We need to exclude a subset of $\mathcal{D}_R$ with measure zero in the case that $\mathrm{m}(\cdot,n)\in\mathcal{P}_R$ and there is some poles in $\mathcal{D}_R$.) The $*$-algebraic structure is defined by

$$\begin{aligned}
(\mathrm{m}_1\cdot\mathrm{m}_2)(z,n) &:= \sum_{l\in\mathbb{Z}^\nu}\mathrm{m}_1(z,l)\mathrm{m}_2(z-l\omega,n-l), \\
\mathrm{m}^*(z,n) &:= \overline{\mathrm{m}(\bar{z}-n\omega,-n)}.
\end{aligned}$$

Then the norm is defined by

$$\|\mathrm{m}\|_{R,r} = \max\{\|\mathrm{m}\|'_{R,r}, \|\mathrm{m}^*\|'_{R,r}\}.$$

For example, if $g \in \mathcal{H}_R$(or $g \in \mathcal{P}_R$) then g can be considered as an element of $\mathcal{U}^{\omega}_{R,r}$, by putting:

$$\mathrm{g}(z,n) := g(z)\delta_{n,0}.$$

Such a kernel is called *diagonal.* If $e \in \mathbb{Z}^{\nu}$, u_e is the kernel

$$\mathrm{u}_e(z,n) := \delta_{n,e}.$$

One can easily see that u_0 is an identity and

$$\mathrm{u}_e^*\mathrm{u}_e = \mathrm{u}_e\mathrm{u}_e^* = \mathrm{u}_0, \quad \forall e \in \mathbb{Z}^{\nu}.$$

The Laplace kernel is then given by

$$\Delta = \sum_{|e|=1} \mathrm{u}_e.$$

A canonical set of representations of $\mathcal{U}^{\omega}_{R,r}$ in $\ell^2(\mathbb{Z}^{\nu})$ is given by

$$[\Pi_z(\mathrm{m})\psi](n) = \sum_{l\in\mathbb{Z}^{\nu}} \mathrm{m}(z - n\omega, l-n)\psi(l),$$

where $\psi \in \ell^2(\mathbb{Z}^{\nu})$, $z \in \mathcal{D}_R$ and $m \in \mathcal{U}^{\omega}_{R,r}$. Actually, $\Pi_z(\mathrm{m})$ can be seen as an infinite matrix, with its matrix elements $[\Pi_z(\mathrm{m})]_{k,l} = \mathrm{m}(z - l\omega, k-l)$.

In this set-up, the Schrödinger operator given by (5.3) can be seen as the operator $\Pi_z(\varepsilon\Delta + V)$. For the sake of completeness, we restate the theorem of Bellissard–Lima–Scoppola in [BLS] as follows:

Theorem 4 (Theorem 1 of Bellissard–Lima–Scoppola [BLS]) *Given $R > 0$, $r > 0$, and $\omega \in \mathbb{R}^{\nu}$ satisfying the diophantine condition:*

$$|\omega \cdot n|_1 \geq \frac{\gamma}{|n|^{\sigma}} \text{ with } \gamma > 0,\ \sigma > \nu,\ \forall 0 \neq n \in \mathbb{Z}^{\nu}. \tag{5.5}$$

If $V \in \mathcal{P}_R$, there is a positive constant ε_c, depending on R, r, γ, σ and $|V|_R$, such that if $\mathrm{m} \in \mathcal{U}^{\omega}_{R,r}$, and $\|\mathrm{m}\|_{R,r} \leq \varepsilon_c$, there exists an invertible element $\mathrm{u} \in \mathcal{U}^{\omega}_{R,r}$ and $\hat{V} \in \mathcal{P}_{r/2}$ with

1. $\mathrm{u}(V + \mathrm{m})\mathbf{u}^{-1} = \hat{V}$,

2. $\max(\|\mathrm{u} - 1\|_{R/2,r/2}, \|\mathrm{u}^{-1} - 1\|_{R/2,r/2}) \leq K_1\|\mathrm{m}\|_{R,r}$,

3. $V - \hat{V} \in \mathcal{H}_{R/2}$ and $\|V - \hat{V}\|_{R/2} \leq K_2\|\mathrm{m}\|_{R,r}$, $|\hat{V}|_{R/2} \geq \frac{1}{2}|V|_R$.

If in addition $\mathrm{m} + V$ is self-adjoint, then u is unitary and $\hat{V} = \hat{V}^*$.

Now we take $V(z) = \tan \pi z \in \mathcal{P}_R$ and $\Pi_z(\mathrm{m}) = \varepsilon\Delta$. The theorem is applicable to the Maryland model provided $\varepsilon\|\Delta\|_{R,r} < \varepsilon_c$. Since $\mathrm{u} \in \mathcal{U}^{\omega}_{R,r}$, the infinite matrix $U = \Pi_x(\mathrm{u})$ has off-diagonal decay, i.e. the matrix elements $U_{m,n}$ satisfy

$$|U_{m,n}| = |\mathrm{u}(x - n\omega, m - n)| \le e^{-r|m-n|}$$

for each $(m, n) \in \mathbb{Z}^\nu \times \mathbb{Z}^\nu$. Thus with all parameters needed above fixed, $U = \Pi_x(\mathrm{u})$ will be the unitary operator needed in Lemma 2.1 for a.e. $x \in \mathbb{R}/\mathbb{Z}$.

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