The Lüroth problem and the Cremona group

Arnaud Beauville

Université de Nice

Tokyo, January 2013
The Lüroth problem

Definitions

A variety $V$ is unirational if $\exists$ generically surjective rational map $\mathbb{P}^n \to V$.

Equivalently, $\mathbb{C}(V) \hookrightarrow \mathbb{C}(t_1, \ldots, t_n)$.

$V$ is rational if $\exists$ birational map $\mathbb{P}^n \sim \mathbb{P}^1 \to V$.

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Lüroth problem: unirational $\Rightarrow$ rational?

Lüroth (1875): yes for curves.

Quite easy with Riemann surface theory; but Lüroth's proof is algebraic.

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Actually Enriques proves unirationality, and relies on an earlier paper of Fano (1908) for the non-rationality. But Fano's analysis is incomplete. Fano made further attempts (1915, 1947), but not acceptable by modern standards.

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Threefolds $V$ with $-K_V$ very ample, $\text{Pic}(V) = \mathbb{Z}[K_V]$. (Fano threefolds of the first species: modern classification due to Iskovskikh.)
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The main result

Theorem

The threefold \( \sum X_i = \sum X_2 \sum X_3 \) in \( \mathbb{P}_6 \) is not rational.

What is the point of giving one more counter-example? This gives one specific example of a non-rational \( V_2, V_3 \).

The proof is very simple – maybe the simplest non-rationality proof available.

Real motivation: it completes the work of Prokhorov on the finite simple subgroups of \( \text{Cr}_3 \).

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So the situation is quite satisfactory, except for $V_{2,3}$ and $V_{10}$. 

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The threefold $\sum X_i = \sum X_{2i} = \sum X_{3i} = 0$ in $P^6$ is not rational.

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$$H^1(C, Z) \subset H^1(C, C) = H^{1,0} \oplus H^{0,1},$$

The image of $H^1(C, Z)$ in $H^{0,1}$ is a lattice, so get complex torus $JC := H^{0,1}/H^1(C, Z)$.

The cup-product defines a unimodular skew-symmetric form $E: H^1(C, Z) \times H^1(C, Z) \to \mathbb{Z}$ such that $E(x, iy) = E(x, y)$, $E(x, ix) > 0$ for $x \neq 0$.

$\Rightarrow \ \theta \in H^2(JC, Z) \cap H^1,1$, hence $\theta = c_1(L)$, ample, $h^0(L) = 1$:

This is a principal polarization on $JC$: we say that $JC$ is a p.p.a.v. Defines unique divisor on $JC$ (up to translation), the theta divisor.
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\[ E : H^1(C, \mathbb{Z}) \times H^1(C, \mathbb{Z}) \to \mathbb{Z} \]
The intermediate Jacobian

Recall the definition of the Jacobian of a curve $C$:

$$H^1(C, \mathbb{Z}) \subset H^1(C, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$$

The image of $H^1(C, \mathbb{Z})$ in $H^{0,1}$ is a lattice, so get complex torus

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$$\exists \theta \in H^2(JC, \mathbb{Z}) \cap H^{1,1},$$
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The Lüroth problem and the Cremona group
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Defines unique divisor on $JC$ (up to translation), the theta divisor.
The Clemens-Griffiths criterion

If $V$ is rational, $J^V$ is a Jacobian or a product of Jacobians.

Sketch of proof: Assume $\exists u: P^3 \sim \mathcal{K}_V$. Hironaka gives

$\begin{array}{c}
\text{composition of blow-ups of points and smooth curves } C_1, \ldots, C_p; \\
\text{v birational morphism. Then:}
\end{array}$

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The Lüroth problem and the Cremona group
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$V$ Fano threefold, completely analogous Hodge decomposition

\[ H^3(V, \mathbb{Z}) \subset H^3(V, \mathbb{C}) = H^{2,1} \oplus H^{1,2} \]
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![Diagram](attachment:image.png)
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\downarrow \ b \quad \downarrow \ v \quad \\
\mathbb{P}^3 \quad \sim \quad u \quad \sim \quad V
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If $V$ is rational, $JV$ is a Jacobian or a product of Jacobians.

