The stable Lüroth problem

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The classical Lüroth problem

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\[ X \text{ complex variety} \]
The classical Lüroth problem

$X$ complex variety

$(\mathbb{P}^n \dashrightarrow X)$

$X$ rational

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\( X \) complex variety

\((\mathbb{P}^n \overset{\sim}{\longrightarrow} X)\) \hspace{2cm} \((\mathbb{P}^n \overset{-}{\longrightarrow} X)\)

\( X \) rational \hspace{2cm} \( X \) unirational

\( (\mathbb{P}^n \overset{\sim}{\longrightarrow} X) \) gives examples in dimension \( \geq 3 \), and quite particular.

\( (\mathbb{P}^n \overset{-}{\longrightarrow} X) \) gives counter-examples (1971), with 3 different methods, at least in dimension 3.

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\( X \) complex variety

\[ (\mathbb{P}^n \sim \to X) \quad \quad \quad \quad \quad (\mathbb{P}^n - - \to X) \]

\( X \) rational \quad \Rightarrow \quad \quad \quad \quad X \) unirational

\( \text{(Lüroth problem)} \)

3 counter-examples (1971), with 3 different methods

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$X$ complex variety

$\mathbb{P}^n \sim X$  \quad  $\mathbb{P}^n \rightarrow X$

$X$ rational  \quad  $\leftrightarrow$  \quad  $X$ unirational

(Lüroth problem)
The classical Lüroth problem

\( X \) complex variety

\[(\mathbb{P}^n \cong X) \quad \quad \quad \quad (\mathbb{P}^n \rightarrow X)\]

\( X \) rational \quad \xrightarrow{\sim} \quad X \) unirational

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$X$ complex variety

$\mathbb{P}^n \cong X$ \quad $\mathbb{P}^n \dashrightarrow X$

$X$ rational \quad $\Leftrightarrow$ \quad $X$ unirational

3 counter-examples (1971), with 3 different methods

$\therefore$ many examples, at least in dimension 3.
The classical Lüroth problem

\begin{align*}
X \text{ complex variety} \\
(\mathbb{P}^n \xrightarrow{\sim} X) & \quad (\mathbb{P}^n \xrightarrow{-} X) \\
X \text{ rational} & \quad \leftrightarrow \quad X \text{ unirational}
\end{align*}

3 counter-examples (1971), with 3 different methods

\xrightarrow[]{-} \text{ many examples, at least in dimension 3.}

(Only the Artin-Mumford method gives examples in dimension > 3, and quite particular.)
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Hence, search for intermediate notion:

$X$ stably rational

if $X \subset \mathbb{P}^m$ rational for some $m$. 

\[\text{rational} \rightarrow \text{unirational} \rightarrow \text{stably rational} \rightarrow \text{Artin-Mumford : stably rational} \rightarrow \text{Tors} H^3 P X, Z_{q \neq 0}:\]

Construct quartic double solid $X$: branched along $\Delta$ quartic symmetroid: defined by $\det p_{ij}$ symmetric 4 4 matrix of linear forms.
Hence, search for intermediate notion:

- \( X \) stably rational if \( X \cong \mathbb{P}^m \) for some \( m \).

\[ \xymatrix{ \text{rational} & \text{unirational} \ar@{<->}[r] \ar@{<->}[d] & \text{stably rational} \ar@{<->}[d] \ar@{<->}[l] \\
\text{Artin-Mumford: stably rational} & \text{Tors} \ar@{<->}[r] & \text{H}_3(\mathbb{P}^3, \mathbb{Z}) \qquad \text{q}=0 }
\]

Construct quartic double solid \( X \) with \( \text{Tors} \mathbb{H}_3(\mathbb{P}^3, \mathbb{Z}) \quad \text{q}=0 \):

- branched along \( \Delta \) quartic symmetroid: defined by \( \det L_{ij} \quad \text{symmetric 4x4 matrix of linear forms.} \)
Hence, search for intermediate notion:

\[ X \text{ stably rational if } X \times \mathbb{P}^m \text{ rational for some } m. \]
Hence, search for intermediate notion:

* $X$ stably rational if $X \times \mathbb{P}^m$ rational for some $m$.

Diagram:

- Rational
- Unirational
- Stably rational

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Hence, search for intermediate notion:

$X$ stably rational if $X \times \mathbb{P}^m$ rational for some $m$. 

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Hence, search for intermediate notion:

\( X \) *stably rational* if \( X \times \mathbb{P}^m \) rational for some \( m \).

\[ \text{rational} \quad \xrightarrow{\text{unirational}} \quad \text{unirational} \]

\[ \text{stably rational} \]

\[ [B-C-S-S] \quad \xleftrightarrow{\text{X}} \]

Artin-Mumford: stably rational \( \Rightarrow \) Tors \( H^3(X, \mathbb{Z}) \\) 0: branched along \( \Delta \) quartic symmetroid: defined by \( \det p_{ij}^{q''} \), \( p_{ij} \) symmetric 4 \( \times \) 4 matrix of linear forms.
Hence, search for intermediate notion:

\( X \text{ stably rational} \) if \( X \times \mathbb{P}^m \) rational for some \( m \).

\[ \text{rational} \quad \overset{\text{unirational}}{\longrightarrow} \quad \text{stably rational} \]

\[ X \text{ stably rational} \quad \overset{\text{A-M}}{\longrightarrow} \quad [A-M] \]

\[ \text{[B-C-S-S]} \quad \overset{\text{X}}{\longrightarrow} \quad \text{[A-M]} \]
Hence, search for intermediate notion:

\[ X \text{ stably rational if } X \times \mathbb{P}^m \text{ rational for some } m. \]

\[
\begin{array}{c}
\text{rational} \quad \leftrightarrow \quad \text{unirational} \\
\text{stably rational} \\
\text{[B-C-S-S]} \quad \leftrightarrow \quad X \quad \leftrightarrow \quad X \\
\text{Arta-Mumford: stably rational} \quad \iff \quad \text{Tors } H^3(X, \mathbb{Z}) = 0
\end{array}
\]
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Hence, search for intermediate notion:

\( X \) \textit{stably rational} if \( X \times \mathbb{P}^m \) rational for some \( m \).

