V. Further developments

Arnaud Beauville

Université de Nice

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Proposition (Bogomolov-Verbitsky)

X hyperkähler, $A$ sub-algebra of $H^\ast(X, \mathbb{Q})$ spanned by $H^2(X, \mathbb{Q})$.

Then $A$ satisfies Poincaré duality; $H^\ast(X, \mathbb{Q}) = A \oplus A^\perp$;

$A = S^\ast H^2(X, \mathbb{Q}) / J$ with $J = \langle x^r+1 | x \in H^2(X, \mathbb{Q}), q(x) = 0 \rangle$

Corollary

$Sp H^2(X, \mathbb{Q}) \to H^2_p(X, \mathbb{Q})$ injective for $p \leq r$.

Arnaud Beauville

V. Further developments
Proposition (Bogomolov-Verbitsky)

Let $X$ be hyperkähler, $A$ a sub-algebra of $H^*(X, \mathbb{Q})$ spanned by $H^2(X, \mathbb{Q})$.

Then $A$ satisfies Poincaré duality; $H^*(X, \mathbb{Q}) = A \oplus A^\perp$;

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Corollary

$$S^p H^2(X, \mathbb{Q}) \rightarrow H^{2p}(X, \mathbb{Q}) \text{ injective for } p \leq r.$$
Proof.

1. Geometric input:

   \[ q(\alpha) = 0 \implies \alpha r + 1 = 0; \exists \omega \in H_2(X, Q), \omega_2 \neq 0. \]

2. Put \( H = H_2(X, Q), B = S^*H/J \). Then \( S^*H \to H^*(X, Q) \) maps \( J \) to 0, hence factors as \( \lambda: B \twoheadrightarrow A \), with \( \lambda(B_2r) \neq 0 \).

3. Representation theory of \( O(H, q) \) \( \Rightarrow \) \( B \) Gorenstein, i.e. \( B_p \times B_2r - p \to B_2r = Q \) perfect \( \forall p \).

4. If \( \ker \lambda \neq 0 \), contains \( B_2r \), contradiction.

Remark: \( A \) depends only on \((H, q)\) and \( r \).
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Arnaud Beauville  
V. Further developments
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Remark : \( A \) depends only on \( (H, q) \) and \( r. \)
Lagrangian fibrations

$X$ hyperkähler, $\dim X = 2r$. **Lagrangian fibration**:

$f : X \to B$ with connected fibres, $B$ Kähler of dimension $r$, smooth fibres Lagrangian (i.e. $\sigma|_{X_b} = 0$).

 Proposition (Arnold-Liouville)

The smooth fibres of $f$ are complex tori.

Proof.  

$$0 \to \mathcal{T} X \to \mathcal{T} X \to f^* \mathcal{T} B \to f^* \Omega^1 B \to \Omega^1 X \to \Omega^1 X \to 0 \implies \Omega^1 X_b \cong O_{X_b} \implies X_b \text{ complex torus.}$$
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Proof.

\[
\begin{array}{cccccc}
0 & \longrightarrow & T_{X/B} & \longrightarrow & T_X & \longrightarrow & f^* T_B & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & f^* \Omega^1_B & \longrightarrow & \Omega^1_X & \longrightarrow & \Omega^1_{X/B} & \longrightarrow & 0 \\
\end{array}
\]

$\Rightarrow \quad \Omega^1_{X_b} \cong \mathcal{O}^r_{X_b} \Rightarrow X_b$ complex torus.
Remark: Lagrangian fibrations correspond to completely integrable hamiltonian system in symplectic geometry.
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Theorem (Matsushita + Hwang)

$X$ hyperkähler, $B$ Kähler with $0 < \dim B < 2r$, $f : X \to B$ with connected fibers. Then:

1. $f$ is a Lagrangian fibration;
2. $B$ Fano with $b_2 = 1$ (and $\dim B = r$);
3. If $X$ projective, $B \sim = P^r$. 

Arnaud Beauville

V. Further developments
**Remark**: Lagrangian fibrations correspond to **completely integrable hamiltonian system** in symplectic geometry.

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3. If \( X \) projective, \( B \cong \mathbb{P}^r \).
Proof.

For $\alpha \in H^2(B, C)$, $\alpha^2 r = 0 \Rightarrow (f^* \alpha)^2 r = 0 \Rightarrow (f^* \alpha)^{r+1} = 0 \Rightarrow \alpha^{r+1} = 0 \Rightarrow \dim B \leq r$ (take $\alpha$ Kähler).

$\alpha \neq 0 \Rightarrow f^* \alpha \neq 0 \Rightarrow (f^* \alpha)^r \neq 0 \Rightarrow \alpha^r \neq 0 \Rightarrow \dim B \geq r$.

$f^* : H^2(B, C) \to H^2(X, C)$ injective $\Rightarrow H^2(B, 0) = 0$.

$f^*(H^2(B, C)) \subset H^1(X)$ totally isotropic for $q$; signature $q \mid H^1 = (1, h_1 - 1) \Rightarrow \dim H^2(B, C) \leq 1$.

$\text{Pic}(B) = \mathbb{Z} \cdot [L]$, $K_B = L \otimes n$.

Idea: $H^r(B, 0) = 0$ (as above) $\Rightarrow n \neq 0$, more work $\Rightarrow n < 0$.

Proof that $X_b$ Lagrangian: Skip proof.

Arnaud Beauville  V. Further developments
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3. $f^* : H^2(B, \mathbb{C}) \to H^2(X, \mathbb{C})$ injective $\Rightarrow H^{2,0}(B) = 0$. 

