Existence result for a two-dimensional system of conservation laws with unilateral constraints

F. Berthelin

Université Côte d'Azur
INRIA Sophia Antipolis, Project Team COFFEE
Laboratoire J. A. Dieudonné, UMR 7351 CNRS,
Université de Nice Sophia-Antipolis, Parc Valrose,
06108 Nice cedex 2, France
e-mail: Florent.Berthelin@unice.fr

Abstract

The aim of this paper is to study a pressureless model with unilateral constraint in two dimensions. The corresponding one-dimensional case is a traffic flow model and was studied in [8]. The two-dimensional extension is linked to pedestrian flow. Several difficulties, geometric and analytical, appear with respect to the one-dimensional case. We get the existence of weak solutions for general initial data and the constraints in a strong sense.

Key-words: conservation laws with constraint – pressureless gas – sticky blocks – splitting dynamics

Mathematics Subject Classification: 35L87, 35L65, 76N10, 35A35, 35D30, 76T10

Contents

1 Introduction

1.1 Context

In this paper, we consider a system of conservation laws with constraints in two dimensions and we prove existence and stability of weak solutions. We start by introducing the studied model. For this, we must remember the Aw-Rascle-Zhang model

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho(u + \overline{p}(\rho))) + \partial_x(\rho u(u + \overline{p}(\rho))) = 0, \end{cases}$$

which is a very well accepted model for traffic flow [1, 29] where ρ is the density of cars, u there velocity and $\overline{p}(\rho)$ is a function of the density. We observe that, in this model, upper bounds on the density are not necessarily preserved through the time evolution of the solution. In practice, the density of cars is bounded from above by a maximal density ρ^* corresponding to a bumper to bumper situation. However, the Aw-Rascle-Zhang model does not exclude cases where, depending on the smallest invariant region which contains the initial data, solutions satisfy the maximal density constraint $\rho \leq \rho^*$ initially but evolve in finite time to a state, still uniformly bounded, but which violates this constraint. Then paper [8] presents a model which improve the Aw-Rascle-Zhang model and preserves the constraints. In order to obtain this, we take in the Aw-Rascle-Zhang model, the pressure

$$\overline{p}_\varepsilon(\rho) = \varepsilon \left(\frac{1}{\rho} - \frac{1}{\rho^*} \right)^{-\gamma} \mathbb{1}_{\rho \leq \rho^*}$$

and assuming that this term have a limit p when $\varepsilon \to 0$, which acts only when $\rho = \rho^*$, it leads to the system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho(u + p)) + \partial_x(\rho u(u + p)) = 0, \end{cases} \tag{1.1}$$

with the constraints

$$0 \le \rho \le \rho^*, \quad u \ge 0, \quad p \ge 0, \quad (\rho^* - \rho)p = 0. \tag{1.2}$$

The term p represents the speed capability which is not used if the road is overloaded or if the cars in front impose a reduced speed than that desired. We proved in particular existence and stability of solutions in [8]. After this paper, some improvements of the model have been completed in [9] for ρ^* depending on u (case where the maximum density of cars depends on the cars velocities) and in [7] for ρ^* depending on x (multi-lines case). Particle approximation of this constrained model was obtained in [10]. Let's mention also [19] for other congested model and self-organization. We remark that the important property of the contraint $(\rho^* - \rho)p = 0$ is its link with the property

$$\mathrm{Supp}\, p \subset \mathrm{Supp}\, (\rho - \rho^*).$$

Notice that the structure of the system is related to the pressureless gases system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0. \end{cases} \tag{1.3}$$

The system (1.3) was studied in [11], [14], [20], [22], [13]. It is known that this system gives Dirac distributions on ρ in finite time, even for smooth initial data. It is clearly incompatible with a constraint for the density. In [12], a system arises in the modeling of two-phase flows as

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \pi) = 0, \end{cases} \tag{1.4}$$

with constraint and pressure Lagrange multiplier

$$0 \le \rho \le 1, \qquad \pi \ge 0, \tag{1.5}$$

and extremality relation

$$(1 - \rho)\pi = 0. \tag{1.6}$$

The model (1.4)-(1.6) is an hyperbolic constraint model which corresponds to gas dynamics when $\pi = 0$ and gives a bound for the density. Existence and weak stability of suitable weak solutions was obtained in [3]. There are now a lot of domains in which constraints models take place. We could find other hyperbolic problems with constraints in [2], [24] and [25]. In [6], the isentropic case of the problem (1.4)-(1.6) was studied with other constraints. See also [5] for a numerical version of this kind of problems. The case with viscosity was studied in [26]. In that direction, the limit of barotropic compressible Navier-Stokes to constraint Navier-Stokes was proved in [15] for the one-dimensional case and in [27] for the multi-dimensional case. In the paper [4], we studied the multi-dimensional extension of this pressureless model for a gas system with unilateral constraint. In two-dimension, this system is written

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \pi_1) + \partial_y(\rho u v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho u v) + \partial_y(\rho v^2 + \pi_2) = 0, \end{cases} \tag{1.7}$$

with the constraints

$$0 \le \rho \le 1, \quad \pi_1 \ge 0, \quad \pi_2 \ge 0, \tag{1.8}$$

and the exclusion relations

$$\rho\pi_1 = \pi_1, \quad \rho\pi_2 = \pi_2. \tag{1.9}$$

By the way, these exclusion relations were in this context only taken by a reformulation in a very weak sense. Thus it is not the most satisfying study. This is why we turn our attention to the model studied in the present paper: in order to get solutions in most classical spaces and with contraints in the classical sense.

In the present paper, we want to study the two-dimensional extension of the model (1.1)-(1.2), that is to say the system

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ \partial_t(\rho(u+p)) + \partial_x(\rho u(u+p)) + \partial_y(\rho(u+p)v) = 0, \\ \partial_t(\rho(v+q)) + \partial_x(\rho u(v+q)) + \partial_y(\rho v(v+q)) = 0, \end{cases} \tag{1.10}$$

with the constraints

$$0 \le \rho \le \rho^*, \quad u \ge 0, \quad v \ge 0, \quad p \ge 0, \quad q \ge 0, \tag{1.11}$$

and the exclusion relations

$$(\rho - \rho^*)p = 0, \quad (\rho - \rho^*)q = 0. \tag{1.12}$$

It can be viewed as the modelization of a pedestrian flow coming from West and South and going to East and North and crossing together. The term (p, q) represents here the speed capability of a pedestrian, which is not used if a pedestrian in the same direction imposes a speed smaller than that desired. The constant $\rho^* > 0$ is the maximal density of pedestrians. This model has different properties and requires a different approach than the one from (1.7)-(1.9). In the present paper, we will get the exclusion relations in a strong sence and for all initial data unlike the case of (1.7)-(1.9). Notice also that we have to get a more robust block approximation which is Proposition 3.2. In this result, we have to perform a nice BV approximation and also to keep the contraints. In order to get all these things together, we will have only (3.47) but it will be enough to get the constraints at the limit. For pedestrian flow model, we refer for exemple to the articles [16, 17, 18] and references inside.

1.2 Main result

In the present paper, we are focusing on extending the existence and stability result of [8] for this system (1.10)-(1.12) in two dimensions.