\[ \begin{align*}
\text{rational} & \quad \longrightarrow \quad \text{unirational} \\
\downarrow & \quad \downarrow & \quad \uparrow \\
\text{stably rational} & \quad \langle \quad \langle & \quad \rangle \quad \rangle \quad \langle & \quad \rangle \\
[B-C-S-S] & \quad \langle & \quad \rangle \quad \langle & \quad \rangle & \quad \langle & \quad \rangle \\
X & \quad \langle & \quad \rangle \quad \langle & \quad \rangle & \quad \langle & \quad \rangle \quad \langle & \quad \rangle & \quad \langle & \quad \rangle \\
X & \quad \langle & \quad \rangle \quad \langle & \quad \rangle & \quad \langle & \quad \rangle \quad \langle & \quad \rangle & \quad \langle & \quad \rangle \quad \langle & \quad \rangle \\
[A-M] & \quad \langle & \quad \rangle \quad \langle & \quad \rangle & \quad \langle & \quad \rangle 
\end{align*} \]

Artin-Mumford: \( \text{stably rational} \implies \text{Tors } H^3(X, \mathbb{Z}) = 0 \)

Construct \textit{quartic double solid} \( X \xrightarrow{2:1} \mathbb{P}^3 \) with \( \text{Tors } H^3(X, \mathbb{Z}) \neq 0 \):
The stable Lüroth problem

Hence, search for intermediate notion:

$X$ stably rational if $X \times \mathbb{P}^m$ rational for some $m$.

\[ \text{rational} \quad \xrightarrow{\text{unirational}} \quad \text{stably rational} \]

Artin-Mumford: stably rational $\implies$ $\text{Tors } H^3(X, \mathbb{Z}) = 0$

Construct \textit{quartic double solid} $X \xrightarrow{2:1} \mathbb{P}^3$ with $\text{Tors } H^3(X, \mathbb{Z}) \neq 0$:

- branched along $\Delta = \text{quartic symmetroid}$: defined by $\det(L_{ij}) = 0$,
- $(L_{ij})$ symmetric $4 \times 4$ matrix of linear forms.
New results

Theorem (Voisin)
A double covering of $\mathbb{P}^3$ branched along a very general quartic surface is not stably rational.

very general := outside countable union of strict subvarieties of the moduli space

Known to be unirational, not rational (AB 77, Voisin 86)

Theorem (AB)
A double covering of $\mathbb{P}^4$ or $\mathbb{P}^5$ branched along a very general quartic hypersurface is not stably rational.

Unirational; rationality was not known.

First example of a prime Fano manifold ($b_2^+ = 1$) of dimension $\geq 3$, unirational but not rational.

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New results

**Theorem (Voisin)**

A double covering of $\mathbb{P}^3$ branched along a very general quartic surface is **not** stably rational.

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**Theorem (AB)**

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- very general := outside countable union of strict subvarieties of the moduli space
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Theorem (AB)

*A double covering of $\mathbb{P}^4$ or $\mathbb{P}^5$ branched along a very general quartic hypersurface is not stably rational.*

- Unirational; rationality was not known.
- First example of a prime Fano manifold ($b_2 = 1$) of dimension $> 3$, unirational but not rational.
Other results

A very general quartic threefold is not stably rational (C-P).

A double covering of $\mathbb{P}^3$ branched along a very general sextic surface is not stably rational (AB).

A very general hypersurface of degree $\ell^2 R_n^3 V_{\infty} \in \mathbb{P}^{n+1}$ is not stably rational (Totaro; applies in particular to quartic threefolds and fourfolds).

But: not expected to be unirational. Already known to be non-rational (Iskovskikh, Manin, Kollár).
A very general quartic threefold is not stably rational (C-P).
A very general quartic threefold is not stably rational (C-P).

A double covering of $\mathbb{P}^3$ branched along a very general sextic surface is not stably rational (AB).
Other results

- A very general quartic threefold is not stably rational (C-P).
- A double covering of $\mathbb{P}^3$ branched along a very general sextic surface is not stably rational (AB).
- A very general hypersurface of degree $\geq 2 \left\lceil \frac{n + 2}{3} \right\rceil$ in $\mathbb{P}^{n+1}$ is not stably rational (Totaro; applies in particular to quartic threefolds and fourfolds).
A very general quartic threefold is not stably rational (C-P).

A double covering of $\mathbb{P}^3$ branched along a very general sextic surface is not stably rational (AB).

A very general hypersurface of degree $\geq 2 \left\lceil \frac{n + 2}{3} \right\rceil$ in $\mathbb{P}^{n+1}$ is not stably rational (Totaro; applies in particular to quartic threefolds and fourfolds).

But: not expected to be unirational.
Other results

- A very general quartic threefold is not stably rational (C-P).
- A double covering of $\mathbb{P}^3$ branched along a very general sextic surface is not stably rational (AB).
- A very general hypersurface of degree $\geq 2 \left\lfloor \frac{n + 2}{3} \right\rfloor$ in $\mathbb{P}^{n+1}$ is not stably rational (Totaro; applies in particular to quartic threefolds and fourfolds).

But: not expected to be unirational.
- Already known to be non-rational (Iskovskikh, Manin, Kollár).
Voisin’s idea:

Degenerate general quartic into symmetroid.

But: $T_{pX, Z}^q H^3$ for a smooth double cover of $\mathbb{P}^n$.

Voisin: stably rational $\Rightarrow CH^0$-trivial $\Rightarrow T_{pX, Z}^q H^3$ and $CH^0$-trivial behaves well under mild degeneration.

Recall: $CH^0_{\mathbb{P}^X}$ Chow group of 0-cycles on $X$.

Proposition (Bloch) $X$ smooth projective of dimension $n$.

If this holds, we say that $X$ is $CH^0$-trivial.

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Voisin’s idea:

Degenerate general quartic into symmetroid.
Voisin’s idea:

Degenerate general quartic into symmetroid.

**But**: $\text{Tors } H^3(X, \mathbb{Z}) = 0$ for a smooth double cover of $\mathbb{P}^n$. 

Recall: $\text{CH}_0$ is the Chow group of 0-cycles on $X$.
Voisin's idea:

Degenerate general quartic into symmetroid.

**But**: $\text{Tors } H^3(X, \mathbb{Z}) = 0$ for a smooth double cover of $\mathbb{P}^n$.

Voisin: stably rational $\iff$ "$CH_0$-trivial" $\implies$ Tors $H^3(X, \mathbb{Z}) = 0$;
Voisin's idea:

Degenerate general quartic into symmetroid.

But: $\text{Tors } H^3(X, \mathbb{Z}) = 0$ for a smooth double cover of $\mathbb{P}^n$.

Voisin: stably rational $\implies "CH_0$-trivial" $\implies \text{Tors } H^3(X, \mathbb{Z}) = 0$;
and "$CH_0$-trivial" behaves well under mild degeneration.
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**But:** $\text{Tors } H^3(X, \mathbb{Z}) = 0$ for a smooth double cover of $\mathbb{P}^n$.

