What is truth?

Benedikt Ahrens

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Contents

The tasks

Syntax - formal language

- express statements unambiguously
- make statements understandable for machines
- for us: at least two types of expressions
  - terms
  - formulae (whose truth is to be defined)

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Syntax - formal language

- express statements unambiguously
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  - terms
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Semantics

- connection: language $\longrightarrow$ universe of objects
- determine validity/truth of formulae

What we are going to learn today

- Syntax - the language
  - how to handle types
  - how to handle the binding effect of quantifiers
  - example: language $\Sigma_{\mathcal{T}}$ of a topos $\mathcal{T}$ - suitable for set theory
- Topos - categorical universe of sets
- Semantics - interpretation of the topos language
  - arrows of $\mathcal{T}$ as little helpers for interpretation
  - validity of formulae of $\Sigma_{\mathcal{T}}$

first thoughts about a language for mathematics

Goals:

- avoid ambiguity
- internationality
- machine aid

What we need

- alphabet: suitable for expressing mathematical ideas
  - logical symbols $\Rightarrow, \wedge, \vee, \neg$
  - quantifiers $\forall, \exists$
  - theory-specific symbols (e.g. $\in$, $+$)
- grammar rules
  - number of arguments
  - type of arguments

two phenomena: types & bound variables

example - types & binding effect

constructor $\forall_{\mathbb{N}}$

- takes a *formula* containing a free variable n of type $\mathbb{N}$, e.g. $n \geq 5$
- returns a formula where the variable is bound, $\forall_{\mathbb{N}} n \geq 5$

We will work with a set of types T.

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We will work with a set of types T.

Definition (category T-Set)

- objects: T-indexed families of sets
- morphisms: T-indexed families of morphisms of sets

T-signature

Definition (T-typed signature)

- set Σ of symbols
- function a, called arity, assigning to each symbol s of S a tuple $a(s) = (s_0; \bar{s}_1, \ldots, \bar{s}_n)$

where $s_0 \in T$ and $\bar{s}_i = (s_{i,0}; s_{i,1}, \ldots, s_{i,n_i})$ $(i = 1, \ldots, n)$ are nonempty finite sequences in T.

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Example

Types $T = \{\Omega \text{ (formula)}, \mathbb{N}\}$

- $\forall : (\Omega; (\Omega; \mathbb{N}))$
  - argument: a formula with a free variable of type $\mathbb{N}$
  - result: a formula where the variable is bound
- $\Rightarrow: (\Omega; (\Omega; \emptyset), (\Omega; \emptyset))$
  - two formulae as arguments, no variable-binding

Expressions = Trees

- mathematical structure for expressions: trees!
- signature = rules for number and type of sons

Definition

- $X = (X_t)_{t \in T}$ element of T-**Set**
- Σ a T-typed signature

$\hat{\Sigma}X = ((\hat{\Sigma}X)_t)_{t \in T}$ the typed set of trees constructible from symbols of Σ and the typed set X of variables as leaves.

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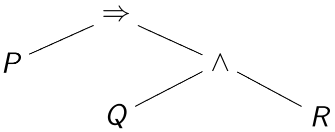
- $T = \{\Omega, \ldots\}$
- Σ contains logical symbols with usual arity
- $P, Q, R \in X_\Omega$ variables of type formula



adding a new element to a (typed) set of variables

Definition

- $$X^* := X \coprod \{\infty_X\}$$
- $u \in T$ and $X = (X_t)_{t \in T}$ in T-**Set**; we define X^{*_u} s.t.
  - $(X^{*_u})_t = X_t$ for all $t \neq u$
  - $(X^{*_u})_u = (X_u)^*$

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Definition

- $\bar{s} = (s_0; s_1, \ldots, s_n)$ nonempty finite sequence of types
- X in T-**Set** a typed set of variables

