The stable Lüroth problem

Arnaud Beauville

Université de Nice

New York, May 2015
The classical Lüroth problem

3 counter-examples (1971), with 3 different methods:

- **Clemens-Griffiths**
  - Example: cubic
  - Method: Hodge theory

- **Iskovskikh-Manin**
  - Example: some quartics
  - Method: Fano's idea

- **Artin-Mumford**
  - Example: specific
  - Method: Algebra (Brauer group)

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The classical Lüroth problem

$X$ complex variety

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The stable Lüroth problem
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$X$ complex variety

$(\mathbb{P}^n \simto X)$

$X$ rational

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$X$ complex variety

\[(\mathbb{P}^n \simrightarrow X) \quad \text{and} \quad (\mathbb{P}^n \twoheadrightarrow X)\]

- $X$ rational
- $X$ unirational

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$X$ complex variety

\[ (\mathbb{P}^n \simrightarrow X) \quad (\mathbb{P}^n \dashrightarrow X) \]

$X$ rational $\quad \rightarrow \quad X$ unirational

---

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The classical Lüroth problem

$X$ complex variety

$\left( \mathbb{P}^n \simarrow X \right) \quad \left( \mathbb{P}^n \dasharrow X \right)$

$X$ rational $\quad \Rightarrow \quad X$ unirational

(Lüroth problem)

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The stable Lüroth problem
The classical Lüroth problem

\[ X \] complex variety

\[ (\mathbb{P}^n \xrightarrow{\sim} X) \quad \text{ (\mathbb{P}^n \dashrightarrow X) } \]

\[ X \text{ rational} \quad \xrightarrow{\sim} \quad X \text{ unirational} \]

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The stable Lüroth problem
The classical Lüroth problem

$\mathcal{X}$ complex variety

$\mathbb{P}^n \dasharrow \mathcal{X}$

$\mathcal{X}$ rational $\iff$ $\mathcal{X}$ unirational

3 counter-examples (1971), with 3 different methods:
The classical Lüroth problem

$X$ complex variety

$\begin{align*}
(\mathbb{P}^n &\sim X) \quad (\mathbb{P}^n &\rightarrow X) \\
X \text{ rational} &\quad \iff \quad X \text{ unirational}
\end{align*}$

3 counter-examples (1971), with 3 different methods:
The classical Lüroth problem

\( \mathbb{P}^n \sim \rightarrow X \) \( \mathbb{P}^n \rightarrow \rightarrow X \)

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The classical Lürroth problem

\[ X \text{ complex variety} \]

\[
\begin{align*}
\mathbb{P}^n \xrightarrow{\sim} X & \quad \text{and} \quad \mathbb{P}^n \dashrightarrow X \\
X \text{ rational} & \quad \iff \quad X \text{ unirational}
\end{align*}
\]

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The classical Lüroth problem

$X$ complex variety

$\mathbb{P}^n \sim \rightarrow X$  \hspace{1cm} \hspace{1cm} \hspace{1cm} $\mathbb{P}^n \dashrightarrow X$

$X$ rational  \hspace{1cm} \hspace{1cm} \hspace{1cm} $\rightarrow$ \hspace{1cm} $\rightarrow$ \hspace{1cm} $\rightarrow$

$X$ unirational

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The classical Lüroth problem

$X$ complex variety

$\left( \mathbb{P}^n \dashrightarrow X \right)$

$X$ rational $\quad \leftrightarrow \quad X$ unirational

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Many examples, at least in dimension 3 (for instance, among Iskovskikh's list of Fano threefolds). Only the Artin-Mumford method gives examples in dimension $\leq 3$, and quite particular.
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many examples, at least \textbf{in dimension 3} (for instance, among Iskovskikh’s list of Fano threefolds).

Only the Artin-Mumford method gives examples in dimension $> 3$, and quite particular.
The stable Lüroth problem

Hence, search for intermediate notion:

$X$ stably rational

if $X \in P^m$ rational for some $m$.

\[ \Downarrow \]

$\Downarrow \rightarrow \uparrow \leftarrow \Uparrow$

\[ \Uparrow \rightarrow \downarrow \rightarrow \leftarrow \Uparrow \]

$X \in [A-M] \in [B-C-S-S]$.

Artin-Mumford: stably rational

$\check{\text{Tors}} H_3 p X, Z q^0$.

Construct quartic double solid $X_{2:1} \check{\text{Ý ÝÝ Ñ}} P^3$ with $\check{\text{Tors}} H_3 p X, Z q^0$:

branched along $\Delta$ quartic symmetroid: defined by $\text{det} p L_{ij} q^0, p L_{ij} q$ symmetric 4 $\hat{\text{4}}$ matrix of linear forms.

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Hence, search for intermediate notion:

\[
\text{rational} \quad \rightarrow \quad \text{unirational} \quad \rightarrow \quad \text{stably rational} \quad \rightarrow \quad \uparrow
\]

\[X \subseteq \text{[A-M]} \quad X \subseteq \text{[B-C-S-S]} \quad \text{Tors} H^3 p, Z q \neq 0 \]

Construct quartic double solid \(X\) of index 2:1 \(\mathbb{P}^3\) with Tors \(H^3 p, Z q \neq 0\):

branched along \(\Delta\) quartic symmetroid: defined by \(\det p_{ij} q \neq 0\), \(p_{ij}\) symmetric 4 \(\times\) 4 matrix of linear forms.
Hence, search for intermediate notion:

\[ X \text{ stably rational if } X \times \mathbb{P}^m \text{ rational for some } m. \]
Hence, search for intermediate notion:

\( X \text{ stably rational} \) if \( X \times \mathbb{P}^m \) rational for some \( m \).

\[
\text{rational} \quad \xrightarrow{\text{unirational}} \quad \text{stably rational}
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Hence, search for intermediate notion:

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\[
\text{rational} \quad \overset{\text{\textbullet}}{\leftrightarrow} \quad \text{unirational} \quad \overset{\text{\textbullet}}{\leftrightarrow} \quad \text{stably rational}
\]

\[ [B-C-S-S] \quad \overset{\text{\textbullet}}{\leftrightarrow} \quad X \quad \overset{\text{\textbullet}}{\leftrightarrow} \quad \text{Artin-Mumford: stably rational} \]

Construct quartic double solid \( X \) :

quotient by \( \mathbb{P}^3 \) with \( \text{Tors} H_3 \),

branched along \( \Delta \) quartic symmetroid: defined by \( \det \mathbf{L} \),

\( \mathbf{L} \) symmetric \( 4 \times 4 \) matrix of linear forms.
The stable Lüroth problem

Hence, search for intermediate notion:

$X$ stably rational if $X \times \mathbb{P}^m$ rational for some $m$.

\[ \text{rational} \rightarrow \text{unirational} \rightarrow \text{stably rational} \rightarrow \text{rational} \]

\[ [B-C-S-S] \leftrightarrow X \leftrightarrow [A-M] \]
The stable Lüroth problem

Hence, search for intermediate notion:

\( X \) stably rational if \( X \times \mathbb{P}^m \) rational for some \( m \).

\[
\begin{array}{ccc}
\text{rational} & \longrightarrow & \text{unirational} \\
\downarrow & & \downarrow \\
\text{stably rational} & \leftarrow & \text{stably rational} \\
\end{array}
\]

\([B-C-S-S]\) \[\longleftrightarrow\] \[X\] \[\longleftrightarrow\] \[A-M]\]

Artin-Mumford: stably rational \[\Longleftrightarrow\] \( \text{Tors } H^3(X, \mathbb{Z}) = 0 \)
Hence, search for intermediate notion:

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\]

Artin-Mumford: stably rational \( \implies \) \( \text{Tors } H^3(X, \mathbb{Z}) = 0 \)

Construct \textit{quartic double solid} \( X \to^{2:1} \mathbb{P}^3 \) with \( \text{Tors } H^3(X, \mathbb{Z}) \neq 0 \):
The stable Lüroth problem

Hence, search for intermediate notion:

\( X \) stably rational if \( X \times \mathbb{P}^m \) rational for some \( m \).

\[
\text{rational} \quad \xrightarrow{\sim} \quad \text{unirational} \\
\Downarrow \Downarrow \Downarrow \Downarrow \quad \Downarrow \Downarrow \Downarrow \Downarrow \quad \Downarrow \Downarrow \Downarrow \Downarrow
\]

\( \text{stably rational} \)

\[
\text{[B-C-S-S]} \quad \longleftrightarrow \quad X \quad \longleftrightarrow \quad [A-M]
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Artin-Mumford: stably rational \( \iff \) Tors \( H^3(X, \mathbb{Z}) = 0 \)

Construct \emph{quartic double solid} \( X \xrightarrow{2:1} \mathbb{P}^3 \) with Tors \( H^3(X, \mathbb{Z}) \neq 0 \):

branched along \( \Delta = \text{quartic symmetroid} \): defined by \( \det(L_{ij}) = 0 \),

\( (L_{ij}) \) symmetric \( 4 \times 4 \) matrix of linear forms.
New results

Theorem (Voisin)
A double covering of $\mathbb{P}^3$ branched along a very general quartic surface is not stably rational.

Very general := outside countable union of strict subvarieties of the moduli space

Known to be unirational, not rational (AB 77, Voisin 86)

Theorem (AB)
A double covering of $\mathbb{P}^4$ or $\mathbb{P}^5$ branched along a very general quartic hypersurface is not stably rational.

Unirational; rationality was not known.

First example of a prime Fano manifold ($b_2^1$) of dimension $\geq 3$, unirational but not rational.
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Theorem (AB)

A double covering of $\mathbb{P}^4$ or $\mathbb{P}^5$ branched along a very general quartic hypersurface is not stably rational.

- Unirational; rationality was not known.
- First example of a prime Fano manifold ($b_2 = 1$) of dimension > 3, unirational but not rational.
Elaborations of Voisin's idea give the non-stable rationality of the very general (in chronological order):

1. Quartic threefold (Colliot-Thélène-Pirutka).
2. Double sextic solid (AB).
3. Hypersurface of degree $e^2 R^n_2 V_{P^n_1}$ (Totaro); in particular, quartic threefold and fourfold.
4. Conic bundle over $P^2$ with discriminant curve of degree $e^6$ (Hassett-Kresch-Tschinkel, with a more general statement).

But: 

'Not expected to be unirational, except some cases in 4.' 

'Already known to be non-rational, except hypersurfaces of degree 2$^{p}$ and 2$^{p-1}$ in $P^3_p$ in 3.' 

'All of dimension 3 except in 3.' 

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The stable Lüroth problem
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Other results

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The stable Lüroth problem
Other results

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- All of dimension 3 except in 3.
The degeneration argument

Idea: degenerate general quartic into symmetroid. But

\[ Tors H_3^p X, Z^q \neq 0 \] for a smooth double cover of \( \mathbb{P}^n \).

Proposition (Voisin, Colliot-Thélène-Pirutka)

\[ X \rightarrow B \] flat projective, \( B \) smooth, general fiber smooth, \( o \mathbb{P}^B \).

Assume \( X \) admits a desingularization \( \sigma: \tilde{X} \rightarrow X \) with:

- \( a \): \( Tors H_3^p \tilde{X}, Z^q \neq 0 \);
- \( b \): \( \sigma \) is \( 1 \)-rational over \( \kappa_x \) for all \( x \in X \).

Then \( X \) is not stably rational for very general \( b \).

For Voisin’s theorem:

\( X \) is a double \( \mathbb{P}^3 \) branched along symmetroid. \( X \) has 10 ordinary double points.