Voisin: stably rational $\implies "CH_0\text{-trivial}" \implies \text{Tors } H^3(X, \mathbb{Z}) = 0$; and "$CH_0\text{-trivial}" behaves well under mild degeneration.

Recall: $CH_0(X) = \text{Chow group of 0-cycles on } X = \mathbb{Z}(X)/\sim_{\text{rat}}$. 
Voisin’s idea:

Degenerate general quartic into symmetroid.

**But** : $\text{Tors } H^3(X, \mathbb{Z}) = 0$ for a smooth double cover of $\mathbb{P}^n$.

Voisin : stably rational $\implies$ "$CH_0$-trivial" $\implies$ $\text{Tors } H^3(X, \mathbb{Z}) = 0$ ; and "$CH_0$-trivial" behaves well under mild degeneration.

Recall : $CH_0(X) = \text{Chow group of 0-cycles on } X = \mathbb{Z}(X) / \sim_{\text{rat}}$.

**Proposition (Bloch)**

$X$ smooth projective of dimension $n$. 

Voisin’s idea:

Degenerate general quartic into symmetroid.

**But**: $\text{Tors } H^3(X, \mathbb{Z}) = 0$ for a smooth double cover of $\mathbb{P}^n$.

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**Proposition (Bloch)**

$X$ smooth projective of dimension $n$.

$(i)$ $CH_0(X_K) = \mathbb{Z}$ for all extensions $\mathbb{C} \to K$;
Voisin’s idea:

Degenerate general quartic into symmetroid.

**But**: Tors $H^3(X, \mathbb{Z}) = 0$ for a smooth double cover of $\mathbb{P}^n$.

Voisin: stably rational $\implies "CH_0\text{-trivial}" \implies$ Tors $H^3(X, \mathbb{Z}) = 0$; and "$CH_0\text{-trivial}"$ behaves well under mild degeneration.

Recall: $CH_0(X) =$ Chow group of 0-cycles on $X = \mathbb{Z}(X)/\sim_{rat}$.

### Proposition (Bloch)

$X$ smooth projective of dimension $n$.

1. $CH_0(X_K) = \mathbb{Z}$ for all extensions $\mathbb{C} \to K$;
2. $\Delta = X \times \{p\} + Z$ in $CH^n(X \times X)$, $\text{Supp}(Z) \subset D \times X$. 

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Voisin’s idea:

Degenerate general quartic into symmetroid.

But: Tors $H^3(X, \mathbb{Z}) = 0$ for a smooth double cover of $\mathbb{P}^n$.

Voisin: stably rational $\implies$ "$CH_0$-trivial" $\implies$ Tors $H^3(X, \mathbb{Z}) = 0$ ;
and "$CH_0$-trivial" behaves well under mild degeneration.

Recall: $CH_0(X) = $ Chow group of 0-cycles on $X = \mathbb{Z}(X)/\sim_{rat}$.

Proposition (Bloch)

$X$ smooth projective of dimension $n$.

(i) $CH_0(X_K) = \mathbb{Z}$ for all extensions $\mathbb{C} \rightarrow K$ ;

(ii) $\Delta = X \times \{p\} + Z$ in $CH^n(X \times X)$, $\text{Supp}(Z) \subset D \times X$

If this holds, we say that $X$ is $CH_0$-trivial.
$CH_0$-trivial $\Rightarrow$ Tors $H^3(X, \mathbb{Z}) = 0$
Proposition

$X$ stably rational $\iff X$ CH$_0$-trivial $\iff \text{Tors } H^3(X, \mathbb{Z}) = 0.$
$CH_0$-trivial $\Rightarrow$ Tors $H^3(X, \mathbb{Z}) = 0$

Proposition

$X$ stably rational $\iff$ $X$ $CH_0$-trivial $\iff$ Tors $H^3(X, \mathbb{Z}) = 0$.

Proof: a) $CH_0$ birational invariant, and $CH_0(X \times \mathbb{P}^m) = CH_0(X)$. 

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Proposition

\( X \) stably rational \( \iff \) \( X \) \( CH_0 \)-trivial \( \iff \) \( \text{Tors} \, H^3(X, \mathbb{Z}) = 0 \).

Proof: a) \( CH_0 \) birational invariant, and \( CH_0(X \times \mathbb{P}^m) = CH_0(X) \).

b) \( X \) \( CH_0 \)-trivial: \( \Delta \sim_{\text{rat}} X \times \{p\} + Z \).
Proposition

$X$ stably rational $\iff X$ CH$_0$-trivial $\iff$ Tors $H^3(X, \mathbb{Z}) = 0$.

Proof: a) CH$_0$ birational invariant, and $CH_0(X \times \mathbb{P}^m) = CH_0(X)$.

b) $X$ CH$_0$-trivial : $\Delta \sim_{\text{rat}} X \times \{p\} + \mathbb{Z}$. 

![Diagram](attachment:image.png)
Proposition

\( X \) stably rational \( \iff X \) CH\(_0\)-trivial \( \iff \) Tors \( H^3(X, \mathbb{Z}) = 0 \).

Proof: a) \( CH_0 \) birational invariant, and \( CH_0(X \times \mathbb{P}^m) = CH_0(X) \).

b) \( X \) CH\(_0\)-trivial: \( \Delta \sim_{\text{rat}} X \times \{p\} + Z \).

For \( \delta \in CH^n(X \times X) \), endomorphism \( \delta^* : \alpha \mapsto p_*(q^*\alpha \cdot \delta) \) of \( H^r(X, \mathbb{Z}) \).
$CH_0$-trivial $\Rightarrow$ Tors $H^3(X, \mathbb{Z}) = 0$

**Proposition**

$X$ stably rational $\implies X$ $CH_0$-trivial $\implies$ Tors $H^3(X, \mathbb{Z}) = 0$.

**Proof:**

a) $CH_0$ birational invariant, and $CH_0(X \times \mathbb{P}^m) = CH_0(X)$.

b) $X$ $CH_0$-trivial: $\Delta \sim_{\text{rat}} X \times \{p\} + Z$.

For $\delta \in CH^n(X \times X)$, endomorphism $\delta^* : \alpha \mapsto p_*(q^* \alpha \cdot \delta)$ of $H^r(X, \mathbb{Z})$.

$\Delta^* = \text{Id}$, $[X \times \{p\}]^* = 0$ for $r \neq 0$, and
**Proposition**

$X$ stably rational $\implies X$ CH$_0$-trivial $\implies$ Tors $H^3(X, \mathbb{Z}) = 0$.