We define

$$\hat{\Sigma}^{\bar{s}} X := (\hat{\Sigma}(X^{*_{s_1} *_{s_2} \cdots *_{s_n}}))_{s_0}$$

in the end we have a lot of structure

$X = (X_t)_{t \in T}$, Y objects in T-**Set**, $f : X \to Y$ in T-**Set**

- $\hat{\Sigma} f : \hat{\Sigma} X \to \hat{\Sigma} Y$ renaming of variables.
- $\eta_X : X \to \hat{\Sigma} X$: variable as a tree
- $\mu_X : \hat{\Sigma}\hat{\Sigma} X \to \hat{\Sigma} X$
- for each $s \in \Sigma$ with arity $a(s) = (s_0; \bar{s}_1, \ldots, \bar{s}_n)$ a mapping

$$s : \prod \hat{\Sigma}^{\bar{s}_i} X \to (\hat{\Sigma} X)_{s_0}$$

which builds trees with root s, arguments = subtrees

example for representation in $\hat{\Sigma}$

Example

- $T = \{\Omega, \ldots\}$
- $\Rightarrow : (\Omega; (\Omega; \emptyset), (\Omega; \emptyset))$

$$\begin{aligned} \Rightarrow_X : (\hat{\Sigma}X)_\Omega \times (\hat{\Sigma}X)_\Omega &\to (\hat{\Sigma}X)_\Omega, \\ (P, Q) &\mapsto (P \Rightarrow Q) \end{aligned}$$

creating a universe of sets within category theory

universe of sets

- collection of sets
- maps between sets
  - composition of maps (with associativity)
  - identity map

is realized in a category

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special sets and maps

- power set $\mathcal{P}B$, hom set B^A, truth values $\Omega = \{0, 1\}$
- characteristic function of $S \subset B$

Goal: imitate a universe of sets with categorical means

- describe special sets by universal properties

definition of a subobject classifier

Definition (subobject classifier)

A subobject classifier in a category **C** (w. fib. products and 1) consists of

- an object Ω of **C**
- a monic $\text{true} : 1 \to \Omega$

s. t. for any mono $S \rightarrowtail B$ there exists a unique char. function $\phi : B \to \Omega$, i. e. a unique ϕ giving a fibered product:

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \text{true}} \\ B & \xrightarrow{\phi} & \Omega. \end{array}$$

Example in **Set**: $\Omega = \{0, 1\}$, $\text{true} : * \mapsto 1$, $\phi = \chi_{(S,B)}$

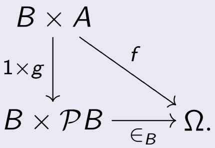
universal property of $\mathcal{P}B$ - power object in **Set** and **C**

membership predicate in **Set**

$$\in_B : B \times \mathcal{P}B \to \Omega, \quad (b, S) \mapsto \begin{cases} 1, & \text{if } b \in S, \\ 0 & \text{otherwise.} \end{cases}$$

universal property of $(\mathcal{P}B, \in_B)$

For any $f : B \times A \to \Omega$ there is a unique arrow $g : A \to \mathcal{P}B$ which makes commute:



In the same way we can postulate the existence of a power object $(PB, \in_B)$ in any category **C** with products and Ω!

Definition of a topos

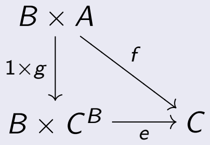
Definition (elementary topos $\mathcal{T}$)

A topos is a category $\mathcal{T}$ with

- a fibered product for every diagram $X \to B \leftarrow Y$;
- a terminal object 1;
- a subobject classifier $(\Omega, \mathrm{true} : 1 \to \Omega)$;
- for every object B a power object PB with $\in_B$: $B \times PB \to \Omega$.

Theorem ($\mathcal{T}$ has exponentials)

For any B and C there is an object C^B and $e = e_{B,C} : B \times C^B \to C$ s.t. for any $f : B \times A \to C$ there is a unique $g : A \to C^B$ which makes commute



some important arrows of $\mathcal{T}$

$P : B \mapsto PB$ can be made a contravariant functor

Consider $k : A \to B$ in $\mathcal{T}$. Define $Pk : PB \to PA$ as the transpose of $\in_B \circ(k \times 1)$:

$$\begin{array}{ccccc} A \times PB & \xrightarrow{k\times 1} & B \times PB & \xrightarrow{\in_B} & \Omega \\ {\scriptstyle 1\times Pk}\downarrow & & & & \| \\ A \times PA & & \xrightarrow[\in_A]{\qquad\qquad} & & \Omega. \end{array}$$

This makes $P : \mathcal{T} \to \mathcal{T}$ a contravariant functor.