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The stable Lüroth problem
The degeneration argument

Idea: degenerate general quartic into symmetroid.

Proposition (Voisin, Colliot-Thélène-Pirutka)

Assume $X \to B$ flat projective, $B$ smooth, general fiber smooth, $o \to B$.

Then $X$ not stably rational for very general $b$.

For Voisin's theorem: $X$, $double \ P^3$ branched along symmetroid.

$X$ has 10 ordinary double points $\eta$, Artin-Mumford $\eta$.

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Idea: degenerate general quartic into symmetroid.

But: $\text{Tors } H^3(X, \mathbb{Z}) = 0$ for a smooth double cover of $\mathbb{P}^n$. 
The degeneration argument

Idea: degenerate general quartic into symmetroid.

**But**: $\text{Tors } H^3(X, \mathbb{Z}) = 0$ for a smooth double cover of $\mathbb{P}^n$.

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**Proposition (Voisin, Colliot-Thélène-Pirutka)**

$\mathcal{X} \to B$ flat projective, $B$ smooth, general fiber smooth, $o \in B$.
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Idea: degenerate general quartic into symmetroid.

**But**: \( \text{Tors } H^3(X, \mathbb{Z}) = 0 \) for a smooth double cover of \( \mathbb{P}^n \).

---

**Proposition (Voisin, Colliot-Thélène-Pirutka)**

\( \mathcal{X} \rightarrow B \) flat projective, \( B \) smooth, general fiber smooth, \( \circ \in B \).

Assume \( X := \mathcal{X}_\circ \) admits a desingularization \( \sigma : \tilde{X} \rightarrow X \) with:

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Idea: degenerate general quartic into symmetroid.

**But**: \( \text{Tors } H^3(X, \mathbb{Z}) = 0 \) for a smooth double cover of \( \mathbb{P}^n \).

**Proposition (Voisin, Colliot-Thélène-Pirutka)**

\( \mathcal{X} \to B \) flat projective, \( B \) smooth, general fiber smooth, \( o \in B \).

Assume \( X := \mathcal{X}_o \) admits a desingularization \( \sigma : \tilde{X} \to X \) with :

a) \( \text{Tors } H^3(\tilde{X}, \mathbb{Z}) \neq 0 \);
The degeneration argument

Idea: degenerate general quartic into symmetroid.

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a) $\text{Tors } H^3(\tilde{X}, \mathbb{Z}) \neq 0$;

b) $\sigma^{-1}(x)$ rational over $\kappa(x)$ for all $x \in X$. 

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The degeneration argument

Idea: degenerate general quartic into symmetroid.

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Assume \( \mathcal{X} := \mathcal{X}_o \) admits a desingularization \( \sigma : \tilde{\mathcal{X}} \to \mathcal{X} \) with :

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**Proposition (Voisin, Colliot-Thélène-Pirutka)**

$\mathcal{X} \to B$ flat projective, $B$ smooth, general fiber smooth, $b \in B$.

Assume $X := \mathcal{X}_b$ admits a desingularization $\sigma : \tilde{X} \to X$ with :

a) $\text{Tors } H^3(\tilde{X}, \mathbb{Z}) \neq 0$;

b) $\sigma^{-1}(x)$ rational over $\kappa(x)$ for all $x \in X$.

Then $\mathcal{X}_b$ not stably rational for very general $b$.

For Voisin’s theorem: $X = \text{double } \mathbb{P}^3$ branched along symmetroid.
The degeneration argument

Idea: degenerate general quartic into symmetroid.

But: $\text{Tors } H^3(X, \mathbb{Z}) = 0$ for a smooth double cover of $\mathbb{P}^n$.

Proposition (Voisin, Colliot-Thélène-Pirutka)

$X \rightarrow B$ flat projective, $B$ smooth, general fiber smooth, $\circ \in B$.

Assume $X := X_\circ$ admits a desingularization $\sigma : \tilde{X} \rightarrow X$ with:

a) $\text{Tors } H^3(\tilde{X}, \mathbb{Z}) \neq 0$;

b) $\sigma^{-1}(x)$ rational over $\kappa(x)$ for all $x \in X$.

Then $X_b$ not stably rational for very general $b$.

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The degeneration argument

Idea: degenerate general quartic into symmetroid.

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**Proposition (Voisin, Colliot-Thélène-Pirutka)**

$\mathcal{X} \to B$ flat projective, $B$ smooth, general fiber smooth, $o \in B$.

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For Voisin’s theorem: $X = \text{double } \mathbb{P}^3$ branched along symmetroid. $X$ has 10 ordinary double points $\Rightarrow b)$, Artin-Mumford $\Rightarrow a)$. ■
How to prove the degeneration argument?

Idea: use \( CH_0 \) of 0-cycles on \( X \).

Proposition (Colliot-Thélène)

\[ CH_0(X) \rightarrow \text{K-thy for all extensions } C \rightarrow K \]

\[ \Delta \rightarrow X^{\tilde{t}} \]

Supp \( Z \subset D^{\tilde{X}} \)

If this holds, we say that \( X \) is \( CH_0 \)-trivial.

This is relevant because of:

Proposition

\[ X \text{ stably rational } \Leftrightarrow X \text{ CH}_0 \text{-trivial} \]

\[ \text{Tors} H^3(X, \mathbb{Z}) = 0. \]

Proof:

a) \( CH_0 \) birational invariant, and

\[ CH_0(X^{\tilde{t}}) = CH_0(X). \]

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How to prove the degeneration argument?

Idea: use $CH_0(X) = \text{Chow group of 0-cycles on } X = \mathbb{Z}(X)/\sim_{\text{rat}}$.
How to prove the degeneration argument?

Idea: use $\text{CH}_0(X) = \text{Chow group of 0-cycles on } X = \mathbb{Z}(X)/\sim_{\text{rat}}$.