An important tool for this result is the sticky block dynamics. In dimension one, the density and momentum of blocks are sum of terms of the form

$$(\rho(t), \rho(t)u(t)) = \rho^*(1, u_i(t))\mathbb{1}_{a_i^l(t) \le x \le a_i^r(t)}$$

with a density equals to ρ^* and a velocity $u_i(t)$ constant on the block $a_i^l(t) \le x \le a_i^r(t)$. The time evolution is defined as follows. The number of blocks n indeed depends on t, but is piecewise constant. As long as the blocks do not meet, they move at constant velocity $u_i(t)$. When two or more blocks collide, they get stuck, building a new block. Then, in dimension one, the blocks dynamics is easy because after a collision, we still have a single block.

In multi-dimension, we extend the notion of blocks as

$$(\rho(t), \rho(t)u(t)\rho(t)v(t)) = \rho^*(1, u_i(t), v_i(t))\mathbb{1}_{a_i^l(t) \le x \le a_i^r(t)}\mathbb{1}_{b_i^l(t) \le y \le b_i^r(t)}$$

with density equals to ρ^* and velocity $(u_i, v_i)(t)$, constant on the block $a_i^l(t) \le x \le a_i^r(t)$, $b_i^l(t) \le y \le b_i^r(t)$. Then a geometric problem appears since when two rectangular parallelepipeds collide, they do not form a rectangular parallelepiped. The idea presented in [4] to pass over this difficulty is to approximate the motion of each block by discrete jumps in all the directions separately in consecutive time steps. In other words, we make a splitting with respect to the various directions of space on consecutive time steps and then, on each time interval, we do vary only one direction then, on the next interval, another direction and so on to keep the geometry at each collision. Then by letting the time step going to 0 and thereby forcing the splitting to be more rapid, we hope to find the limit of the speed on any directions.

In this paper, we have to consider more complex blocks since we also need ρp and ρq to be of this kind. Then, we will call block data sum of terms of the form

$$\begin{aligned} & (\rho(t), \rho(t)u(t)\rho(t)v(t), \rho p(t), \rho q(t)) \\ = \; & \rho^*(1, u_i(t), v_i(t), p_i(t), q_i(t)) \mathbb{1}_{a_i^l(t) \le x \le a_i^r(t)} \mathbb{1}_{b_i^l(t) \le y \le b_i^r(t)} \end{aligned}$$

with density equals to ρ^*, velocity $(u_i, v_i)(t)$ and speed capacity $(p_i, q_i)(t)$ constant on the block $a_i^l(t) \le x \le a_i^r(t)$, $b_i^l(t) \le y \le b_i^r(t)$.

The purpose of this paper is to achieve this approach in the present case and prove that it works.

Furthermore for block initial data in the one-dimensional case, we get explicit solutions. Here, in the multi-dimensional case, we will only get approximations of solutions for these special initial data. Then, the stability and existence of solutions will require additional steps to work.

Let us also consider initial data

$$\begin{cases} \rho(0, x, y) = \rho^0(x, y), \\ \rho(0, x, y)u(0, x, y) = \rho^0(x, y)u^0(x, y), \\ \rho(0, x, y)v(0, x, y) = \rho^0(x, y)v^0(x, y), \\ \rho(0, x, y)p(0, x, y) = \rho^0(x, y)p^0(x, y), \\ \rho(0, x, y)q(0, x, y) = \rho^0(x, y)q^0(x, y), \end{cases} \tag{1.13}$$

with the regularities

$$\rho^0 \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2), \quad u^0, v^0, p^0, q^0 \in L^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2). \tag{1.14}$$

Let us define precisely the weak solutions we shall consider. We are looking for solutions satisfying

$$\rho \in L_t^\infty(]0, +\infty[, L_{xy}^\infty(\mathbb{R}^2) \cap L_{xy}^1(\mathbb{R}^2)) \cap C_t([0, +\infty[, L_{w*}^\infty(\mathbb{R}^2)), \tag{1.15}$$

$$u, v, p, q \in L_t^\infty(]0, +\infty[, L_{xy}^\infty(\mathbb{R}^2)) \cap L^\infty(]0, T[, BV_{loc}(\mathbb{R}^2)). \tag{1.16}$$

for any $T > 0$.

Hence, (1.10), (1.13) must be satisfied in the sense of distributions:
for all $\varphi \in C_c^\infty([0, +\infty[\times\mathbb{R}^2)$,

$$\begin{aligned} \int_{[0,+\infty[} \iint_{\mathbb{R}^2} & (\rho \partial_t \varphi + \rho u \partial_x \varphi + \rho v \partial_y \varphi) \, dx \, dy \, dt \\ & + \iint_{\mathbb{R}^2} \rho^0(x, y) \varphi(0, x, y) \, dx \, dy = 0, \end{aligned} \tag{1.17}$$

$$\int_{[0,+\infty[} \iint_{\mathbb{R}^2} (\rho(u+p)\partial_t\varphi + \rho u(u+p)\partial_x\varphi + \rho(u+p)v\partial_y\varphi)\,dx\,dy\,dt$$
$$+ \iint_{\mathbb{R}^2} (\rho^0(u^0+p^0))(x,y)\varphi(0,x,y)\,dx\,dy = 0, \tag{1.18}$$

and

$$\int_{[0,+\infty[} \iint_{\mathbb{R}^2} (\rho(v+q)\partial_t\varphi + \rho u(v+q)\partial_x\varphi + \rho v(v+q)\partial_y\varphi)\,dx\,dy\,dt$$
$$+ \iint_{\mathbb{R}^2} (\rho^0(v^0+p^0))(x,y)\varphi(0,x,y)\,dx\,dy = 0. \tag{1.19}$$

The main result we get in this paper is the following.

Theorem 1.1 (Existence of solutions) *Let us consider $\rho^* > 0$ and initial data $(\rho^0, u^0, v^0, p^0, q^0)$ with regularities (1.14) and satisfying*

$$0 \le \rho \le \rho^*, \quad u^0 \ge 0, \quad v^0 \ge 0, \quad p^0 \ge 0, \quad q^0 \ge 0, \tag{1.20}$$
$$(\rho^0 - \rho^*)p^0 = 0, \quad (\rho^0 - \rho^*)q^0 = 0. \tag{1.21}$$