Definition Heyting algebra

- A lattice L is a set with two elements $0 \neq 1$ and two binary operations $\wedge$ and $\vee$ which satisfy
  - commutativity, associativity
  - $x \wedge x = x, \quad x \vee x = x, \quad 1 \wedge x = x, \quad 0 \vee x = x,$
  - $x \wedge (y \vee x) = x = (x \wedge y) \vee x.$

  Induces a partial order by $\quad x \leq y$ iff $x \wedge y = x$.
- A lattice H is a Heyting algebra if for $x, y \in H$ we have an exponential $(x \Rightarrow y)$ s.t.

$$z \leq (x \Rightarrow y) \quad \text{iff} \quad z \wedge x \leq y.$$

- In a Heyting algebra H we define a negation operator

$$\neg x := (x \Rightarrow 0).$$

Equations in H and diagrammatic version

Lemma (Equations holding in Heyting algebra H)

For $x, y, z \in H$ we have

$$(x \Rightarrow x) = 1$$

$$x \wedge (x \Rightarrow y) = x \wedge y, \qquad y \wedge (x \Rightarrow y) = y$$

$$x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z)$$

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Example for diagrammatic version of $x \wedge (x \Rightarrow y) = x \wedge y$

$$\begin{array}{ccc} H \times H \times H & \xrightarrow{1 \times \Rightarrow} & H \times H \\ {\scriptstyle \Delta \times 1} \uparrow & & \downarrow {\scriptstyle \wedge} \\ H \times H & \xrightarrow{\quad \wedge \quad} & H \end{array}$$

PB is an internal Heyting algebra

Theorem

For any B in $\mathcal{T}$ the power object PB is an internal Heyting algebra, i.e. there are arrows

- $\wedge, \vee, \Rightarrow : PB \times PB \to PB$
- $\neg : PB \to PB$
- $0, 1 : 1_{\mathcal{T}} \to PB$ *(for the elements 0 and 1)*

which make commute the diagrams given by the preceding equations of a Heyting algebra.

In particular, $\Omega = P1$ is such.

Internal partial order of a Heyting algebra object

- H an internal Heyting algebra of $\mathcal{T}$. The object $\leq_H$ shall be the equalizer of $\wedge, \pi_1 : H \times H \to H$ as in

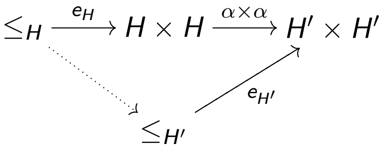
$$\begin{array}{ccc} \leq_H & \longrightarrow & H \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \Delta_H} \\ H \times H & \xrightarrow[\langle \wedge, \pi_1 \rangle]{} & H \times H . \end{array}$$

- Properties reflexivity, transitivity, antisymmetry can be expressed diagrammatically. Example: reflexivity

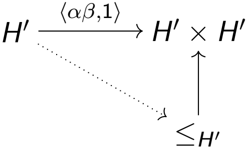
$$\begin{array}{ccc} H & \xrightarrow{\Delta_H} & H \times H \\ & \searrow & \uparrow{\scriptstyle e} \\ & & \leq_H . \end{array}$$

Def: internal adjunction

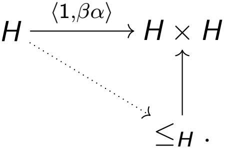
$(H, \leq_H)$ and $(H', \leq_{H'})$ internal partially ordered objects in $\mathcal{T}$.
Suppose $\alpha : H \to H'$ and $\beta : H' \to H$ are monotonous arrows, i. e.



and similarly for β.
α is internally left adjoint to β if we have factorizations as in



and



Pk has internal adjoints

Theorem

For any arrow $k : A \to B$ the arrow $Pk : PB \to PA$ has an

- *internal left adjoint $\exists_k : PA \to PB$;*
- *internal right adjoint $\forall_k : PA \to PB$.*

Arrow δ as equality predicate

Definition (arrows δ_B and true_X)

- $\Delta_B = \langle 1_B, 1_B \rangle : B \rightarrowtail B \times B$ has a characteristic map

$$\delta_B : B \times B \to \Omega.$$

- For any object X let be defined by

$$\text{true}_X : X \to 1 \xrightarrow{\text{true}} \Omega.$$

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Lemma

Let b and b' two generalized elements of the object B, i. e. $b, b' : X \rightrightarrows B$ are two arrows in a topos $\mathcal{T}$. Then

$$\delta_B \langle b, b' \rangle = true_X \quad \textit{iff} \quad b = b'.$$

The signature $\Sigma = \Sigma_{\mathcal{T}}$

- types = objects of $\mathcal{T}$
- formulae = expressions of type Ω
- we omit indices A, B for the constructors