Arnaud Beauville  V. Further developments
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5. Pic$(B) = \mathbb{Z} \cdot [L], K_B = L \otimes n$. Idea : $H^{r,0}(B) = 0$ (as above)

$\Rightarrow n \neq 0$, more work $\Rightarrow n < 0.$
Proof.

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6. Proof that $X_b$ Lagrangian: Skip proof
Proof that the fibres are Lagrangian

Lemma

$\alpha, \beta, \gamma \in H^2(X, C)$ with $q(\alpha) = q(\alpha, \beta) = 0$. Then

$$\int_X \alpha \cdot p \cdot \beta \cdot q \cdot m = 0$$

for $p > m$.

Proof of the lemma.

$\forall \gamma \in H^2(X, C)$, $q(t \alpha + \beta + s \gamma) = c st + P(s)$ \Rightarrow

$$\int_X (t \alpha + \beta + s \gamma)^2 = f_X(c st + P(s)) \Rightarrow$$

$$\int_X \alpha \cdot p \cdot \beta \cdot q \cdot m = 0$$

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Arnaud Beauville

V. Further developments
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\[ \alpha, \beta, \gamma \in H^2(X, \mathbb{C}) \text{ with } q(\alpha) = q(\alpha, \beta) = 0. \text{ Then} \]

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\[ \forall \gamma \in H^2(X, \mathbb{C}), \quad q(t\alpha + \beta + s\gamma) = cst + P(s) \]
\[ \Rightarrow \int_X (t\alpha + \beta + s\gamma)^{2r} = f_X (cst + P(s))^r = \sum_{m \geq p} a_{p,m} t^p s^m \]
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\( \alpha, \beta, \gamma \in H^2(X, \mathbb{C}) \) with \( q(\alpha) = q(\alpha, \beta) = 0 \). Then

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Proof that the fibres are Lagrangian.

- **Apply with**: $\alpha = f^* \alpha_0$ with $\int_B \alpha'_0 = m \neq 0$, $\beta = \sigma + \bar{\sigma}$, $\gamma = \text{Kähler class on } X$. 
Proof that the fibres are Lagrangian.

- **APPLY WITH**: \( \alpha = f^* \alpha_0 \) with \( \int_B \alpha_0^r = m \neq 0 \), \( \beta = \sigma + \bar{\sigma} \), \( \gamma = \text{Kähler class on } X \).

- \( i : X_b \hookrightarrow X \). Then \( \int_X \alpha^r \omega = m \int_{X_b} i^* \omega \). Thus:

\[ i^* \gamma \text{ Kähler} \Rightarrow \text{hermitian form } (\alpha, \beta) \mapsto \int_X \alpha \bar{\beta} (i^* \gamma) > 0 \] on \( H^2, 0 \left( X_b \right) \) and \( \bar{\alpha} = 0 \).
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- **Apply with**: $\alpha = f^* \alpha_0$ with $\int_B \alpha_0^r = m \neq 0$, $\beta = \sigma + \bar{\sigma}$, $\gamma = \text{Kähler class on } X$.

- $i : X_b \hookrightarrow X$. Then $\int_X \alpha^r \omega = m \int_{X_b} i^* \omega$. Thus:

- $0 = \int_X \alpha^r \beta^2 \gamma^{r-2} = m \int_{X_b} i^* (\beta^2 \gamma^{r-2}) = 2m \int_{X_b} (i^* \sigma)(i^* \bar{\sigma})(i^* \gamma)^{r-2}$.
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- $i^* \gamma$ Kähler $\Rightarrow$ hermitian form $(\alpha, \beta) \mapsto \int_X \alpha \bar{\beta}(i^* \gamma)^{r-2} > 0$ on $H^{2,0}(X_b) \Rightarrow i^* \sigma = 0$. 

$$\blacksquare$$
Some open questions

If $f : X \to B$ Lagrangian and $M$ ample on $B$, $f^* M$ nef and $q(f^* M) = 0$. 

Example: $S^3 K_3$ with $\text{Pic}(S^3) = \mathbb{Z}[L]$. Recall: $\text{Pic}(X) = \mathbb{Z}[L], q(L) = L_2$, $q(\delta r) = -2(r - 1)$.

Assume $L_2 = 2(r - 1)n_2$, then $M = L[r](\delta_2 n)$ has $q(M) = 0$.

Theorem (Sawon, Markushevich): $\exists f : S[r] \to P[r]$ with $f^* O_{P[r]}(1) = M$.

Recall $H^*(X, \mathbb{Q}) = A \oplus A^\perp$. What about $A^\perp$?

Known: $4 | b_{2i+1}$ (Wakakuwa).
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If $f : X \to B$ Lagrangian and $M$ ample on $B$, $f^* M$ nef and $q(f^* M) = 0$.

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Arnaud Beauville | V. Further developments
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**Example:** $S$ K3 with $\text{Pic}(S) = \mathbb{Z}[L]$. Recall:

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\text{Pic}(X) = \mathbb{Z}[L^r] \perp \mathbb{Z}[\delta_r], \quad q(L^r) = L^2, \quad q(\delta_r) = -2(r-1).
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Arnaud Beauville | V. Further developments
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Arnaud Beauville | V. Further developments
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3. Recall \( H^*(X, \mathbb{Q}) = A \oplus A^\perp \). What about \( A^\perp \) ?

Known: \( 4 \mid b_{2i+1} \) (Wakakuwa).
Can we say more for hyperkähler 4-folds?

**Theorem** (Guan): either $b_2 = 23$, or $3 \leq b_2 \leq 8$. Improve?
4. Can we say more for hyperkähler 4-folds?

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5. Do they have only finitely many deformation types?
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