Then there exists (ρ, u, v, p, q), with regularities (1.15)-(1.16), which are weak solutions of (1.10) with initial data $(\rho^0, u^0, v^0, p^0, q^0)$, that is to say (1.17)-(1.19), with the constraints (1.11)-(1.12) and satisfying the bounds

$$0 \le \rho \le \rho^*, \quad \iint_{\mathbb{R}^2} \rho(t,x,y)\,dx\,dy \le \iint_{\mathbb{R}^2} \rho^0(x,y)\,dx\,dy, \tag{1.22}$$

$$0 \le u \le esssup\ u^0, \quad 0 \le v \le esssup\ v^0, \tag{1.23}$$
$$0 \le p \le esssup\ u^0 + esssup\ p^0, \quad 0 \le q \le esssup\ v^0 + esssup\ q^0, \tag{1.24}$$
$$\iint_{\mathbb{R}\times[-a,a]} |\partial_x w(t,x,y)| \le (1+a_t^2) \iint_{\mathbb{R}^2} \left|\partial_x w^0(x,y)\right|, \tag{1.25}$$
$$\iint_{[-a,a]\times\mathbb{R}} |\partial_y w(t,x,y)| \le (1+a_t^2) \iint_{\mathbb{R}^2} \left|\partial_y w^0(x,y)\right| \tag{1.26}$$

for $w = u$ and v and

$$\iint_{\mathbb{R}\times[-a,a]} |\partial_x p(t,x,y)| \le (1+a_t^2) \left(\iint_{\mathbb{R}^2} \left|\partial_x u^0(x,y)\right| + \iint_{\mathbb{R}^2} \left|\partial_x p^0(x,y)\right| \right), \tag{1.27}$$

$$\iint_{[-a,a]\times\mathbb{R}} |\partial_y p(t,x,y)| \le (1+a_t^2) \left(\iint_{\mathbb{R}^2} \left|\partial_y u^0(x,y)\right| + \iint_{\mathbb{R}^2} \left|\partial_y p^0(x,y)\right| \right), \tag{1.28}$$

$$\iint\limits_{\mathbb{R}\times[-a,a]} |\partial_x q(t,x,y)| \leq (1+a_t^2) \left(\iint\limits_{\mathbb{R}^2} \left|\partial_x v^0(x,y)\right| + \iint\limits_{\mathbb{R}^2} \left|\partial_x q^0(x,y)\right| \right), \tag{1.29}$$

$$\iint\limits_{[-a,a]\times\mathbb{R}} |\partial_y q(t,x,y)| \leq (1+a_t^2) \left(\iint\limits_{\mathbb{R}^2} \left|\partial_y v^0(x,y)\right| + \iint\limits_{\mathbb{R}^2} \left|\partial_y q^0(x,y)\right| \right) \tag{1.30}$$

for any $a > 0$ *and with* $a_t = a + t\max(\|u^0\|_\infty + \|p^0\|_\infty, \|v^0\|_\infty + \|q^0\|_\infty)$.

An other important result of the paper is the Proposition 3.2 which gives a very general result for approximation and can be use as such, or this kind of proof, in other contexts.

The scheme of the proof we have to keep in mind to read the paper is the following. In order to get the existence of solutions, we approximate the initial data by blocks initial data. For these blocks initial data, we prove the existence of approximations of solutions. We obtain the limit of these approximations to get the existence of solutions for initial data with a block form. Finally, by a stability result, we find a solution for the general initial data. We could draw the scheme of the proof in figure 1.

$(\rho^0, u^0, v^0, p0, q^0)$ initial data

approximation by blocks ↓ Proposition 3.2

$(\rho^0_k, u^0_k, v^0_k, p^0_k, q^0_k)$ approximations by blocks initial data

dynamics of blocks ↓ Proposition 3.9

$(\rho_{kl}, u_{kl}, v_{kl}, p_{kl}, q_{kl})$ approximations of solutions

k fixed, $l \to +\infty$ ↓ Theorem 4.1

$(\rho_k, u_k, v_k, p_k, q_k)$ solutions with initial data (ρ^0_k, u^0_k, v^0_k)

$k \to +\infty$ ↓ Theorem 1.1

(ρ, u, v, p, q) solution with initial data (ρ^0, u^0, v^0)

Figure 1.

The paper is organized as follows. In section 2, we present some particular solutions for the system. In section 3, we prove that we can approximate any initial data by block initial (first arrow of figure 1) and we prove that there exists approximations of solutions in the class for block initial data : first for the particular solutions of section 2 then for any approximated solution (Proposition 3.9, second arrow of figure 1). In section 4, we conclude to the existence result (third and fourth arrows of figure 1).

2 Particular solutions

We first present some particular solutions for the system. It concerns the evolution of solutions which are sum of terms of the form

$$(\rho(t), \rho(t)u(t)\rho(t)v(t)) = \rho^*(1, u_i(t), v_i(t))\mathbb{1}_{a_i^l(t)\leq x\leq a_i^r(t)}\mathbb{1}_{b_i^l(t)\leq y\leq b_i^r(t)}.$$

We start first by studying the dynamics when constraints don't act. In this very particular case, there is no specific dynamic of the system and it is similarly as in [4], it leads to the study of pressureless dynamics equations in dimension two, which are given by

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_y(\rho uv) = 0, \\ \partial_t(\rho v) + \partial_x(\rho uv) + \partial_y(\rho v^2) = 0. \end{cases} \tag{2.31}$$

and in this paper, we have the following result just by adding ρ^* to the proof.

Proposition 2.1 *Let $\tilde{u}, \tilde{v}, a_0, b_0 \in \mathbb{R}$ and $c, d > 0$. The functions*

$$\rho(1, u, v)(t, x, y) = \rho^*(1, \tilde{u}, \tilde{v})\mathbb{1}_{0\leq t}\mathbb{1}_{a(t)\leq x\leq a(t)+c}\mathbb{1}_{b(t)\leq y\leq b(t)+d}, \tag{2.32}$$

where $a(t) = a_0 + \tilde{u}t$ and $b(t) = b_0 + \tilde{v}t$, are solutions of (2.31) in the distributional sense with the initial data

$$\rho^*(1, \tilde{u}, \tilde{v})\mathbb{1}_{a_0\leq x\leq a_0+c}\mathbb{1}_{b_0\leq y\leq b_0+d}.$$

The previous dynamics concern some particular evolutions as long as there is no collision. Now we consider the case with a collision in the x direction at some time t^*. Then the two (or more) blocks collide, they get stuck, building a new block, which volume is the sum of the volumes, and taking the velocity of the block with the smallest velocity. This result is specific to the studied system. We first have the following Lemma.

Lemma 2.2 *Let $u, v \in \mathbb{R}$, $\varphi \in C_c^1([0, +\infty \times \mathbb{R}^2)$, $a_1, a_2, b_1, b_2 \in C^1([0, +\infty[, \mathbb{R})$ such that $a_1'(t) = a_2'(t) = u$ et $b_1'(t) = b_2'(t) = v$. Then, for any $\sigma, s \in [0, +\infty[$, we have*

$$\begin{aligned} &\left[\int_{a_1(t)}^{a_2(t)}\int_{b_1(t)}^{b_2(t)} \varphi(t, x, y)\, dy\, dx\right]_\sigma^s \\ = &\int_\sigma^s\int_{a_1(t)}^{a_2(t)}\int_{b_1(t)}^{b_2(t)} \left(\partial_t\varphi(t, x, y) + u\partial_x\varphi(t, x, y) + v\partial_y\varphi(t, x, y)\right)\, dy\, dx\, dt. \end{aligned}$$

The proof of this result can be found in Annex. We have now the result.