**Proposition (Colliot-Thélène)**

$X$ smooth projective of dimension $n$. 

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Proposition (Colliot-Thélène)

$X$ smooth projective of dimension $n$.

(i) $CH_0(X_K) = \mathbb{Z}$ for all extensions $\mathbb{C} \to K$.
How to prove the degeneration argument?

Idea: use $CH_0(X) = \text{Chow group of 0-cycles on } X = \mathbb{Z}(X) / \sim_{\text{rat}}$.

**Proposition (Colliot-Thélène)**

$X$ smooth projective of dimension $n$.

$\Leftrightarrow$

(i) $CH_0(X_K) = \mathbb{Z}$ for all extensions $\mathbb{C} \to K$;

(ii) $\Delta = X \times \{p\} + Z$ in $CH^n(X \times X)$, $\text{Supp}(Z) \subset D \times X$.
How to prove the degeneration argument?

Idea: use $\text{CH}_0(X) = \text{Chow group of 0-cycles on } X = \mathbb{Z}(X)/\sim_{\text{rat}}$.

**Proposition (Colliot-Thélène)**

$X$ smooth projective of dimension $n$.

\begin{align*}
(i) \quad & \text{CH}_0(X_K) = \mathbb{Z} \quad \text{for all extensions } \mathbb{C} \to K ; \\
(ii) \quad & \Delta = X \times \{p\} + Z \text{ in } \text{CH}^n(X \times X), \text{Supp}(Z) \subset D \times X
\end{align*}

If this holds, we say that $X$ is $\text{CH}_0$-trivial.
How to prove the degeneration argument?

Idea: use \( CH_0(X) = \text{Chow group of 0-cycles on } X = \mathbb{Z}(X)/\sim_{\text{rat}} \).

**Proposition (Colliot-Thélène)**

\( X \) smooth projective of dimension \( n \).

- (i) \( CH_0(X_K) = \mathbb{Z} \) for all extensions \( \mathbb{C} \to K \);
- (ii) \( \Delta = X \times \{p\} + Z \) in \( CH^n(X \times X) \), \( \text{Supp}(Z) \subset D \times X \)

If this holds, we say that \( X \) is \( CH_0\)-trivial.

This is relevant because of:
How to prove the degeneration argument?

Idea: use $CH_0(X) = \text{Chow group of 0-cycles on } X = \mathbb{Z}^n(X)/\sim_{\text{rat}}$.

Proposition (Colliot-Thélène)

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(i) & \quad CH_0(X_K) = \mathbb{Z} \quad \text{for all extensions } \mathbb{C} \to K; \\
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\end{align*}

If this holds, we say that $X$ is $CH_0$-trivial.

This is relevant because of:

Proposition

$X$ stably rational $\implies$ $X$ $CH_0$-trivial $\implies$ $\text{Tors } H^3(X, \mathbb{Z}) = 0.$
How to prove the degeneration argument?

Idea: use $CH_0(X) = \text{Chow group of 0-cycles on } X = \mathbb{Z}(X)/\sim_{\text{rat}}$.

**Proposition (Colliot-Thélène)**

$X$ smooth projective of dimension $n$.

\(\begin{align*}
(i) & \quad CH_0(X_K) = \mathbb{Z} \quad \text{for all extensions } \mathbb{C} \to K; \\
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\end{align*}\)

If this holds, we say that $X$ is $CH_0$-trivial.

This is relevant because of:

**Proposition**

$X$ stably rational $\implies X$ $CH_0$-trivial $\implies$ Tors $H^3(X, \mathbb{Z}) = 0$.

**Proof:** a) $CH_0$ birational invariant, and $CH_0(X \times \mathbb{P}^m) = CH_0(X)$. 

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$CH_0$ trivial $\implies H^3(X,\mathbb{Z}) = 0$
$CH_0$ trivial $\Rightarrow H^3(X, \mathbb{Z}) = 0$

b) $X$ $CH_0$-trivial : $\Delta \sim_{\text{rat}} X \times \{p\} + Z$, $\text{Supp}(Z) \subset D \times X$;
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b) $X$ $\text{CH}_0$-trivial: $\Delta \sim_{\text{rat}} X \times \{p\} + Z$, $\text{Supp}(Z) \subset D \times X$; assume $D$ smooth for simplicity.
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b) $X$ $\text{CH}_0$-trivial: $\Delta \sim_{\text{rat}} X \times \{p\} + Z$, $\text{Supp}(Z) \subset D \times X$; assume $D$ smooth for simplicity.

For $\mathfrak{z} \in CH^n(X \times X)$, endomorphism $\mathfrak{z}^*: \alpha \mapsto p_*(q^* \alpha \cdot \mathfrak{z})$ of $H^r(X, \mathbb{Z})$. 
$CH_0$ trivial $\Rightarrow H^3(X, \mathbb{Z}) = 0$

b) $X$ $CH_0$-trivial: $\Delta \sim_{\text{rat}} X \times \{p\} + Z$, $\text{Supp}(Z) \subset D \times X$; assume $D$ smooth for simplicity.

\[
\begin{array}{ccc}
X \times X & \xrightarrow{p} & X \\
\downarrow & & \downarrow \\
X & \xrightarrow{q} & X
\end{array}
\]

For $\mathcal{Z} \in CH^n(X \times X)$, endomorphism $\mathcal{Z}^* : \alpha \mapsto p_*(q^* \alpha \cdot \mathcal{Z})$ of $H^r(X, \mathbb{Z})$.

$\Delta^* = \text{Id}$, $[X \times \{p\}]^* = 0$ for $r \neq 0$, and
b) $X$ $CH_0$-trivial: $\Delta \sim_{\text{rat}} X \times \{p\} + Z$, $\text{Supp}(Z) \subset D \times X$; assume $D$ smooth for simplicity.