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$$
\begin{aligned}
\Sigma_{\mathcal{T}} = \{ \quad \langle , \rangle &: \big(A \times B; (A;\emptyset), (B;\emptyset)\big),\\
= &: \big(\Omega; (B;\emptyset), (B;\emptyset)\big),\\
app &: \big(B; (B^A;\emptyset), (A;\emptyset)\big),\\
\in &: \big(\Omega; (B;\emptyset), (\Omega^B;\emptyset)\big),\\
\lambda &: \big(B^A; (B;A)\big),\\
\wedge &: \big(\Omega; (\Omega;\emptyset), (\Omega;\emptyset)\big),\\
\vee &: \big(\Omega; (\Omega;\emptyset), (\Omega;\emptyset)\big),\\
\Rightarrow &: \big(\Omega; (\Omega;\emptyset), (\Omega;\emptyset)\big),\\
\neg &: \big(\Omega; (\Omega;\emptyset)\big),\\
\forall &: \big(\Omega; (\Omega;B)\big),\\
\exists &: \big(\Omega; (\Omega;B)\big) \quad \}
\end{aligned}
$$

Arrows as helpers for interpretation

An arrow for every expression

- Z an object of $\mathcal{T}$-**Set**
- define for every expression $E \in (\hat{\Sigma}Z)_B$ an arrow $a(E)$ of $\mathcal{T}$
  - target$(a(E)) = B =$ type of E
  - $$\text{source}(a(E)) = \prod_{\substack{B \in \mathcal{T} \\ |Z_B| = i}} B^i$$
    where we suppose that $|Z_B| < \infty$ and only finitely many $Z_B \neq \emptyset$

- variable b of type B interpreted by $1_B : B \to B$
- $\phi(a, b, b') \in \hat{\Sigma}(0_{\mathcal{T}\text{-}\mathbf{Set}}{}^{*A*B*B})_\Omega$ interpreted by an arrow

$$A \times B \times B \to \Omega$$

Expanding the set of free variables by combining expressions

Constructors with several arguments

- for $Z \leq Y$ we have $\hat{\Sigma} Z \leq \hat{\Sigma} Y$
- necessity of adjusting the source of interpreting arrow

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Definition (enlarging the source of an interpreting arrow)

- expression σ interpreted by $\sigma : U \to B$
- σ is also interpreted by

$$\sigma \circ \pi_1 : U \times A \to B$$

Given two expressions, we can hence suppose that the sources of their interpreting arrows coincide.

Recursive definition of the interpreting arrow I

- variable b of type B interpreted by $1_B : B \longrightarrow B$
- interpreting arrow for an equality predicate
    - E a formula $(\sigma = \tau) \in \hat{\Sigma} Z_\Omega$
    - σ and τ of type B interpreted by $\sigma, \tau : U \to B$

  E is interpreted by

$$(\sigma = \tau) : U \xrightarrow{\langle \sigma,\tau \rangle} B \times B \xrightarrow{\delta_B} \Omega,$$

- E a formula $\phi \Rightarrow \psi$,
  ϕ and ψ formulae interpreted by $\phi, \psi : U \to \Omega$

$$\phi \Rightarrow \psi : U \xrightarrow{\langle \phi,\psi \rangle} \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$$

Example

- $(b = b)$ a formula in $\hat{\Sigma}(0_{\mathcal{T}\text{-}\mathbf{Set}}{}^{*B})_\Omega$ interpreted by arrow

$$B \xrightarrow{\langle 1,1\rangle = \Delta_B} B \times B \xrightarrow{\quad\delta_B\quad} \Omega$$

- $(b = b')$ in $\hat{\Sigma}(0_{\mathcal{T}\text{-}\mathbf{Set}}{}^{*B*B})_\Omega$
  - we regard b, b' as elements in $\hat{\Sigma}(0_{\mathcal{T}\text{-}\mathbf{Set}}{}^{*B*B})_B$
  - b interpreted by $1_B \circ \pi_1 : B \times B \to B$
  - b' interpreted by $1_B \circ \pi_2 : B \times B \to B$

  $(b = b')$ interpreted by the arrow

$$B \times B \xrightarrow{\langle \pi_1,\pi_2\rangle = 1} B \times B \xrightarrow{\quad\delta_B\quad} \Omega.$$