Proposition 2.3 *Let $t^*, \mu > 0$, $x^*, u_l, u_r, p_l, p_r, c, d, v_l, v_r, q_l, q_r \in \mathbb{R}$ with $u_l > u_r > 0$ and $v_l, v_r, p_l, p_r, q_l, q_r \geq 0$. The functions*

$$\begin{aligned} \rho(1, u, v)(t, x, y) \;=\; &\mathbb{1}_{0\leq t<t^*}\,\rho^*\Big((1, u_l, v_l)\mathbb{1}_{a_l(t)-c\leq x\leq a_l(t)}\mathbb{1}_{b(t)\leq y\leq b(t)+\mu} \\ &\qquad + (1, u_r, v_r)\mathbb{1}_{a_r(t)\leq x\leq a_r(t)+d}\mathbb{1}_{b(t)\leq y\leq b(t)+\mu}\Big) \\ &+ \mathbb{1}_{t^*\leq t}\,\rho^*(1, u_r, v_l)\,\mathbb{1}_{a_r(t)-c\leq x\leq a_r(t)}\mathbb{1}_{b(t)\leq y\leq b(t)+\mu} \\ &+ \mathbb{1}_{t^*\leq t}\,\rho^*(1, u_r, v_r)\,\mathbb{1}_{a_r(t)\leq x\leq a_r(t)+d}\mathbb{1}_{b(t)\leq y\leq b(t)+\mu}, \end{aligned}$$

and

$$\begin{aligned}\rho(1,p,q)(t,x,y) &= \mathbb{1}_{0\le t<t^*}\,\rho^*\Big((1,p_l,q_l)\mathbb{1}_{a_l(t)-c\le x\le a_l(t)}\mathbb{1}_{b(t)\le y\le b(t)+\mu}\\ &\qquad +(1,p_r,q_r)\mathbb{1}_{a_r(t)\le x\le a_r(t)+d}\mathbb{1}_{b(t)\le y\le b(t)+\mu}\Big)\\ &\quad+\mathbb{1}_{t^*\le t}\,\rho^*\Big((1,u_l-u_r+p_l,q_l)\mathbb{1}_{a_r(t)-c\le x\le a_r(t)}\mathbb{1}_{b(t)\le y\le b(t)+\mu}\\ &\qquad +\,(1,p_r,q_r)\,\mathbb{1}_{a_r(t)\le x\le a_r(t)+d}\mathbb{1}_{b(t)\le y\le b(t)+\mu}\Big),\end{aligned}$$

where $a_l(t)=x^*+u_l(t-t^*)$, $a_r(t)=x^*+u_r(t-t^*)$, $b'(t)=v$ (the point x^* being the point of collision), are solution of (1.10)-(1.12) in the distributional sense.

Proof. Let φ be a test function with a support on $[0,+\infty[\times\mathbb{R}^2$ and $S:\mathbb{R}^2\to\mathbb{R}$ be a continuous function. We have

$$< \partial_t(\rho S(u+p,v+q))+\partial_x(\rho S(u+p,v+q)u)+\partial_y(\rho S(u+p,v+q)v),\varphi >$$

$$= -\int_0^{t^*}\int_{a_l(t)-c}^{a_l(t)}\int_{b(t)}^{b(t)+\mu}\rho^*S(u_l+p_l,v_l+q_l)(\partial_t\varphi+u_l\partial_x\varphi+v_l\partial_y\varphi)\,dy\,dx\,dt \tag{2.33}$$

$$-\int_0^{t^*}\int_{a_r(t)}^{a_r(t)+d}\int_{b(t)}^{b(t)+\mu}\rho^*S(u_r+p_r,v_r+q_r)(\partial_t\varphi+u_r\partial_x\varphi+v_r\partial_y\varphi)\,dy\,dx\,dt \tag{2.34}$$

$$-\int_{t^*}^{+\infty}\int_{a_r(t)-c}^{a_r(t)}\int_{b(t)}^{b(t)+\mu}\rho^*S(u_r+u_l-u_r+p_l,v_l+q_l)(\partial_t\varphi+u_r\partial_x\varphi+v_l\partial_y\varphi)\,dy\,dx\,dt \tag{2.35}$$

$$-\int_{t^*}^{+\infty}\int_{a_r(t)}^{a_r(t)+d}\int_{b(t)}^{b(t)+\mu}\rho^*S(u_r+p_r,v_r+q_r)(\partial_t\varphi+u_r\partial_x\varphi+v_r\partial_y\varphi)\,dy\,dx\,dt. \tag{2.36}$$

Using Lemma 2.2 with $a_1(t)=a_l(t)-c$, $a_2(t)=a_l(t)$, $b_1(t)=b(t)$, $b_2(t)=b(t)+\mu$, $\sigma=0$ and $s=t^*$, we have

$$\begin{aligned}&\int_0^{t^*}\int_{a_l(t)-c}^{a_l(t)}\int_{b(t)}^{b(t)+\mu}\rho^*S(u_l+p_l,v_l+q_l)(\partial_t\varphi+u_l\partial_x\varphi+v_l\partial_y\varphi)\,dy\,dx\,dt\\ &= S(u_l+p_l,v_l+q_l)\left[\int_{a_1(t)-c}^{a_l(t)}\int_{b(t)}^{b(t)+\mu}\varphi(t,x,y)\,dy\,dx\right]_0^{t^*}\\ &= \int_{a_l(t^*)-c}^{a_l(t^*)}\int_{b(t^*)}^{b(t^*)+\mu}S(u_l+p_l,v_l+q_l)\varphi(t,x,y)\,dy\,dx\\ &\quad-\int_{a_l(0)-c}^{a_l(0)}\int_{b(0)}^{b(0)+\mu}S(u_l+p_l,v_l+q_l)\varphi(t,x,y)\,dy\,dx.\end{aligned}$$

Using Lemma 2.2 with $a_1(t)=a_r(t)$, $a_2(t)=a_r(t)+d$, $b_1(t)=b(t)$, $b_2(t)=b(t)+\mu$, $\sigma=0$ and $s=t^*$, we have

$$\begin{aligned}&\int_0^{t^*}\int_{a_r(t)}^{a_r(t)+d}\int_{b(t)}^{b(t)+\mu}\rho^*S(u_r+p_r,v_r+q_r)(\partial_t\varphi+u_r\partial_x\varphi+v_r\partial_y\varphi)\,dy\,dx\,dt\\ &= S(u_r+p_r,v_r+q_r)\left[\int_{a_r(t)}^{a_r(t)+d}\int_{b(t)}^{b(t)+\mu}\varphi(t,x,y)\,dy\,dx\right]_0^{t^*}\end{aligned}$$

$$
\begin{aligned}
&= \int_{a_r(t^*)}^{a_r(t^*)+d} \int_{b(t^*)}^{b(t^*)+\mu} S(u_r+p_r, v_r+q_r)\varphi(t,x,y)\,dy\,dx \\
&\quad - \int_{a_r(0)}^{a_r(0)+d} \int_{b(0)}^{b(0)+\mu} S(u_r+p_r, v_r+q_r)\varphi(t,x,y)\,dy\,dx.
\end{aligned}
$$