*Proof*: a) CH$_0$ birational invariant, and $CH_0(X \times \mathbb{P}^m) = CH_0(X)$.

b) $X$ CH$_0$-trivial : $\Delta \sim_{rat} X \times \{p\} + Z$.

\[ \xymatrix{ X \times X & \ar[l]_p \ar[r]^q \ar[d] \ar[u] & X \times X \ar[l] \ar[r] & X \ar[d] \ar[u] } \]

For $\delta \in CH^n(X \times X)$, endomorphism $\delta^* : \alpha \mapsto p_*(q^* \alpha \cdot \delta)$ of $H^r(X, \mathbb{Z})$.

$\Delta^* = \text{Id}$, $[X \times \{p\}]^* = 0$ for $r \neq 0$, and

$\text{Id} = Z^* : H^r(X, \mathbb{Z}) \to H^{r-2}(D, \mathbb{Z}) \xrightarrow{i_*} H^r(X, \mathbb{Z})$. 

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\[ CH_0 \text{-trivial} \Rightarrow \text{Tors } H^3(X, \mathbb{Z}) = 0 \]

**Proposition**

\[ X \text{ stably rational} \implies X \text{ } CH_0 \text{-trivial} \implies \text{Tors } H^3(X, \mathbb{Z}) = 0. \]

**Proof:**

a) \( CH_0 \) birational invariant, and \( CH_0(X \times \mathbb{P}^m) = CH_0(X) \).

b) \( X \) \( CH_0 \)-trivial: \( \Delta \sim_{\text{rat}} X \times \{p\} + Z \).

For \( z \in CH^n(X \times X) \), endomorphism \( z^* : \alpha \mapsto p_*(q^*\alpha \cdot z) \) of \( H^r(X, \mathbb{Z}) \).

\[ \Delta^* = \text{Id}, [X \times \{p\}]^* = 0 \text{ for } r \neq 0, \text{ and} \]

\[ \text{Id} = Z^* : H^r(X, \mathbb{Z}) \rightarrow H^{r-2}(D, \mathbb{Z}) \stackrel{i_*}{\rightarrow} H^r(X, \mathbb{Z}). \]

For \( r = 3 \), \( H^1(D, \mathbb{Z}) \) torsion free \( \Rightarrow \) .
The degeneration argument

Proposition (Voisin, Colliot-Thélène-Pirutka)

$X$ flat projective, $B$ smooth, general fiber smooth, $O_B$.

Assume $X_o$ admits a desingularization $\sigma: \tilde{X} \to X$ with:

- $a$ $q_Tor H^3(p_\tilde{X}, Z_{q\neq 0});$
- $b$ $q_\sigma^{-1}(x)$ rational over $\kappa_x$ for all $x \in X$.

Then $X_b$ not stably rational for very general $b$.

Idea:
- $a$ $q_{\tilde{X}}$ not $CH_0$-trivial;
- $b$ $q_X$ same for $X$.

Specialization argument $q_{\text{generic fiber}}$, then for very general fiber $X_b$ (Baire) $q_X$ not stably rational.

Thus to prove the theorem, we need a double covering $X \to P^n$ ($n = 3, 4, 5$) with a desingularization $\tilde{X} \to X$ satisfying $a$ and $b$.

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The degeneration argument

**Proposition (Voisin, Colliot-Thélène-Pirutka)**

$\mathcal{X} \to B$ flat projective, $B$ smooth, general fiber smooth, $o \in B$. 

Assume $X^0$ admits a desingularization $\sigma : \tilde{X} \to X$ with:

- $a_\text{q} T\text{ors} H^3_p \tilde{X}, Z_{q} \neq 0$;
- $b_\text{q} \sigma^{-1} p_x \text{ rational over } \kappa$ for all $x \in X$.

Then $X^b$ not stably rational for very general $b$.

Idea:
- $a_\text{q}$ not $\text{CH}_0$-trivial;
- $b_\text{q}$ same for $X$.

Specialization argument same for generic fiber, then for very general fiber $X^b (\text{Baire})$ not stably rational.

Thus to prove the theorem, we need a double covering $\mathcal{X} \to \mathbb{P}^n (n = 3, 4, 5)$ with a desingularization $\tilde{X} \to X$ satisfying $a$ and $b$. 

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**The stable Lüroth problem**
The degeneration argument

 Proposition (Voisin, Colliot-Thélène-Pirutka)

\( \mathcal{X} \to B \) flat projective, \( B \) smooth, general fiber smooth, \( o \in B \).

Assume \( X := \mathcal{X}_o \) admits a desingularization \( \sigma : \tilde{X} \to X \) with:

\[ a \in H^3_p(\tilde{X}), Z^q \neq 0; b \in \sigma^{-1}(\mathbb{Q}^r) \] for all \( x \in X \).

Then \( X \) is not stably rational for very general fiber.

Idea:

- \( a \) and \( \tilde{X} \) not \( \text{CH}^0 \)-trivial;
- \( b \) and \( X \) the same for generic fiber,

then for very general fiber \( X \) is not stably rational.

Thus to prove the theorem, we need a double covering \( \mathcal{X} \to \mathbb{P}^n \) (\( n = 3, 4, 5 \)) with a desingularization \( \tilde{X} \to X \) satisfying \( a \) and \( b \).
Proposition (Voisin, Colliot-Thélène-Pirutka)

\( \mathcal{X} \to B \) flat projective, \( B \) smooth, general fiber smooth, \( o \in B \).
Assume \( X := \mathcal{X}_o \) admits a desingularization \( \sigma : \tilde{X} \to X \) with :

a) \( \text{Tors} \ H^3(\tilde{X}, \mathbb{Z}) \neq 0; \)
The degeneration argument

**Proposition (Voisin, Colliot-Thélène-Pirutka)**

\[ \mathcal{X} \to B \] flat projective, \( B \) smooth, general fiber smooth, \( o \in B \).

Assume \( X := \mathcal{X}_o \) admits a desingularization \( \sigma : \tilde{X} \to X \) with:

a) \( \text{Tors} H^3(\tilde{X}, \mathbb{Z}) \neq 0 \);

b) \( \sigma^{-1}(x) \) rational over \( \kappa(x) \) for all \( x \in X \).
The degeneration argument

Proposition (Voisin, Colliot-Thélène-Pirutka)

Let $\mathcal{X} \rightarrow B$ be a flat projective family, $B$ smooth, general fiber smooth, $0 \in B$.