For $\beta \in CH^n(X \times X)$, endomorphism $\beta^*: \alpha \mapsto p_*(q^*\alpha \cdot \beta)$ of $H^r(X, \mathbb{Z})$.

$\Delta^* = \text{Id}$, $[X \times \{p\}]^* = 0$ for $r \neq 0$, and

$\text{Id} = Z^*: H^r(X, \mathbb{Z}) \to H^{r-2}(D, \mathbb{Z}) \xrightarrow{i_*} H^r(X, \mathbb{Z})$. 
\( CH_0 \text{ trivial } \Rightarrow H^3(X, \mathbb{Z}) = 0 \)

b) \( X \) \( CH_0 \)-trivial : \( \Delta \sim_{\text{rat}} X \times \{p\} + Z \), \( \text{Supp}(Z) \subset D \times X \); assume \( D \) smooth for simplicity.

For \( \delta \in CH^n(X \times X) \), endomorphism \( \delta^* : \alpha \mapsto p_*(q^* \alpha \cdot \delta) \) of \( H^r(X, \mathbb{Z}) \).

\[ \Delta^* = \text{Id}, \ [X \times \{p\}]^* = 0 \text{ for } r \neq 0, \text{ and} \]

\[ \text{Id} = Z^* : H^r(X, \mathbb{Z}) \to H^{r-2}(D, \mathbb{Z}) \xrightarrow{i_*} H^r(X, \mathbb{Z}). \]

For \( r = 3 \), \( H^1(D, \mathbb{Z}) \) torsion free \( \Rightarrow \) \( \blacksquare \).
Idea of proof of the degeneration argument

Proposition $X$ is flat projective, $B$ is smooth, the general fiber is smooth, $\sigma_p$.

Assume $X$ admits a desingularization $\sigma: \tilde{X} \to X$ with:

- $a_q \cdot H_3(\tilde{X}), Z_q \not\equiv 0$;
- $\sigma \cdot 1_p$ is rational over $\kappa_p$ for all $x \in X$.

Then $X$ is not stably rational for very general $b$.

Ingredients:
- If not, Baire $\eta$ the generic fiber $X_\eta$ is $CH_0$-trivial.
- Specialization homomorphism $s: CH_0(\tilde{X}), \sigma \to CH_0(X)$.
- Replace $B$ by $\text{Spec} \hat{O}_B$, $\sigma \cdot \text{CH}_0(\tilde{X})$.
- Hensel lemma $\sigma \cdot \text{CH}_0(\tilde{X}) = Z_q$.
- $b_q \cdot \tilde{X}$ is $CH_0$-trivial $\iff$ $Tors H_3(\tilde{X}), Z_q \equiv 0$.

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The stable Lüroth problem
Proposition

\( \mathcal{X} \to B \) flat projective, \( B \) smooth, general fiber smooth, \( o \in B \).

Assume \( X := \mathcal{X}_o \) admits a desingularization \( \sigma : \tilde{X} \to X \) with:

1. \( \text{Tors } H^3(\tilde{X}, \mathbb{Z}) \neq 0 \);
2. \( \sigma^{-1}(x) \) rational over \( \kappa(x) \) for all \( x \in X \).

Then \( \mathcal{X}_b \) not stably rational for very general \( b \).
Idea of proof of the degeneration argument

**Proposition**

$\mathcal{X} \rightarrow B$ flat projective, $B$ smooth, general fiber smooth, $\mathfrak{o} \in B$.

Assume $X := \mathcal{X}_\mathfrak{o}$ admits a desingularization $\sigma : \tilde{X} \rightarrow X$ with:

a) $\text{Tors } H^3(\tilde{X}, \mathbb{Z}) \neq 0$;

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Then $\mathcal{X}_b$ not stably rational for very general $b$.

**Ingredients**: If not, $\xrightarrow{\text{Baire}}$ the generic fiber $\mathcal{X}_\eta$ is $\text{CH}_0$-trivial.
Idea of proof of the degeneration argument

**Proposition**

\( \mathcal{X} \to B \) flat projective, \( B \) smooth, general fiber smooth, \( \circ \in B \).

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Then \( \mathcal{X}_b \) not stably rational for very general \( b \).

**Ingredients:** If not, \( \overset{\text{Baire}}{\implies} \) the generic fiber \( X_\eta \) is \( CH_0 \)-trivial.

Specialization homomorphism \( s : CH_0(X_\eta) \to CH_0(X) \).
Idea of proof of the degeneration argument

**Proposition**

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**Ingredients**: If not, $\text{Baire}$ the generic fiber $\mathcal{X}_\eta$ is $CH_0$-trivial.

Specialization homomorphism $s : CH_0(\mathcal{X}_\eta) \to CH_0(X)$.

Replace $B$ by $\text{Spec } (\hat{O}_{B,o})$. Hensel lemma $\Rightarrow \sigma_* CH_0(\tilde{X}) = \mathbb{Z}$. 

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The stable Lüroth problem
**Idea of proof of the degeneration argument**

**Proposition**

$X \rightarrow B$ flat projective, $B$ smooth, general fiber smooth, $o \in B$.

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Replace $B$ by $\text{Spec}(\hat{O}_{B,o})$. Hensel lemma $\Rightarrow \sigma_* CH_0(\tilde{X}) = \mathbb{Z}$.

$\sigma^{-1}(x)$ rational over $\kappa(x)$ for all $x \in X$.

$\Rightarrow \tilde{X}$ $CH_0$-trivial $\Rightarrow \text{Tors } H^3(\tilde{X}, \mathbb{Z}) = 0$. 

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The stable Lüroth problem
The Brauer group

Thus to prove the theorem, we need a double covering $X \to \mathbb{P}^n(n \geq 3, 4, 5)$ with a desingularization $\tilde{X} \to X$ satisfying $a_q$ and $b_q$.

How to find torsion elements in $H^3_p, \mathbb{Z}^q$?