Using Lemma 2.2 with $a_1(t) = a_r(t) - c$, $a_2(t) = a_r(t)$, $b_1(t) = b(t)$, $b_2(t) = b(t) + \mu$, $\sigma = t^*$ and s tends to $+\infty$, we have

$$
\begin{aligned}
&\int_{t^*}^{+\infty} \int_{a_r(t)-c}^{a_r(t)} \int_{b(t)}^{b(t)+\mu} \rho^* S(u_l+p_l, v_l+q_l)(\partial_t\varphi + u_r\partial_x\varphi + v_l\partial_y\varphi)\,dy\,dx\,dt \\
&= S(u_l+p_l, v_l+q_l) \left[\int_{a_r(t)-c}^{a_r(t)} \int_{b(t)}^{b(t)+\mu} \varphi(t,x,y)\,dy\,dx \right]_{t^*}^{+\infty} \\
&= -\int_{a_r(t^*)-c}^{a_r(t^*)} \int_{b(0)}^{b(0)+\mu} S(u_l+p_l, v_l+q_l)\varphi(t,x,y)\,dy\,dx.
\end{aligned}
$$

Using Lemma 2.2 with $a_1(t) = a_r(t)$, $a_2(t) = a_r(t) + d$, $b_1(t) = b(t)$, $b_2(t) = b(t) + \mu$, $\sigma = t^*$ and s tends to $+\infty$, we have

$$
\begin{aligned}
&\int_{t^*}^{+\infty} \int_{a_r(t)}^{a_r(t)+d} \int_{b(t)}^{b(t)+\mu} \rho^* S(u_l+p_l, v_l+q_l)(\partial_t\varphi + u_r\partial_x\varphi + v_l\partial_y\varphi)\,dy\,dx\,dt \\
&= -\int_{a_r(t^*)}^{a_r(t^*)+d} \int_{b(0)}^{b(0)+\mu} S(u_l+p_l, v_l+q_l)\varphi(t,x,y)\,dy\,dx.
\end{aligned}
$$

Then we get, since $a_l(t^*) = a_r(t^*)$,

$$
\begin{aligned}
&< \partial_t(\rho S(u+p, v+q)) + \partial_x(\rho S(u+p, v+q)u) + \partial_y(\rho S(u+p, v+q)v), \varphi > \\
&= \int_{a_l(0)-c}^{a_l(0)} \int_{b(0)}^{b(0)+\mu} \rho^* S(u_l+p_l, v_l+q_l)\varphi(t,x,y)\,dy\,dx \\
&\quad + \int_{a_r(0)}^{a_r(0)+d} \int_{b(0)}^{b(0)+\mu} \rho^* S(u_r+p_r, v_r+q_r)\varphi(t,x,y)\,dy\,dx.
\end{aligned}
$$

Both terms on the right-hand side correspond to the initial data conditions. If we take a test function with compact support on $]0, +\infty[\times\mathbb{R}^2$, these two terms are zero and for $S(w, z) = 1$, it gives

$$\partial_t\rho + \partial_x(\rho u) + \partial_y(\rho v) = 0,$$

for $S(w, z) = w$, it gives

$$\partial_t(\rho(u+p)) + \partial_x(\rho u(u+p)) + \partial_y(\rho v(u+p)) = 0$$

and for $S(w, z) = z$, we get

$$\partial_t(\rho(v+q)) + \partial_x(\rho u(v+q)) + \partial_y(\rho v(v+q)) = 0. \ \square$$

By a similar proof for the case of a collision in the y direction at some time t^*, we have the following result.

Proposition 2.4 *Let $t^*, \mu > 0$, $x^*, u_l, u_r, p_l, p_r, c, d, v_l, v_r, q_l, q_r \in \mathbb{R}$ with $v_l > v_r > 0$ and $u_l, u_r, p_l, p_r, q_l, q_r \geq 0$. The functions*

$$\begin{aligned}\rho(1,u,v)(t,x,y) &= \mathbb{1}_{0\le t<t^*}\,\rho^*\Big((1,u_l,v_l)\mathbb{1}_{a(t)\le x\le a(t)+\mu}\mathbb{1}_{b_l(t)-c\le y\le b_l(t)}\\ &\qquad +(1,u_r,v_r)\mathbb{1}_{a(t)\le x\le a(t)+\mu}\mathbb{1}_{b_r(t)\le y\le b_r(t)+d}\Big)\\ &\quad +\mathbb{1}_{t^*\le t}\,\rho^*(1,u_l,v_r)\,\mathbb{1}_{a(t)\le x\le a(t)+\mu}\mathbb{1}_{b_r(t)-c\le y\le b_r(t)}\\ &\quad +\mathbb{1}_{t^*\le t}\,\rho^*(1,u_r,v_r)\,\mathbb{1}_{a(t)\le x\le a(t)+\mu}\mathbb{1}_{b_r(t)\le y\le b_r(t)+d},\end{aligned}$$

and

$$\begin{aligned}\rho(1,p,q)(t,x,y) &= \mathbb{1}_{0\le t<t^*}\,\rho^*\Big((1,p_l,q_l)\mathbb{1}_{a(t)\le x\le a(t)+\mu}\mathbb{1}_{b_l(t)-c\le y\le b_l(t)}\\ &\qquad +(1,p_r,q_r)\mathbb{1}_{a(t)\le x\le a(t)+\mu}\mathbb{1}_{b_r(t)\le y\le b_r(t)+d}\Big)\\ &\quad +\mathbb{1}_{t^*\le t}\,\rho^*\Big((1,p_l,v_l-v_r+q_l)\mathbb{1}_{a(t)\le x\le a(t)+\mu}\mathbb{1}_{b_r(t)-c\le y\le b_r(t)}\\ &\qquad +\,(1,p_r,q_r)\,\mathbb{1}_{a(t)\le x\le a(t)+\mu}\mathbb{1}_{b_r(t)\le y\le b_r(t)+d}\Big),\end{aligned}$$

where $b_l(t) = x^* + v_l(t-t^*)$, $b_r(t) = x^* + v_r(t-t^*)$, $a'(t) = u$ (the point x^* being the point of collision), are solutions of (1.10)-(1.12) in the distributional sense.

3 Approximations of solutions

In this section, we prove that for any initial data, we can construct a sequence of blocks which converges in the distributional sense toward this initial data. Then we prove that for each block initial data, we can construct a sequence of approximations of solutions.

3.1 Discretization with blocks

About approximation of data by block data, the first result was given in one dimension in [3]. It was extended for the BV context in [7]. Then a very precise result was obtained in [4] which allows to deal with multi-variable functions instead of real-variable functions. In particular, the arguments with BV functions have to be changed. There are also additional difficulties in the definition of the blocks which approximate the initial data. Here, we extend again the result to get the following approximation by blocks result for initial data. It concerns an extension to functions (u, v, p, q) instead of (u, v) and also the argument for BV approximation for (p, q) is an important difference with respect to the proof of [4]. Finaly, an other important difference is that in order to conserve all the bounds, we will only have (3.47) instead of an egality at the block level.