Assume $X := \mathcal{X}_0$ admits a desingularization $\sigma : \tilde{X} \rightarrow X$ with:

1. $\text{Tors} H^3(\tilde{X}, \mathbb{Z}) \neq 0$;
2. $\sigma^{-1}(x)$ is rational over $\kappa(x)$ for all $x \in X$.

Then $\mathcal{X}_b$ is not stably rational for very general $b$. 

Idea:

- $\text{Tors} H^3(\tilde{X}, \mathbb{Z})$ not $\text{CH}_0$-trivial;
- Same for $X$.

Specialization argument:

- Same for generic fiber;
- Then for very general fiber $\mathcal{X}_b$ (Baire) is not stably rational.

Thus to prove the theorem, we need a double covering $\mathcal{X} \rightarrow \mathbb{P}^n$ ($n = 3, 4, 5$) with a desingularization $\tilde{X} \rightarrow X$ satisfying $a)$ and $b)$. 

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The stable Lüroth problem
Proposition (Voisin, Colliot-Thélène-Pirutka)

$\mathcal{X} \to B$ flat projective, $B$ smooth, general fiber smooth, $0 \in B$.
Assume $X := \mathcal{X}_o$ admits a desingularization $\sigma : \tilde{X} \to X$ with:

a) $\text{Tors } H^3(\tilde{X}, \mathbb{Z}) \neq 0$;

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The degeneration argument

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Assume \( X := \mathcal{X}_\circ \) admits a desingularization \( \sigma : \tilde{X} \to X \) with:

a) \( \text{Tors } H^3(\tilde{X}, \mathbb{Z}) \neq 0 \);

b) \( \sigma^{-1}(x) \) rational over \( \kappa(x) \) for all \( x \in X \).

Then \( \mathcal{X}_b \) not stably rational for very general \( b \).

Idea : a) \( \Rightarrow \tilde{X} \) not \( CH_0 \)-trivial; b) \( \Rightarrow \) same for \( X \).

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The degeneration argument

Proposition (Voisin, Colliot-Thélène-Pirutka)

\( \mathcal{X} \to B \) flat projective, \( B \) smooth, general fiber smooth, \( \mathfrak{o} \in B \).
Assume \( \mathcal{X} := \mathcal{X}_0 \) admits a desingularization \( \sigma : \tilde{\mathcal{X}} \to \mathcal{X} \) with:

\begin{itemize}
  \item[a)] \( \text{Tors } H^3(\tilde{\mathcal{X}}, \mathbb{Z}) \neq 0 \);
  \item[b)] \( \sigma^{-1}(x) \) rational over \( \kappa(x) \) for all \( x \in \mathcal{X} \).
\end{itemize}

Then \( \mathcal{X}_b \) not stably rational for very general \( b \).

\textbf{Idea :} a) \( \Rightarrow \tilde{\mathcal{X}} \) not \( CH_0 \)-trivial; b) \( \Rightarrow \) same for \( \mathcal{X} \).

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Thus to prove the theorem, we need a double covering \( X \to \mathbb{P}^n \).
The degeneration argument

Proposition (Voisin, Colliot-Thélène-Pirutka)
\[ X \to B \] flat projective, \( B \) smooth, general fiber smooth, \( \circ \in B \).
Assume \( X := X_0 \) admits a desingularization \( \sigma : \tilde{X} \to X \) with :

\( a) \) \( \text{Tors} \, H^3(\tilde{X}, \mathbb{Z}) \neq 0; \)

\( b) \) \( \sigma^{-1}(x) \) rational over \( \kappa(x) \) for all \( x \in X \).

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Idea : \( a) \) \( \Rightarrow \) \( \tilde{X} \) not \( CH_0 \)-trivial; \( b) \) \( \Rightarrow \) same for \( X \).
Specialization argument \( \Rightarrow \) same for generic fiber, then for very general fiber \( X_b \) (Baire) \( \Rightarrow \) \( X_b \) not stably rational.

Thus to prove the theorem, we need a double covering \( X \to \mathbb{P}^n \) (\( n = 3, 4, 5 \)) with a desingularization \( \tilde{X} \to X \) satisfying \( a) \) and \( b) \).
The Brauer group

How to find torsion elements in $H^3_p, \mathbb{Z}^q$?

Observation

For $V$ smooth projective with $H^2_p V, \mathcal{O}^q = 0$, $\text{Tors} H^3_p V, \mathbb{Z}^q = \text{Tors} H^2_p V, \mathcal{O}_{\text{hol}}^q$.

Proof: The exponential exact sequence gives an exact sequence $H^2_p V, \mathcal{O}_{\text{hol}}^q \Rightarrow H^2_p V, \mathcal{O}_{\text{hol}}^q \Rightarrow H^3_p V, \mathbb{Z}^q \Rightarrow H^3_p V, \mathcal{O}_{\text{hol}}^q$. 

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The stable Lüroth problem
The Brauer group

How to find torsion elements in $H^3(\ , \mathbb{Z})$?
The Brauer group

How to find torsion elements in $H^3(\, , \mathbb{Z})$?

Observation

For $V$ smooth projective with $H^2(V, \mathcal{O}_V) = 0$, the exponential exact sequence gives an exact sequence:

$$
0 \rightarrow H^2(p, \mathcal{O}_V) \rightarrow H^3(p, \mathcal{O}_V) \rightarrow \text{Br}_p(V) \rightarrow 0
$$

where $\text{Br}_p(V)$ is the Brauer group of $V$. The sheaf of holomorphic functions on $V$. 

Proof: The exponential exact sequence gives an exact sequence:

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The stable Lüroth problem
How to find torsion elements in $H^3(\cdot, \mathbb{Z})$?

**Observation**

For $V$ smooth projective with $H^2(V, \mathcal{O}_V) = 0$,

$$\text{Tors } H^3(V, \mathbb{Z}) = \text{Tors } H^2(V, \mathcal{O}_h^*) := \text{Br}(V)$$

(Brauer group)

where $\mathcal{O}_h :=$ sheaf of holomorphic functions on $V$. 

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The stable Lüroth problem
How to find torsion elements in $H^3(\cdot, \mathbb{Z})$?

**Observation**

For $V$ smooth projective with $H^2(V, \mathcal{O}_V) = 0$, 

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where $\mathcal{O}_h :=$ sheaf of holomorphic functions on $V$.

**Proof**: The exponential exact sequence gives an exact sequence

$$H^2(V, \mathcal{O}_h) = 0 \longrightarrow H^2(V, \mathcal{O}_h^*) \longrightarrow H^3(V, \mathbb{Z}) \longrightarrow H^3(V, \mathcal{O}_h).$$
One way to get classes in the Brauer group $\text{Br}_p\mathbb{V}$ is to consider $\mathbb{P}^m$-bundles on $\mathbb{V}$, that is, smooth fibrations $\mathbb{P} \rightarrow \mathbb{V}$ with fibers isomorphic to $\mathbb{P}^m$ (a.k.a. Severi-Brauer schemes over $\mathbb{V}$).