Observation

For $V$ smooth projective with $H^2_p, O^p \not\sim 0$, $Tors H^3_p, \mathbb{Z}^q = Tors H^2_p, O^\ast$: $Br_p, O^\ast$ (Brauer group)

where $O^\ast$: sheaf of holomorphic functions on $V$.

Proof: The exponential exact sequence gives an exact sequence $H^2_p, O_q \not\sim 0 \to H^2_p, O^\ast \to H^3_p, \mathbb{Z}^q \to H^3_p, O_q$.
Thus to prove the theorem, we need a double covering $X \to \mathbb{P}^n$.
Thus to prove the theorem, we need a double covering $X \to \mathbb{P}^n$ ($n = 3, 4, 5$) with a desingularization $\tilde{X} \to X$ satisfying $a)$ and $b)$. 

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The stable Lüroth problem
Thus to prove the theorem, we need a double covering $X \to \mathbb{P}^n$ ($n = 3, 4, 5$) with a desingularization $\tilde{X} \to X$ satisfying a) and b). How to find torsion elements in $H^3(\mathbb{P}^n, \mathbb{Z})$?
The Brauer group

Thus to prove the theorem, we need a double covering $X \to \mathbb{P}^n$ $(n = 3, 4, 5)$ with a desingularization $\tilde{X} \to X$ satisfying $a)$ and $b)$. How to find torsion elements in $H^3(\cdot, \mathbb{Z})$?

Observation

For $V$ smooth projective with $H^2(V, \mathcal{O}_V) = 0$,
Thus to prove the theorem, we need a double covering \( X \to \mathbb{P}^n \) (\( n = 3, 4, 5 \)) with a desingularization \( \tilde{X} \to X \) satisfying \( a \) and \( b \). How to find torsion elements in \( H^3(\cdot, \mathbb{Z}) \)?

**Observation**

For \( V \) smooth projective with \( H^2(V, \mathcal{O}_V) = 0 \),

\[
\text{Tors } H^3(V, \mathbb{Z}) = \text{Tors } H^2(V, \mathcal{O}_h^*) := \text{Br}(V) \quad \text{(Brauer group)}
\]

where \( \mathcal{O}_h := \) sheaf of holomorphic functions on \( V \).
Thus to prove the theorem, we need a double covering $X \to \mathbb{P}^n$ ($n = 3, 4, 5$) with a desingularization $\tilde{X} \to X$ satisfying a) and b).

How to find torsion elements in $H^3(\cdot, \mathbb{Z})$?

**Observation**

For $V$ smooth projective with $H^2(V, \mathcal{O}_V) = 0$,

$$\text{Tors } H^3(V, \mathbb{Z}) = \text{Tors } H^2(V, \mathcal{O}_h^*) := \text{Br}(V)$$

(Brauer group)

where $\mathcal{O}_h :=$ sheaf of holomorphic functions on $V$.

**Proof** : The exponential exact sequence gives an exact sequence

$$H^2(V, \mathcal{O}_h) = 0 \longrightarrow H^2(V, \mathcal{O}_h^*) \longrightarrow H^3(V, \mathbb{Z}) \longrightarrow H^3(V, \mathcal{O}_h).$$
One way to get classes in the Brauer group $\text{Br}_V$ is to consider $\mathbb{P}^m$-bundles on $V$, that is, smooth fibrations $\mathbb{P}^m \to V$ with fibers isomorphic to $\mathbb{P}^m$ (a.k.a. Severi-Brauer schemes over $V$).

Reminder: A $\mathbb{P}^n$-bundle $\mathbb{P} \to V$ defines a $n$-torsion class $r_{\mathbb{P}} \in \text{Br}_V$, which is trivial if and only if $\mathbb{P}$ is a projective bundle $\mathbb{P}_V^n \to \mathbb{P}^n$.

The exact sequence $1 \to \mathbb{C}^* \to \text{GL}_n \to \text{PGL}_n \to 1$ gives a cohomology exact sequence $H^1(V, \text{GL}_n) \to H^1(V, \text{PGL}_n) \to H^2(V, \mathbb{C}^*)$. 

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One way to get classes in the Brauer group $\text{Br}(V)$ is to consider $\mathbb{P}^m$-bundles on $V$,
One way to get classes in the Brauer group $\text{Br}(V)$ is to consider $\mathbb{P}^m$-bundles on $V$, that is, smooth fibrations $P \to V$ with fibers isomorphic to $\mathbb{P}^m$ (a.k.a. Severi-Brauer schemes over $V$).
One way to get classes in the **Brauer group** \( Br(V) \) is to consider \( \mathbb{P}^m \)-bundles on \( V \), that is, smooth fibrations \( P \to V \) with fibers isomorphic to \( \mathbb{P}^m \) (a.k.a. **Severi-Brauer** schemes over \( V \)).

**Reminder**

A \( \mathbb{P}^{n-1} \)-bundle \( P \to V \) defines a \( n \)-torsion class \([P] \in Br(V)\),
One way to get classes in the Brauer group $\text{Br}(V)$ is to consider $\mathbb{P}^m$-bundles on $V$, that is, smooth fibrations $P \to V$ with fibers isomorphic to $\mathbb{P}^m$ (a.k.a. Severi-Brauer schemes over $V$).

**Reminder**

A $\mathbb{P}^{n-1}$-bundle $P \to V$ defines a $n$-torsion class $[P] \in \text{Br}(V)$, which is trivial if and only if $P$ is a projective bundle $\mathbb{P}_V(E)$. 
One way to get classes in the **Brauer group** $\text{Br}(V)$ is to consider $\mathbb{P}^m$-bundles on $V$, that is, smooth fibrations $P \to V$ with fibers isomorphic to $\mathbb{P}^m$ (a.k.a. Severi-Brauer schemes over $V$).