Definition 3.1 *Let $\rho^0 > 0$. We call block initial data a function $(\rho^0, u^0, v^0, p^0, q^0)$ depending on (x, y) of the form*

$$\begin{aligned}&\rho^0(x,y)(1,u^0(x,y),v^0(x,y),p^0(x,y),q^0(x,y))\\ &\qquad = \sum_{i=-I}^{I'}\sum_{j=-J}^{J'} \rho_{ij}(1,u_{ij},v_{ij},p_{ij},q_{ij})\mathbb{1}_{(x,y)\in P_{ij}},\end{aligned} \tag{3.37}$$

where

$$\mathbb{1}_{(x,y)\in P_{ij}} = \mathbb{1}_{a_{ij}\le x\le b_{ij}}\mathbb{1}_{c_{ij}\le y\le d_{ij}}, \tag{3.38}$$

with $I, I', J, J' \in \mathbb{N}$ *and, for* $-I \le i \le I'$, $-J \le j \le J'$,

$$\rho_{ij} \in \{0, \rho^*\},$$

$a_{ij}, b_{ij}, c_{ij}, d_{ij}, u_{ij}, v_{ij} \in \mathbb{R}$ *such that* $b_{ij} \le a_{i+1,j}$ *and* $d_{ij} \le c_{i,j+1}$.

Proposition 3.2 *Let* $\rho^0 \in L^1(\mathbb{R}^2)$, $u^0, v^0, p^0, q^0 \in L^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$ *such that*

$$0 \le \rho^0 \le \rho^* \text{ and } (\rho^0 - \rho^*)p^0 = (\rho^0 - \rho^*)q^0 = 0.$$

Then, there exists a sequence of block initial data $(\rho_k^0, u_k^0, v_k^0, p_k^0, q_k^0)_{k\ge1}$ *with* ρ_k^0 *of the form*

$$\rho_k^0(x,y) = \sum_{i,j=-k^2}^{k^2} \rho^* \mathbb{1}_{]\frac{i}{k},\frac{i}{k}+r_{ijk}[}(x)\mathbb{1}_{]\frac{j}{k},\frac{j}{k}+r_{ijk}[}(y), \tag{3.39}$$

with $0 \le r_{ijk} \le 1/k$, *and such that, for any* $k \in \mathbb{N}^*$,

$$\rho_k^0 \in L^1(\mathbb{R}^2), \quad u_k^0, v_k^0, p_k^0, q_k^0 \in L^\infty(\mathbb{R}^2) \cap BV_{loc}(\mathbb{R}^2) \tag{3.40}$$

with the bounds

$$0 \le \rho_k^0 \le \rho^*, \quad \iint_{\mathbb{R}^2} \rho_k^0(x,y)\,dx\,dy \le \iint_{\mathbb{R}^2} \rho^0(x,y)\,dx\,dy, \tag{3.41}$$

$$\operatorname{ess\,inf}\ u^0 \le u_k^0 \le \operatorname{ess\,sup}\ u^0, \quad \operatorname{ess\,inf}\ v^0 \le v_k^0 \le \operatorname{ess\,sup}\ v^0, \tag{3.42}$$

$$\operatorname{ess\,inf}\ p^0 \le p_k^0 \le \operatorname{ess\,sup}\ p^0, \quad \operatorname{ess\,inf}\ q^0 \le q_k^0 \le \operatorname{ess\,sup}\ q^0, \tag{3.43}$$

$$\iint_{\mathbb{R}\times[-a,a]} \left|\partial_x w_k^0(x,y)\right| \le (1+a^2) \iint_{\mathbb{R}^2} \left|\partial_x w^0(x,y)\right|, \tag{3.44}$$

$$\iint_{[-a,a]\times\mathbb{R}} \left|\partial_y w_k^0(x,y)\right| \le (1+a^2) \iint_{\mathbb{R}^2} \left|\partial_y w^0(x,y)\right|, \tag{3.45}$$

for any $a > 0$, *and with* $w = u$, v, p *and* q, *and for which the convergences*

$$\rho_k^0 \rightharpoonup \rho^0, \quad \rho_k^0 u_k^0 \rightharpoonup \rho^0 u^0, \quad \rho_k^0 v_k^0 \rightharpoonup \rho^0 v^0, \quad \rho_k^0 p_k^0 \rightharpoonup \rho^0 p^0, \quad \rho_k^0 q_k^0 \rightharpoonup \rho^0 q^0 \tag{3.46}$$

hold in the distributional sense as well as the constraints

$$(\rho_k^0 - \rho^*)p_k^0 \rightharpoonup 0, \quad (\rho_k^0 - \rho^*)q_k^0 \rightharpoonup 0. \tag{3.47}$$

Proof. Let $k \in \mathbb{N}^*$ and set for any $i, j \in \mathbb{Z}$

$$m_{ijk} = \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \rho^0(x,y)\,dx\,dy,$$

$$u^0_{ijk} = k^2 \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} u^0(x,y)\,dx\,dy, \quad v^0_{ijk} = k^2 \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} v^0(x,y)\,dx\,dy, \tag{3.48}$$

$$p^0_{ijk} = k^2 \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} p^0(x,y)\,dx\,dy, \quad q^0_{ijk} = k^2 \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} q^0(x,y)\,dx\,dy, \tag{3.49}$$

Now we set, for any $(x,y) \in \mathbb{R}^2$,

$$\rho^0_k(x,y) = \rho^* \sum_{i,j=-k^2}^{k^2} 1\!\!1_{]\frac{i}{k}, \frac{i}{k}+\sqrt{m_{ijk}}[}(x)\, 1\!\!1_{]\frac{j}{k}, \frac{j}{k}+\sqrt{m_{ijk}}[}(y) \tag{3.50}$$

and for $w = u, v, p$ and q,

$$\begin{aligned} w^0_k(x,y) \quad = \quad & \sum_{i,j=-k^2}^{k^2} w^0_{ijk} 1\!\!1_{]\frac{i}{k},\frac{i+1}{k}[}(x)\, 1\!\!1_{]\frac{j}{k},\frac{j+1}{k}[}(y) \\ & + \sum_{j=-k^2}^{k^2} \Big(w^0_{-k^2,j,k} 1\!\!1_{]-\infty,-k[}(x) + w^0_{k^2,j,k} 1\!\!1_{]k+1/k,+\infty[}(x) \Big)\, 1\!\!1_{]\frac{j}{k},\frac{j+1}{k}[}(y) \\ & + \sum_{i=-k^2}^{k^2} \Big(w^0_{i,-k^2,k} 1\!\!1_{]-\infty,-k[}(y) + w^0_{i,k^2,k} 1\!\!1_{]k+1/k,+\infty[}(y) \Big)\, 1\!\!1_{]\frac{i}{k},\frac{i+1}{k}[}(x) \\ & + \Big(w^0_{k^2,k^2,k} 1\!\!1_{]k+1/k,+\infty[}(y) + w^0_{k^2,-k^2,k} 1\!\!1_{]-\infty,-k[}(y) \Big)\, 1\!\!1_{]k+1/k,+\infty[}(x) \\ & + \Big(w^0_{-k^2,k^2,k} 1\!\!1_{]k+1/k,+\infty[}(y) + w^0_{-k^2,-k^2,k} 1\!\!1_{]-\infty,-k[}(y) \Big)\, 1\!\!1_{]-\infty,-k[}(x) \end{aligned} \tag{3.51}$$