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$$\begin{align*}
P & \xrightarrow{b} \mathbb{P}^3 \xrightarrow{v} \mathbb{P}^3 \\
& \xrightarrow{u} V
\end{align*}$$

$b$: composition of blow-ups of points and smooth curves $C_1, \ldots, C_p$;

$v$: birational morphism. Then:
The Clemens-Griffiths criterion (continued)

\[ P \rightarrow P \text{ blow up} \Rightarrow J_P = J_{1} \times \ldots \times J_{p}, \text{with} J_i := J \mathcal{C}_i; \]

\[ P \rightarrow V \text{ morphism} \Rightarrow H^*(P, Z) \xrightarrow{v^*} H^*(V, Z) \xleftarrow{v^*} \text{with} v^* v^* = \text{Id}, \]

so \[ H^*(P, Z) = H^*(V, Z) \oplus M \Rightarrow J_P \sim = J_V \times A \]

for some p.p.a.v. \( A \).

Miracle The decomposition \( J_P = J_1 \times \ldots \times J_p \) is unique (up to permutation).

This is because \( \Theta_{J_P} = \Theta_{J_1} \times \ldots \times \Theta_{J_p} + \ldots + \Theta_{J_1} \times \ldots \times \Theta_{J_p} - 1 \times \Theta_{J_{p-1}} \) and the theta divisor of a Jacobian is irreducible.

So \( J_P \sim = J_1 \times \ldots \times J_p \sim = J_V \times A = \Rightarrow J_V \sim = J_{k_1} \times \ldots \times J_{k_m} \).

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The Lüroth problem and the Cremona group
\[ b : P \to \mathbb{P}^3 \text{ blow up} \Rightarrow JP = J_1 \times \ldots \times J_p, \text{ with } J_i := JC_i ; \]
The Clemens-Griffiths criterion (continued)

\[ b : P \to \mathbb{P}^3 \text{ blow up} \implies JP = J_1 \times \ldots \times J_p, \text{ with } J_i := JC_i \; ; \]

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\[ b : P \rightarrow \mathbb{P}^3 \text{ blow up} \Rightarrow JP = J_1 \times \ldots \times J_p, \text{ with } J_i := JC_i; \]
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The Clemens-Griffiths criterion (continued)

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The Clemens-Griffiths criterion (continued)

\(b : P \to \mathbb{P}^3\) blow up \(\Rightarrow\) \(JP = J_1 \times \ldots \times J_p\), with \(J_i := JC_i\); 

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So \( JP \cong J_1 \times \ldots \times J_p \cong JV \times A \implies JV \cong J_{k_1} \times \ldots \times J_{k_m}. \)
Proof of the theorem

How can one prove that $J^V \not\sim J_1 \times \ldots \times J_p$?

Usually by studying the geometry of the theta divisor (singular locus, Gauss map, ...).

I will use instead the action of $A_7$.

**Proof of the theorem:** $V$ defined by

\[ \sum X_i = \sum X_{2i} = \sum X_{3i} = 0 \text{ in } P_6 : \]

action of $S_7$, hence of $A_7$.

Thus $A_7$ acts on $J^V$. Non-trivially?

**Lemma** $J^V$ contains no abelian subvariety fixed by $A_7$.

**Proof:** analyze the action of $A_7$ on $T_0(J^V) = \mathcal{H}^1, 2 \sim \mathcal{H}^2(V, \Omega^1_V)$.

Find: $T_0(J^V) = V_6 \oplus V_{14}$, both faithful.
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Proof of the theorem

How can one prove that \( JV \not\sim J_1 \times \ldots \times J_p \)?