**Reminder**

A $\mathbb{P}^{n-1}$-bundle $P \to V$ defines a $n$-torsion class $[P] \in \text{Br}(V)$, which is trivial if and only if $P$ is a projective bundle $\mathbb{P}_V(E)$.

\[
\left( \text{the exact sequence } 1 \to \mathbb{C}^* \to \text{GL}_n(\mathbb{C}) \to \text{PGL}_n(\mathbb{C}) \to 1 \text{ gives a cohomology exact sequence} \right.
\]

\[
H^1(V, \text{GL}_n(\mathcal{O}_h)) \to H^1(V, \text{PGL}_n(\mathcal{O}_h)) \to H^2(V, \mathcal{O}_h^*)
\]
The construction

...
The construction

$L = \text{general linear system of quadrics in } \mathbb{P}^3 \text{ of dimension } n \ (n \leq 9)$. 

Proposition

The $\mathbb{P}^1$-bundle $p: \mathbb{P} \to X_{\text{sm}}$ is not a projective bundle.

Proof: Suppose $p$ has a rational section \( \sigma \), \( \pi_q : \mathbb{P} \to L \setminus Q \).

For general $q$, 2 systems $\sigma, \sigma_1 \to \mathbb{P}$, parametrized by $\mathbb{P}^1$.

Get rational section of quadric family $Q \to L$. 

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The stable Lüroth problem
The construction

$L = \text{general linear system of quadrics in } \mathbb{P}^3 \text{ of dimension } n \ (n \leq 9).$

\[ \Sigma \subset \Delta \subset L \cong \mathbb{P}^n \]

\[ \text{rk} \leq 2 \quad \text{rk} \leq 3 \]
The construction

$L = \text{general linear system of quadrics in } \mathbb{P}^3 \text{ of dimension } n \ (n \leq 9)$. \\
$\Sigma \subset \Delta \subset L \cong \mathbb{P}^n$ \\
rk $\leq 2$ \quad \text{rk} \leq 3 \\

$\Delta$ quartic hypersurface, $\text{Sing}(\Delta) = \Sigma$, $\text{dim}(\Sigma) = n - 3$. \\

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The stable Lüroth problem
$L =$ general linear system of quadrics in $\mathbb{P}^3$ of dimension $n$ ($n \leq 9$).

$$\Sigma \subset \Delta \subset L \cong \mathbb{P}^n$$

$\text{rk} \leq 2$ \hspace{1cm} $\text{rk} \leq 3$

$\Delta$ quartic hypersurface, $\text{Sing}(\Delta) = \Sigma$, $\text{dim}(\Sigma) = n - 3$.

$\pi : X \to L :=$ double cover branched along $\Delta$. $X_{sm} = \pi^{-1}(L \setminus \Sigma)$
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\[ = \{(q, \sigma) \mid q \in L, \text{rk}(q) = 3 \text{ or } 4, \sigma = \text{system of generatrices of } q\}.\]
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$\sigma$ parametrized by $\mathbb{P}^1 \rightsquigarrow \mathbb{P}^1$-bundle $p : P \to X_{sm}$.
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Proposition

The $\mathbb{P}^1$-bundle $p : P \to X_{sm}$ is not a projective bundle.
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**Proposition**

The \( \mathbb{P}^1\text{-bundle } p : P \to X_{sm} \) is not a projective bundle.

**Proof**: Suppose \( p \) has a rational section : \( (q, \sigma) \dashrightarrow \mathcal{L}(q, \sigma) \in \sigma. \)

For general \( q, 2 \) systems \( \sigma, \sigma' \hookrightarrow \mathcal{L}(q, \sigma) \cap \mathcal{L}(q, \sigma') = s(q) \in q. \)
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Get rational section of quadric family \( Q \to L. \)
Universal family of hypersurfaces

Lemma
$L \mid O_{P^n}$

The universal family $p : H \to L$ has no rational section.

Proof: rational section $u \in L_{P^n} \to H$

By Lefschetz, $H_{2n-2} p \mid \hat{P^n}$

Easy calculation gives $p \cdot \alpha_{H_{2n-2} H, \hat{P^n}} \in \alpha_{Z, \hat{P^n}}$, contradiction.

Conclusion: get a nontrivial 2-torsion class in $H_3 p \mid X_{sm}, Z$.
Lemma

\[ L \subseteq |O_{\mathbb{P}^n}(d)| \text{ base point free linear system, } d \geq 2. \]
Lemma

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The universal family $p : \mathcal{H} \to L$ has no rational section.

\[ \mathcal{H} := \{(H, x) \in L \times \mathbb{P}^n \mid x \in H\} \]
Lemma

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smooth hypersurface in $L \times \mathbb{P}^n$. 

\[ \begin{xy}
  0;/r1cm/: 0;/r2cm/: *++[F.](H) / (p) / (q) / \mathbb{P}^n \ar@{->}[dl] \ar@{->}[dr] \\
  L \ar@{->}[r] & \mathcal{H} 
\end{xy} \]
Lemma

Let \( L \subset |\mathcal{O}_{\mathbb{P}^n}(d)| \) be a base point free linear system, \( d \geq 2 \).

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Proof: rational section \( z \in H^{2n-2}(\mathcal{H}, \mathbb{Z}) \) with \( p_*z = 1 \).
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By Lefschetz, \( H^{2n-2}(L \times \mathbb{P}^n, \mathbb{Z}) \sim H^{2n-2}(\mathcal{H}, \mathbb{Z}) \).
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**Lemma**

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\[ \square \]
Lemma

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By Lefschetz, $H^{2n-2}(L \times \mathbb{P}^n, \mathbb{Z}) \xrightarrow{\sim} H^{2n-2}(\mathcal{H}, \mathbb{Z})$. Easy calculation gives $p_*H^{2n-2}(\mathcal{H}, \mathbb{Z}) \subset d\mathbb{Z}$, contradiction.