We prove only the properties which are not obvious. Notice also that $\sqrt{m_{ijk}} \le \dfrac{1}{k} - \dfrac{1}{k^2} < \dfrac{1}{k}$. This element is crucial for blocks to be disjoint. We have (3.41), in particular since

$$\begin{aligned} \iint_{\mathbb{R}^2} \rho^0_k(x,y)\,dx\,dy \quad &= \quad \sum_{i,j=-k^2}^{k^2} m_{ijk} \\ &= \quad \sum_{i,j=-k^2}^{k^2} \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \int_{\frac{j}{k}}^{\frac{j+1}{k}-\frac{1}{k^2}} \rho^0(x,y)\,dx\,dy \\ &\le \quad \iint_{\mathbb{R}^2} \rho^0(x,y)\,dx\,dy. \end{aligned}$$

For $w = u$, v, p or q, we have the following BV estimate. For $y \in]\frac{j}{k}, \frac{j+1}{k}[$, the function w_k^0 takes the value

$$w_k^0(x,y) = \sum_{i=-k^2}^{k^2} w_{ijk}^0 \mathbb{1}_{]\frac{i}{k},\frac{i+1}{k}[}(x) + \left(w_{-k^2,j,k}^0 \mathbb{1}_{]-\infty,-k[}(x) + w_{k^2,j,k}^0 \mathbb{1}_{]k,+\infty[}(x) \right),$$

then we get, for such a y,

$$\int_{\mathbb{R}_x} \left| \partial_x w_k^0(x,y) \right| = \sum_{i=-k^2}^{k^2-1} |w_{i+1,jk}^0 - w_{ijk}^0|.$$

For $y \in]-\infty, -k[$, the function w_k^0 takes the value

$$w_k^0(x,y) = \sum_{i=-k^2}^{k^2} w_{i,-k^2,k}^0 \mathbb{1}_{]\frac{i}{k},\frac{i+1}{k}[}(x) + w_{k^2,-k^2,k}^0 \mathbb{1}_{]k+1/k,+\infty[}(x) + w_{-k^2,-k^2,k}^0 \mathbb{1}_{]-\infty,-k[}(x),$$

which gives, for such a y,

$$\int_{\mathbb{R}_x} \left| \partial_x w_k^0(x,y) \right| = \sum_{i=-k^2}^{k^2-1} |w_{i+1,-k^2,k}^0 - w_{i,-k^2,k}^0|.$$

Similarly, for $y \in]k+1/k, +\infty[$, we have

$$\int_{\mathbb{R}_x} \left| \partial_x w_k^0(x,y) \right| = \sum_{i=-k^2}^{k^2-1} |w_{i+1,k^2,k}^0 - w_{i,k^2,k}^0|.$$

Then, for any $y \in \mathbb{R}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}_x} \left| \partial_x w_k^0(x,y) \right| \quad = \quad & \sum_{i,j=-k^2}^{k^2-1} |w_{ijk}^0 - w_{i-1,jk}^0| \, \mathbb{1}_{]\frac{j}{k},\frac{j+1}{k}[}(y) \\ & + \sum_{i=-k^2}^{k^2-1} |w_{i+1,-k^2,k}^0 - w_{i,-k^2,k}^0| \, \mathbb{1}_{]-\infty,-k[}(y) \\ & + \sum_{i=-k^2}^{k^2-1} |w_{i+1,k^2,k}^0 - w_{i,k^2,k}^0| \, \mathbb{1}_{]k+1/k,+\infty[}(y), \end{aligned}$$

and we get

$$\begin{aligned} \iint_{\mathbb{R}\times[-a,a]} \left| \partial_x w_k^0(x,y) \right| \quad \leq \quad & \sum_{i,j=-k^2}^{k^2} |w_{ijk}^0 - w_{i-1,jk}^0| \frac{1}{k} \\ & + \sum_{i=-k^2}^{k^2-1} |w_{i+1,-k^2,k}^0 - w_{i,-k^2,k}^0| \, (a-k) \mathbb{1}_{a>k} \\ & + \sum_{i=-k^2}^{k^2-1} |w_{i+1,k^2,k}^0 - w_{i,k^2,k}^0| \, (a-k) \mathbb{1}_{a>k}. \end{aligned} \tag{3.52}$$

Now

$$
\begin{aligned}
|w^0_{ijk} - w^0_{i-1,jk}| &= k^2 \left| \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} w^0(x,y)\,dx\,dy - \int_{\frac{i-1}{k}}^{\frac{i}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} w^0(x,y)\,dx\,dy \right| \\
&= k^2 \left| \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} w^0(x,y) - w^0(x-\frac{1}{k},y)\,dx\,dy \right| \\
&\leq k^2 \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \left| w^0(x,y) - w^0(x-\frac{1}{k},y) \right| dx\,dy,
\end{aligned}
$$

therefore

$$
\begin{aligned}
\sum_{i,j=-k^2}^{k^2-1} |w^0_{ijk} - w^0_{i-1,jk}| \frac{1}{k} &\leq \sum_{i,j=-k^2}^{k^2} k \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \left| w^0(x,y) - w^0(x-\frac{1}{k},y) \right| dx\,dy \\
&\leq \iint\limits_{\mathbb{R}^2} \left| \frac{w^0(x,y) - w^0(x-\frac{1}{k},y)}{1/k} \right| dx\,dy \\
&\leq \iint\limits_{\mathbb{R}^2} \left| \partial_x w^0(x,y) \right|.
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
&\sum_{i=-k^2}^{k^2-1} |w^0_{i+1,-k^2,k} - w^0_{i,-k^2,k}|\,(a-k)\mathbb{1}_{a>k} \\
\leq\ & a\mathbb{1}_{a>k} \sum_{i=-k^2}^{k^2-1} k^2 \int_{\frac{i+1}{k}}^{\frac{i+2}{k}} \int_{-k}^{-k+1/k} \left| w^0(x,y) - w^0(x-\frac{1}{k},y) \right| dx\,dy \\
\leq\ & a^2 \int_{-k+\frac{1}{k}}^{k+\frac{1}{k}} \int_{-k}^{-k+1/k} \left| \frac{w^0(x,y) - w^0(x-\frac{1}{k},y)}{1/k} \right| dx\,dy
\end{aligned}
$$

and then

$$
\begin{aligned}
&\sum_{i=-k^2}^{k^2-1} |w^0_{i+1,-k^2,k} - w^0_{i,-k^2,k}|\,(a-k)\mathbb{1}_{a>k} + \sum_{i=-k^2}^{k^2-1} |w^0_{i+1,k^2,k} - w^0_{i,k^2,k}|\,(a-k)\mathbb{1}_{a>k} \\
&\leq a^2 \iint\limits_{\mathbb{R}^2} \left| \frac{w^0(x,y) - w^0(x-\frac{1}{k},y)}{1/k} \right| dx\,dy \\
&\leq a^2 \iint\limits_{\mathbb{R}^2} \left| \partial_x w^0(x,y) \right|.
\end{aligned}
$$

Finally we get

$$
\iint\limits_{\mathbb{R}\times[-a,a]} \left| \partial_x w^0_k(x,y) \right| \leq (1+a^2) \iint\limits_{\mathbb{R}^2} \left| \partial_x w^0(x,y) \right|.
$$