Reminder: An $\mathbb{P}^n$-bundle $\mathbb{P} \rightarrow \mathbb{V}$ defines an $n$-torsion class $r_{\mathbb{P}} \in \text{Br}_p\mathbb{V}$, which is trivial if and only if $\mathbb{P}$ is a projective bundle $\mathbb{P}_p\mathbb{V}/\mathbb{P}$. (follows from the cohomology exact sequence associated to $1 \rightarrow \mathbb{C}^* \rightarrow \text{GL}_n(\mathbb{C}) \rightarrow \text{PGL}_n(\mathbb{C}) \rightarrow 1$.)

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One way to get classes in the Brauer group $\text{Br}(V)$ is to consider $\mathbb{P}^m$-bundles on $V$. 
One way to get classes in the Brauer group $\text{Br}(V)$ is to consider $\mathbb{P}^m$-bundles on $V$, that is, smooth fibrations $P \to V$ with fibers isomorphic to $\mathbb{P}^m$ (a.k.a. Severi-Brauer schemes over $V$).
One way to get classes in the Brauer group $\text{Br}(V)$ is to consider $\mathbb{P}^m$-bundles on $V$, that is, smooth fibrations $P \to V$ with fibers isomorphic to $\mathbb{P}^m$ (a.k.a. Severi-Brauer schemes over $V$).

**Reminder**

A $\mathbb{P}^{n-1}$-bundle $P \to V$ defines a $n$-torsion class $[P] \in \text{Br}(V)$. 
One way to get classes in the Brauer group $\text{Br}(V)$ is to consider $\mathbb{P}^m$-bundles on $V$, that is, smooth fibrations $P \to V$ with fibers isomorphic to $\mathbb{P}^m$ (a.k.a. Severi-Brauer schemes over $V$).

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A $\mathbb{P}^{n-1}$-bundle $P \to V$ defines a $n$-torsion class $[P] \in \text{Br}(V)$, which is trivial if and only if $P$ is a projective bundle $\mathbb{P}_V(E)$. 

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The stable Lüroth problem
One way to get classes in the **Brauer group** $\text{Br}(V)$ is to consider $\mathbb{P}^m$-bundles on $V$, that is, smooth fibrations $P \to V$ with fibers isomorphic to $\mathbb{P}^m$ (a.k.a. **Severi-Brauer** schemes over $V$).

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The construction

\[ \text{The construction} \]

\[ \Sigma \Delta L - P^n \text{rk} \geq 2 \text{rk} \geq 3 \text{∆ quartic hypersurface,} \]

\[ \text{Sing}_p \Delta q'' \Sigma, \dim_p \Delta q'' n - 3. \]

\[ \pi: X \rightarrow L := \text{double cover branched along } \Delta. \]

\[ X_{sm} \pi' 1_p L \setminus \Sigma \text{tp}_q \sigma_q|_q P L, \text{rk}_p q \sigma_q'' 3 \text{ or 4,} \sigma_q \text{system of generatrices of } q_u. \]

\[ \sigma \text{parametrized by } P^1 \text{ù } \text{P}^1 \text{-bundle } p: P \rightarrow X_{sm}. \]

\[ \text{Proposition} \]

\[ \text{The } P^1 \text{-bundle } p: P \rightarrow X_{sm} \text{ is not a projective bundle.} \]

\[ \text{Proof:} \text{ Suppose } p \text{ has a rational section:} p q \ell_p q, \sigma_q p \sigma_q X \ell_p q, \sigma_1 q''. \]

\[ \text{For general } q, 2 \text{ systems } \sigma, \sigma_1 \ell_p q, \sigma_q X \ell_p q, \sigma_1 q''. \]

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The construction

$L = \text{general linear system of quadrics in } \mathbb{P}^3 \text{ of dimension } n.$
The construction

\( L = \text{general linear system of quadrics in } \mathbb{P}^3 \text{ of dimension } n. \)

\[ \Sigma \subset \Delta \subset L \cong \mathbb{P}^n \]
\[ \text{rk } \leq 2 \quad \text{rk } \leq 3 \]
$L = \text{general linear system of quadrics in } \mathbb{P}^3 \text{ of dimension } n.$

$\Sigma \subset \Delta \subset L \cong \mathbb{P}^n$

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$\Delta \text{ quartic hypersurface, } \text{Sing}(\Delta) = \Sigma, \dim(\Sigma) = n - 3.$
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The construction

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The stable Lüroth problem
The construction

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**Proposition**

The \(\mathbb{P}^1\)-bundle \(p : P \to X_{sm}\) is not a projective bundle.
$L$ = general linear system of quadrics in $\mathbb{P}^3$ of dimension $n$.

$\Sigma \subset \Delta \subset L \cong \mathbb{P}^n$

$\text{rk} \leq 2 \quad \text{rk} \leq 3$

$\Delta$ quartic hypersurface, $\text{Sing}(\Delta) = \Sigma$, $\dim(\Sigma) = n - 3$.

$\pi : X \to L := $ double cover branched along $\Delta$. $X_{sm} = \pi^{-1}(L \setminus \Sigma) = \{(q, \sigma) \mid q \in L, \text{rk}(q) = 3 \text{ or } 4, \sigma = \text{system of generatrices of } q\}$. $\sigma$ parametrized by $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1$-bundle $p : P \to X_{sm}$.

**Proposition**

The $\mathbb{P}^1$-bundle $p : P \to X_{sm}$ is not a projective bundle.

**Proof** : Suppose $p$ has a rational section : $(q, \sigma) \dashrightarrow \ell(q, \sigma) \in \sigma$.
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For general \( q \), 2 systems \( \sigma, \sigma' \overset{\sim}{\mapsto} \ell(q, \sigma) \cap \ell(q, \sigma') = s(q) \in q. \)
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Get rational section of quadric family \( Q \to L \).
Lemma

The universal family $p: H \to \mathbb{P}^n$ has no rational section.

Proof: rational section $u: \mathbb{Z} \to H$, $r \mapsto \hat{u}(r)$ smooth hypersurface in $H \to \mathbb{P}^n$.

Conclusion: get a nontrivial 2-torsion class in $\text{Br}_p X_{\text{sm}}$.
Lemma

$L \subset |\mathcal{O}_{\mathbb{P}^n}(d)|$ base point free linear system, $d \geq 2$. 