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\[ V \text{ defined by } \sum X_i = \sum X_{i+2} = \sum X_{i+3} = 0 \text{ in } \mathbb{P}^6; \]

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The Lüroth problem and the Cremona group
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Lemma

$JV$ contains no abelian subvariety fixed by $A_7$.

Proof : analyze the action of $A_7$ on $T_0(JV) = H^{1,2} \cong H^2(V, \Omega^1_V)$. 

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The Lüroth problem and the Cremona group
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Find: $T_0(JV) = V_6 \oplus V_{14}$, both faithful.
In particular, $\mathfrak{A}_7 \subset \text{Aut}(JV)$. Note: $\dim JV = 20$. 

Step 2: Assume $JV = J_1 \times \ldots \times J_n$.

(more subtle: e.g. $\text{Aut}(E_{20}) \supset S_{20}$).
In particular, $\mathcal{A}_7 \subset \text{Aut}(J\!V)$. Note: $\dim J\!V = 20$.

**Step 1:** If $\mathcal{A}_7 \subset \text{Aut}(J\!C)$, $g(C) \geq 31$
Step 1: \( JV \neq JC \)

In particular, \( \mathcal{A}_7 \subset \text{Aut}(JV) \). Note: \( \dim JV = 20 \).

**Step 1:** If \( \mathcal{A}_7 \subset \text{Aut}(JC) \), \( g(C) \geq 31 \) (hence \( JV \neq JC \)).
In particular, $\mathcal{A}_7 \subset \text{Aut}(JV)$. Note: $\dim JV = 20$.

**Step 1:** If $\mathcal{A}_7 \subset \text{Aut}(JC)$, $g(C) \geq 31$ (hence $JV \neq JC$).

**Torelli:**

$$\text{Aut}(JC) = \begin{cases} 
\text{Aut}(C) & \text{if } C \text{ hyperelliptic} \\
\text{Aut}(C) \times \mathbb{Z}/2 & \text{otherwise.}
\end{cases}$$
In particular, $\mathcal{A}_7 \subset \text{Aut}(J_V)$. Note: $\dim J_V = 20$.

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Thus $\mathcal{A}_7 \hookrightarrow \text{Aut}(C) \implies \frac{1}{2} 7! \leq 84(g - 1)$,
Step 1: $JV \neq JC$

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\text{Aut}(C) & \text{if } C \text{ hyperelliptic} \\
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(more subtle: e.g. \( \text{Aut}(E^{20}) \supset \mathfrak{S}_{20} \)).
Assume \( JV \cong J_1 \times \ldots \times J_n \)

Unicity of the decomposition \( \Rightarrow A_7 \) permutes the \( J_i \)'s:

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*If $\mathfrak{A}_7$ acts transitively on a set $S$, then $\# S = 1, 7, 15$ or $\geq 21$.***
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Thus $p, q, \cdots = 7$ or $15$; contradiction!
The method applies to other threefolds:

• $V_2, 3$:
  \[ \sum X^2_i = \sum X^3_i = 0 \text{ in } \mathbb{P}^5, \text{ with group } S_6; \text{ more difficult.} \]

• Klein cubic:
  \[ \sum_{i \in \mathbb{Z}/5} X^2_i X_i^4 + 1 = 0 \text{ in } \mathbb{P}^4, \text{ with group } \text{PSL}(2, F_{11}). \]

• The $S_6$-invariant quartic threefolds $X_t$:
  \[ \sum x_i = 0, t \sum x_i^4 - (\sum x_i^2)^2 = 0 \text{ in } \mathbb{P}^5, t \in \mathbb{P}^1. \]

$X_2$ is the Burkhardt quartic, $X_4$ the Igusa quartic.