**Conclusion**: get a nontrivial 2-torsion class in $H^3(X_{sm}, \mathbb{Z})$. 

Arnaud Beauville | The stable Lüroth problem
The desingularization

Recall: $\Sigma \Delta \Lambda L - P_{n^{rk}} \geq 2 \text{rk} - 3$

...quartic hypersurface, \text{Sing} \notin \Sigma, \text{dim} \notin \Sigma^q \leq 3.

1) Baby case: $n = 3$ surface, $\Sigma^t P_1, \ldots, P_{10}$ ordinary double points: near $p_i$, $\Delta - \text{loc} Q$ quadratic cone $x^2`y^2`z^2 = 0$ in $\mathbb{C}^3$.

\text{Sing} \notin X^{q^t P_1, \ldots, P_{10}} \text{ordinary double points}

\text{Blow up} \notin p_i \sigma \notseq \tilde{X} \xrightarrow{} X \text{smooth}, \sigma \leq 1 p_i^q \text{smooth quadric}.

Need to check $a^q$ and $b^q$:

$b^q$ immediate; for $a^q$, exact sequence $0 \notseq H^1 p_Qi, \mathcal{Z}^q \notseq H^3 p_{\tilde{X}}, \mathcal{Z}^q \notseq H^3 p_X, \mathcal{Z}^q \notseq H^2 p_Qi, \mathcal{Z}^q \notseq H^2 p_X$.
Recall: $\Sigma \subset \Delta \subset L \cong \mathbb{P}^n$

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Blow up \( p_i \) \( \leadsto \sigma: \tilde{X} \to X \) smooth, \( \sigma^{-1}(p_i) = Q_i \) smooth quadric.
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Need to check $a$) and $b$): $b$) immediate; for $a$), exact sequence

$$0 = \bigoplus H^1(Q_i, \mathbb{Z}) \to H^3(\tilde{X}, \mathbb{Z}) \to H^3(X_{sm}, \mathbb{Z}) \to \bigoplus H^2(Q_i, \mathbb{Z})$$
Higher dimension

Arnaud Beauville  The stable Lüroth problem
2) \( n = 4 \) or \( 5 \)
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\( \Sigma \) smooth; near \( q \in \Sigma, \; \Delta \cong_{loc} \Sigma \times Q \) (\( Q \) quadratic cone in \( \mathbb{C}^3 \)).
2) \( n = 4 \text{ or } 5 \)

\( \Sigma \) smooth; near \( q \in \Sigma \), \( \Delta \cong_{loc} \Sigma \times \mathcal{Q} \) (\( \mathcal{Q} \) quadratic cone in \( \mathbb{C}^3 \)).

Blow up \( \Sigma \) in \( X \):

\[
\begin{array}{ccc}
E & \xleftarrow{\sigma_E} & \tilde{X} \\
\downarrow & & \downarrow \\
\Sigma & \xleftarrow{\sigma} & X
\end{array}
\]

\( \tilde{X} \) and \( E \) smooth, \( \sigma_E : E \to \Sigma \) quadric bundle.
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**Key point** for \( a \) and \( b \): the fibration \( \sigma_E \) is *Zariski locally trivial*. 
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Blow up \( \Sigma \) in \( X \):

\[ \begin{array}{ccc}
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\Sigma & \subset & X
\end{array} \]

\( \tilde{X} \) and \( E \) smooth, \( \sigma_E : E \rightarrow \Sigma \) quadric bundle.

**Key point** for a) and b): the fibration \( \sigma_E \) is Zariski locally trivial. (follows from: the projective normal cone to \( \Sigma \) in \( \Delta \) is Zariski locally trivial over \( \Sigma \).)
Higher dimension

2) $n = 4$ or $5$

$\Sigma$ smooth; near $q \in \Sigma$, $\Delta \cong_{loc} \Sigma \times \mathcal{Q}$ ($\mathcal{Q}$ quadratic cone in $\mathbb{C}^3$).

Blow up $\Sigma$ in $X$:

$$
\begin{array}{c}
E \leftarrow \tilde{X} \\
\downarrow \sigma_E \quad \downarrow \sigma \\
\Sigma \leftarrow X
\end{array}
$$

$\tilde{X}$ and $E$ smooth, $\sigma_E : E \rightarrow \Sigma$ quadric bundle.

**Key point** for $a)$ and $b)$: the fibration $\sigma_E$ is *Zariski locally trivial*. (follows from: the projective normal cone to $\Sigma$ in $\Delta$ is Zariski locally trivial over $\Sigma$.) $b)$ follows.
Tors $H^3(\tilde{X}, \mathbb{Z})$
For $a)$, use factorization

\[ E \xleftarrow{\sigma_E} \tilde{X} \xrightarrow{\sigma} X' \]

\[ E' \xleftarrow{\sigma} \tilde{X} \xrightarrow{\sigma} X' \]

\[ \Sigma \xleftarrow{\sigma} \tilde{X} \xrightarrow{\sigma} X' \]

with $E'$ smooth of codimension 2 in $X'$, $\tilde{X} = \text{Bl}_{E'}(X')$. 

Tors $H^3(\tilde{X}, \mathbb{Z})$
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E' & \longrightarrow & X'
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\[
0 \rightarrow H^3(X', \mathbb{Z}) \rightarrow H^3(X_{sm}, \mathbb{Z}) \rightarrow H^0(E', \mathbb{Z})
\]
For $a$), use factorization

$$
\begin{array}{ccc}
E & \xrightarrow{\sigma_E} & \tilde{X} \\
\downarrow & & \downarrow \\
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$$0 \rightarrow H^3(X', \mathbb{Z}) \rightarrow H^3(X_{sm}, \mathbb{Z}) \rightarrow H^0(E', \mathbb{Z})$$

and $\text{Tors } H^3(X', \mathbb{Z}) \cong \text{Tors } H^3(\tilde{X}, \mathbb{Z})$. 

THE END
Happy birthday, Bob!