Proof: rational section $u : \mathbb{P}^n \to \mathcal{O}_{\mathbb{P}^n}(d)$.

Conclusion: get a nontrivial 2-torsion class in $\text{Br}_{\mathbb{P}^n}$. Then:

$\text{Br}_{\mathbb{P}^n} \tilde{\mathcal{X}}_\mathbb{P}^n = \text{Br}_{\mathbb{P}^n} \mathcal{X}_\mathbb{P}^n$. 

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The stable Lüroth problem
Lemma

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The universal family $p : \mathcal{H} \to L$ has no rational section.
Lemma

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The universal family $p : \mathcal{H} \to L$ has no rational section.

$\mathcal{H} := \{(H, x) \in L \times \mathbb{P}^n \mid x \in H\}$

![Diagram](image)
Lemma

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\[ \mathcal{H} := \{ (H, x) \in L \times \mathbb{P}^n \mid x \in H \} \]

smooth hypersurface in $L \times \mathbb{P}^n$. 

\[ \begin{array}{ccc}
\mathcal{H} & \xrightarrow{p} & L \\
\downarrow & & \downarrow \\
\mathbb{P}^n & \xrightarrow{q} & \mathbb{P}^n
\end{array} \]
Lemma

\[ L \subset |\mathcal{O}_{\mathbb{P}^n}(d)| \text{ base point free linear system, } d \geq 2. \]

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**Proof**: rational section \( Z \subset \mathcal{H} \), \( [Z] \cdot p^*[\text{pt}] = 1. \)
**Lemma**

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**Proof:**

rational section \( \sim \) \( Z \subset \mathcal{H} \), \( [Z] \cdot p^*[\text{pt}] = 1 \).

\( [Z] \in H^{2n-2}(\mathcal{H}, \mathbb{Z}) \): by Lefschetz, \( [Z] = \sum_i n_i p^* h^i_L \cdot q^* h^{n-1-i}_{\mathbb{P}^n} \).
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\( [Z] \cdot p^* [pt] = n_0 p^* [pt] \cdot q^* h^{n-1}_{\mathbb{P}^n} = n_0 d \quad \Rightarrow \quad \boxed{\text{}}. \)
Lemma

$L \subset |\mathcal{O}_{\mathbb{P}^n}(d)|$ base point free linear system, $d \geq 2$.

The universal family $p : \mathcal{H} \to L$ has no rational section.

$\mathcal{H} := \{(H, x) \in L \times \mathbb{P}^n \mid x \in H\}$
smooth hypersurface in $L \times \mathbb{P}^n$.

**Proof:** rational section $\rightsquigarrow Z \subset \mathcal{H}$, $[Z] \cdot p^*[\text{pt}] = 1$.

$[Z] \in H^{2n-2}(\mathcal{H}, \mathbb{Z})$ : by Lefschetz, $[Z] = \sum n_i p^* h_L^i \cdot q^* h_{\mathbb{P}^n}^{n-1-i}$

$[Z] \cdot p^*[\text{pt}] = n_0 p^*[\text{pt}] \cdot q^* h_{\mathbb{P}^n}^{n-1} = n_0 d \implies \square$.

**Conclusion:** get a nontrivial 2-torsion class in $\text{Br}(X_{sm})$. 
Lemma

$L \subset |O_{\mathbb{P}^n}(d)|$ base point free linear system, $d \geq 2$.

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smooth hypersurface in $L \times \mathbb{P}^n$.

\[ \text{Proof: rational section } Z \subset \mathcal{H}, \ [Z] \cdot p^*[\text{pt}] = 1. \]

\[ [Z] \in H^{2n-2}(\mathcal{H}, \mathbb{Z}) : \text{by Lefschetz, } [Z] = \sum_i n_i p^* h_i^L \cdot q^* h_{\mathbb{P}^n}^{n-1-i} \]

\[ [Z] \cdot p^*[\text{pt}] = n_0 p^*[\text{pt}] \cdot q^* h_{\mathbb{P}^n}^{n-1} = n_0 d \implies \square \]

**Conclusion:** get a nontrivial 2-torsion class in $\text{Br}(X_{sm})$. Then:

\[ \text{Br}(\tilde{X}) \twoheadrightarrow \text{Br}(X_{sm}) \]
The desingularization

Recall: \( \Sigma \)

\[ \Delta \]

\( L - P \)

\( n \rightarrow 2 \rightarrow 3 \)

\( \Delta \) quartic hypersurface,

\( \text{Sing} \)

\( p \)

\( \Delta q'' \)

\( \Sigma, \dim \)

\( p \)

\( \Sigma q'' n \rightarrow 3. \)

Assume \( n \rightarrow 5 \):

Then \( \Sigma \) smooth; locally at \( q \)

\( P \)

\( \Delta - \Sigma \hat{Q} \)

\( Q \)

quadratic cone in \( P_3 \).

Blow up \( L \) along \( \Sigma \):

\[ \mathcal{L} \]

\( \mathcal{E} \)

\( \mathcal{L} \tilde{\Delta} \)

\( \mathcal{Q} \)

\( \mathcal{E} \rightarrow \mathcal{L} \mathcal{Q} \)

\( \tilde{\Delta} \)

\( \mathcal{Q} \)

\( \mathcal{E} \rightarrow \mathcal{L} \mathcal{C} \)

\( \mathcal{C} \)

smooth conic.

Key fact

True locally for the Zariski topology.
The desingularization

Recall: \( \Sigma \subset \Delta \subset L \cong \mathbb{P}^n \)

\( \text{rk} \leq 2 \quad \text{rk} \leq 3 \)
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\[
\text{rk} \leq 2 \quad \text{rk} \leq 3
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\( \Delta \) quartic hypersurface, \( \text{Sing}(\Delta) = \Sigma, \dim(\Sigma) = n - 3 \).
Recall: \( \Sigma \subset \Delta \subset L \cong \mathbb{P}^n \)
\( \text{rk } \leq 2 \quad \text{rk } \leq 3 \)

\( \Delta \) quartic hypersurface, \( \text{Sing}(\Delta) = \Sigma, \dim(\Sigma) = n - 3 \).

**Assume** \( n \leq 5 \): Then \( \Sigma \) smooth; locally at \( q \in \Sigma \), \( \Delta \cong \Sigma \times Q \),
\( Q \) quadratic cone in \( \mathbb{P}^3 \).
Recall: $\Sigma \subset \Delta \subset L \cong \mathbb{P}^n$

$\text{rk} \leq 2 \quad \text{rk} \leq 3$

$\Delta$ quartic hypersurface, $\text{Sing}(\Delta) = \Sigma$, $\text{dim}(\Sigma) = n - 3$.