For $t \neq 0$, $2, 4, 6, 10, 7$, $X_t$ has 30 nodes:

\[ \text{Sing}(X_t) = S_6\text{-orbit of } (1, 1, \rho, \rho, \rho_2, \rho_2), \quad \rho = e^{2\pi i/3}. \]

\[ \dim J^{\hat{}}X_t = 5, \text{ action of } S_6 \text{ nontrivial} \Rightarrow X_t \text{ not rational.} \]

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The Cremona group

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$\text{Cr} := \{\text{birational automorphisms of } \mathbb{P}^n\}.$

The finite subgroups of $\text{Cr}_2$ are known (Kantor, Wiman, Dolgachev-Iskovskikh); very long list.

The simple (non-cyclic) finite subgroups of $\text{Cr}_2$ are much easier to classify: $A_5, A_6$ and $\text{PSL}(2, F_7)$.

Theorem (Prokhorov)
The simple finite subgroups of $\text{Cr}_3$ not contained in $\text{Cr}_2$ are $A_7, \text{SL}(2, F_8)$ and $\text{PSp}(4, F_3)$.

Up to conjugacy, $\text{SL}(2, F_8)$ admits only one embedding in $\text{Cr}_3$, and $\text{PSp}(4, F_3)$ two.
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*Up to conjugacy, \( SL(2, \mathbb{F}_8) \) admits only one embedding in \( Cr_3 \), and \( PSp(4, \mathbb{F}_3) \) two.*
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Proof: Prokhorov classifies (up to birational equivalence) all $G \subset Aut(V)$, $G$ finite simple, $V$ rationally connected 3-fold.
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$\mathfrak{A}_7$ appears twice: action on $\mathbb{P}^3$ above, and action on $V$:

$$\sum X_i = \sum X_i^2 = \sum X_i^3 = 0 \quad \text{in} \quad \mathbb{P}^6.$$
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Since $V$ is not rational, only one embedding $\mathfrak{A}_7 \subset Cr_3$. 

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Another corollary

Proposition

The group $S_7$ does not embed in $Cr_3$. 

Definition:

$$crdim(G) := \min\{n | \exists G \to Cr^n\}.$$ 

Proposition

For $n \geq 4$, $crdim(S_n) \leq n - 3$, with equality for $4 \leq n \leq 7$. 

Proof:

$S_n$ acts on the quadric $Q_{n-3}$: $\sum X_i = \sum X_i^2 = 0$ in $P_{n-1}$. 

$S_5 \not\subset Cr_1$, $S_6 \not\subset Cr_2$, $S_7 \not\subset Cr_3$. 

Question: Is it true that $crdim(S_n) = n - 3$?
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Idea of the proof: extend Prokhorov’s method to $\mathfrak{S}_7 \hookrightarrow$ any rationally connected 3-fold with an action of $\mathfrak{S}_7$ is birational to $V$, hence not rational.
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**Proposition**

The group $\mathcal{S}_7$ does not embed in $Cr_3$.

*Idea of the proof:* extend Prokhorov’s method to $\mathcal{S}_7 \sim\rightarrow$ any rationally connected 3-fold with an action of $\mathcal{S}_7$ is birational to $V$, hence not rational.

**Definition:** $cr\dim(G) := \min\{n \mid \exists \ G \hookrightarrow Cr_n\}$.
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The group $\mathfrak{S}_7$ does not embed in $Cr_3$.

*Idea of the proof*: extend Prokhorov’s method to $\mathfrak{S}_7 \hookrightarrow$ any rationally connected 3-fold with an action of $\mathfrak{S}_7$ is birational to $V$, hence not rational.

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For $n \geq 4$, $crdim(\mathfrak{S}_n) \leq n - 3$, with equality for $4 \leq n \leq 7$. 

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Arnaud Beauville

The Lüroth problem and the Cremona group
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\( S_5 \nsubseteq Cr_1 \), \( S_6 \nsubseteq Cr_2 \), \( S_7 \nsubseteq Cr_3 \).

Question: Is it true that \( \text{crdim}(S_n) = n - 3 \)?
The end
THE END