Recursive definition of the interpreting arrow II

variable b of type B interpreted by $1_B : B \to B$

$$\langle\sigma,\tau\rangle : U \xrightarrow{\langle\sigma,\tau\rangle} A \times B$$

$$\sigma(\tau) : U \xrightarrow{\langle\sigma,\tau\rangle} A \times B^A \xrightarrow{e_{A,B}} B$$

$$(\sigma \in \tau) : U \xrightarrow{\langle\sigma,\tau\rangle} B \times \Omega^B \xrightarrow{\in_B} \Omega$$

$$(\lambda_{A,B,z})\sigma : U \xrightarrow{\text{transpose}(\sigma)} B^A$$

$$\phi \vee \psi : U \xrightarrow{\langle\phi,\psi\rangle} \Omega \times \Omega \xrightarrow{\vee} \Omega$$

$$\neg\phi : U \xrightarrow{\phi} \Omega \xrightarrow{\neg} \Omega$$

$$\forall_B(\lambda_{B,\Omega,z})\phi : U \xrightarrow{\text{transpose}(\phi)} \Omega^B \xrightarrow{\forall_{(\bullet_B)}} \Omega$$

where $\bullet_B : B \to 1$

How a formula defines an object

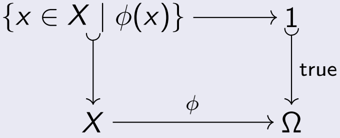
Definition (object defined by a formula)

- $\phi \in Z\hat{\Sigma}_\Omega$ a formula
- in free variables x_i of type X_i
- interpreting arrow $\phi : X = \prod X_i \to \Omega$

The object

$$\{x \in X \mid \phi(x)\}$$

is defined to be the fibered product



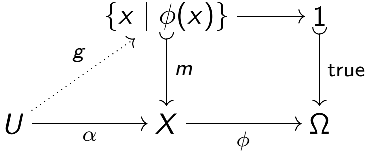
Validity of a formula

- $\phi = \phi(x_1, \ldots, x_n)$ a formula in $Z\hat{\Sigma}_\Omega$ with free variables x_i of type X_i $(i = 1, \ldots n)$
- $\alpha_i : U \to X_i$ generalized elements of the X_i

ϕ is true for the α_i,

$$U \Vdash \phi(\alpha_1, \ldots, \alpha_n),$$

if $\alpha = \langle \alpha_1, \ldots, \alpha_n \rangle : U \to X = \prod X_i$ factors through $\{x \mid \phi(x)\}$



iff

$$\phi \circ \alpha = true_U.$$

Special case: universal validity

- $\phi(x)$ a formula with interpreting arrow $\phi : X \to \Omega$

is universally valid if one of the following equivalent holds:

$$\alpha := 1_X : X \to X \text{ factors through } \{x \mid \phi(x)\} \tag{1}$$

$$\phi = true_X : X \longrightarrow 1 \xrightarrow{\text{true}} \Omega \tag{2}$$

$$\{x \mid \phi(x)\} \cong X \tag{3}$$

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Example

The formula $(b = b)$ in $(0_{\mathcal{T}\text{-}\mathbf{Set}}{}^{*B})\hat{\Sigma}_\Omega$ interpreted by arrow

$$B \xrightarrow{\langle 1,1\rangle = \Delta_B} B \times B \xrightarrow{\delta_B} \Omega$$

is universally valid by def. of the arrow δ_B as char. map of Δ_B.

Why the strange choice of arrows?

The Beth-Kripke-Joyal theorem tells us that the validity we defined behaves as we would expect. Examples:

- $A \Vdash \phi(a) \wedge \psi(a)$ iff $A \Vdash \phi(a)$ and $A \Vdash \psi(a)$
- $A \Vdash \forall y \phi(a, y)$ iff for any object B, for any arrow $h : B \to A$ and every generalized element $b : B \to Y$ it holds $B \Vdash \phi(ah, b)$.

What was this good for?

- equiconsistency WPT $\longleftrightarrow$ RZC (c.f. [MLM92])
  - WPT the axioms defining a well-pointed topos (terminal object generates the topos)
  - RZC the axioms of restricted Zermelo set theory
- form instead of substance (c.f. [Law05])
  - set theory based on isomorphism-invariant structure (e.g. universal properties) instead of membership relation

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