**Assume $n \leq 5$**: Then $\Sigma$ smooth; locally at $q \in \Sigma$, $\Delta \cong \Sigma \times Q$, $Q$ quadratic cone in $\mathbb{P}^3$. Blow up $L$ along $\Sigma$:
Recall: \( \Sigma \subset \Delta \subset L \cong \mathbb{P}^n \)
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\( \Delta \) quartic hypersurface, \( \text{Sing}(\Delta) = \Sigma \), \( \dim(\Sigma) = n - 3 \).

**Assume** \( n \leq 5 \): Then \( \Sigma \) smooth; locally at \( q \in \Sigma \), \( \Delta \cong \Sigma \times Q \), \( Q \) quadratic cone in \( \mathbb{P}^3 \). Blow up \( L \) along \( \Sigma \):

\[
\begin{array}{ccc}
E & \hookrightarrow & \tilde{L} \\
\downarrow & & \downarrow b \\
\Sigma & \hookrightarrow & L
\end{array}
\]
The desingularization

Recall: \( \Sigma \subset \Delta \subset L \cong \mathbb{P}^n \)

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\downarrow & & \downarrow \\
\tilde{L} & \leftarrow & L
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The desingularization

Recall: \( \Sigma \subset \Delta \subset L \cong \mathbb{P}^n \)
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\begin{array}{ccc}
E & \hookrightarrow & \tilde{L} & \leftarrow & \tilde{\Delta} \\
\downarrow & & \downarrow b & & \downarrow \\
\Sigma & \hookrightarrow & L & \leftarrow & \Delta
\end{array}
\]

locally over \( q \in \Sigma \): \( \tilde{\Delta} \cong \Sigma \times \tilde{Q} \), \( \tilde{\Delta} \cap E \cong \Sigma \times C \), \( C \) smooth conic.
The desingularization

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**Key fact**

True locally for the Zariski topology.
Double coverings

π: X → L branched along ∆; define ˜π: ˜X → ˜L branched along ˜∆ (note that ˜∆ = ∆ + ∆ | b δ ∆ 2 E | )

˜Σ ↘ ↙ → ↓ ↓ ˜X ˜π → → ˜Σ ↓ ↓ ˜L b ↓ ↓ Σ ↘ ↙ → X π → → L

"double cover of E branched along ˜∆ X E."

˜Σ → Zar X ˜Σ → Zar X ˆt smooth quadric u.

ù condition a q for σ.
\[ \pi : X \rightarrow L \text{ branched along } \Delta; \]
Double coverings

\[ \pi : X \rightarrow L \text{ branched along } \Delta; \text{ define } \tilde{\pi} : \tilde{X} \rightarrow \tilde{L} \text{ branched along } \tilde{\Delta} \]

Arnaud Beauville
The stable Lüroth problem
$\pi : X \to L$ branched along $\Delta$; define $\tilde{\pi} : \tilde{X} \to \tilde{L}$ branched along $\tilde{\Delta}$
(note that $\tilde{\Delta} \in |b^*\Delta - 2E|$)
Double coverings

$\pi : X \rightarrow L$ branched along $\Delta$; define $\tilde{\pi} : \tilde{X} \rightarrow L$ branched along $\tilde{\Delta}$
(note that $\tilde{\Delta} \in |b^*\Delta - 2E|$)

$\quad \begin{array}{c}
\tilde{\Sigma} \rightarrow \tilde{X} \xrightarrow{\tilde{\pi}} \tilde{L} \\
\downarrow \quad \downarrow \sigma \quad \downarrow b \\
\Sigma \rightarrow X \xrightarrow{\pi} L
\end{array}$

Arnaud Beauville
The stable Lüroth problem
Double coverings

$\pi : X \to L$ branched along $\Delta$; define $\tilde{\pi} : \tilde{X} \xrightarrow{2:1} \tilde{L}$ branched along $\tilde{\Delta}$
(note that $\tilde{\Delta} \in |b^* \Delta - 2E|$)

\[ \begin{array}{ccc}
\tilde{\Sigma} & \subset & \tilde{X} \\
\downarrow & & \downarrow \sigma \\
\Sigma & \subset & X \\
\downarrow & & \downarrow \pi \\
\Sigma & \rightarrow & L \\
\end{array} \]

$\tilde{\Sigma} =$ double cover of $E$ branched along $\tilde{\Delta} \cap E$. 
Double coverings

\( \pi : X \to L \) branched along \( \Delta \); define \( \tilde{\pi} : \tilde{X} \xrightarrow{2:1} \tilde{L} \) branched along \( \tilde{\Delta} \)

(note that \( \tilde{\Delta} \in |b^*\Delta - 2E| \))

\[ \begin{array}{ccc}
\Sigma & \xhookrightarrow{} & \tilde{X} & \xrightarrow{\tilde{\pi}} & \tilde{L} \\
\downarrow & & \downarrow_{\sigma} & & \downarrow_{b} \\
\Sigma & \xhookrightarrow{} & X & \xrightarrow{\pi} & L
\end{array} \]

\( \tilde{\Sigma} = \) double cover of \( E \) branched along \( \tilde{\Delta} \cap E \).

\( \tilde{\Delta} \cap E \cong_{\text{Zar}} X \times C \Rightarrow \tilde{\Sigma} \cong_{\text{Zar}} X \times \{\text{smooth quadric}\}. \)
\[ \pi : X \to L \text{ branched along } \Delta; \text{ define } \tilde{\pi} : \tilde{X} \xrightarrow{2:1} \tilde{L} \text{ branched along } \tilde{\Delta} \]

(note that \( \tilde{\Delta} \in |b^*\Delta - 2E| \))

\[
\begin{array}{c}
\tilde{\Sigma} \hookrightarrow \tilde{X} \xrightarrow{\tilde{\pi}} \tilde{L} \\
\downarrow \quad \downarrow \sigma \quad \downarrow b \\
\Sigma \hookrightarrow X \xrightarrow{\pi} L
\end{array}
\]

\( \tilde{\Sigma} = \text{double cover of } E \text{ branched along } \tilde{\Delta} \cap E \).

\( \tilde{\Delta} \cap E \cong_{Zar} X \times C \Rightarrow \tilde{\Sigma} \cong_{Zar} X \times \{\text{smooth quadric}\} \).

\( \leadsto \) condition a) for \( \sigma \).
THE END
THE END

Happy birthday, Fabrizio!