We turn now to convergences (3.46)-(3.47). Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ and let $k_0 \in \mathbb{N}$ such that $supp\ \varphi \subset [-k_0, k_0]^2$. For $k > k_0$, we have

$$\begin{aligned}
&\left| \iint\limits_{\mathbb{R}^2} p_k^0(x,y)\varphi(x,y)\,dx\,dy - \iint\limits_{\mathbb{R}^2} p^0(x,y)\varphi(x,y)\,dx\,dy \right| \\
\leq\ & \sum_{i,j=-kk_0}^{kk_0-1} \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} |p_{ijk}^0 - p^0(x,y)|\,|\varphi(x,y)|\,dx\,dy \\
\leq\ & \sum_{i,j=-kk_0}^{kk_0-1} \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \left|\Delta_{ijk}^{p^0}(x,y)\right| \|\varphi\|_\infty\ dx\,dy,
\end{aligned}$$

where

$$\Delta_{ijk}^{p^0}(x,y) = k^2 \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} (p^0(\tilde{x},\tilde{y}) - p^0(x,y))\,d\tilde{x}\,d\tilde{y}.$$

We have

$$\begin{aligned}
|\Delta_{ijk}^{p^0}(x,y)| &\leq k^2 \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} |p^0(\tilde{x},\tilde{y}) - p^0(x,y)|\,d\tilde{x}\,d\tilde{y} \\
&\leq k^2 \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \Big(|p^0(\tilde{x},\tilde{y}) - p^0(x,\tilde{y})| + |p^0(x,\tilde{y}) - p^0(x,y)|\Big)\ d\tilde{x}\,d\tilde{y} \\
&\leq k^2 \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \left(\left|\int_x^{\tilde{x}} \partial_x p^0(z,\tilde{y})\,dz\right| + \left|\int_y^{\tilde{y}} \partial_y p^0(\tilde{x},z)\,dz\right|\right)\ d\tilde{x}\,d\tilde{y} \\
&\leq k^2 \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \left(\int_{\frac{i}{k}}^{\frac{i+1}{k}} |\partial_x p^0(z,\tilde{y})|\,dz + \int_{\frac{j}{k}}^{\frac{j+1}{k}} |\partial_y p^0(\tilde{x},z)|\,dz\right)\ d\tilde{x}\,d\tilde{y},
\end{aligned}$$

thus

$$\begin{aligned}
&\left| \iint\limits_{\mathbb{R}^2} p_k^0(x,y)\varphi(x,y)\,dx\,dy - \iint\limits_{\mathbb{R}^2} p^0(x,y)\varphi(x,y)\,dx\,dy \right| \\
\leq\ & \|\varphi\|_\infty\, k^2 \sum_{i,j=-kk_0}^{kk_0-1} \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \int_{\frac{i}{k}}^{\frac{i+1}{k}} |\partial_x p^0(z,\tilde{y})|\,dz\,d\tilde{x}\,d\tilde{y}\,dx\,dy \\
& + \|\varphi\|_\infty\, k^2 \sum_{i,j=-kk_0}^{kk_0-1} \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} |\partial_y p^0(\tilde{x},z)|\,dz\,d\tilde{x}\,d\tilde{y}\,dx\,dy \\
\leq\ & \|\varphi\|_\infty\, k^2 \sum_{i,j=-kk_0}^{kk_0-1} \frac{1}{k^3} \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} |\partial_x p^0(z,\tilde{y})|\,dz\,d\tilde{y} \\
& + \|\varphi\|_\infty\, k^2 \sum_{i,j=-kk_0}^{kk_0-1} \frac{1}{k^3} \int_{\frac{i}{k}}^{\frac{i+1}{k}} \int_{\frac{j}{k}}^{\frac{j+1}{k}} |\partial_y p^0(\tilde{x},z)|\,dz\,d\tilde{x} \\
\leq\ & \|\varphi\|_\infty \frac{1}{k} \left(\iint\limits_{\mathbb{R}^2} |\partial_x p^0(x,y)| + \iint\limits_{\mathbb{R}^2} |\partial_y p^0(x,y)| \right) \underset{k\to+\infty}{\to} 0.
\end{aligned}$$

It gives the limit of the new term and we get that $p_k^0 \rightharpoonup p^0$ holds in the distributional sense. For convergences (3.46), we refer to the appendix A of [4] adapting the end of the proof with the argument just above. Since $\rho_k^0 p_k^0 \rightharpoonup \rho^0 p^0$ and $\rho^* p_k^0 \rightharpoonup \rho^* p^0$ and get $(\rho_k^0 - \rho^*) p_k^0 \rightharpoonup (\rho^0 - \rho^*) p^0 = 0$. Similarly we obtain $(\rho_k^0 - \rho^*) q_k^0 \rightharpoonup 0$. Finally, we have (3.47). □

Remark 3.1 In the paper [4], we use a different approximation :

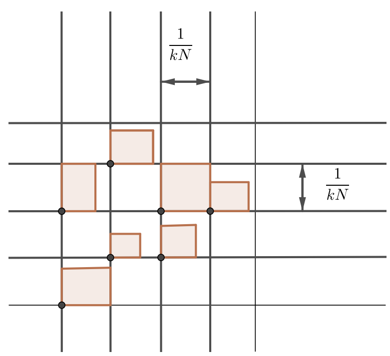
$$m_{ijk} = \int_{\frac{i}{k}}^{\frac{i+1}{k} - \frac{1}{k^2}} \int_{\frac{j}{k}}^{\frac{j+1}{k} - \frac{1}{k^2}} \rho^0(x, y)\, dx\, dy,$$

because we need that no collision happens to time $t = 0$.

Remark 3.2 In particular if we assume that u^0, v^0, p^0, q^0 are non-negative functions, then using (3.42)-(3.43), the blocks $u_k^0, v_k^0, p_k^0, q_k^0$ are non-negative functions.

3.2 Approximations of solutions for block initial data

First we present how we approximate the particular solutions of Section 2. This way, it gives an idea on how to deal with the general case which will be a succession of such local situations.



We define now the notion of discrete blocks we are going to use. The definition is more precise than in [4].

Definition 3.3 *Let $\Delta t, \Delta x, \Delta y > 0$. We call discrete block a function (ρ, u, v, p, q) depending on (t, x, y) of the form*

$$\rho(1, u, v, p, q)(t, x, y) = \sum_{i=-I}^{I'} \sum_{j=-J}^{J'} \sum_{l=0}^{+\infty} \rho_{ijl}(1, u_{ijl}, v_{ijl}, p_{ijl}, q_{ijl}) \mathbb{1}_{(t,x,y)\in P_{ijl}}, \tag{3.53}$$

at level of discretization $(\Delta t, \Delta x, \Delta y)$, where

$$\mathbb{1}_{(t,x,y)\in P_{ijl}} = \mathbb{1}_{l\Delta t \le t < (l+1)\Delta t} \mathbb{1}_{i\Delta x \le x < i\Delta x + a_{ijl}} \mathbb{1}_{j\Delta y \le y < j\Delta y + b_{ijl}}, \tag{3.54}$$

with $I, I', J, J' \in \mathbb{N}$ and, for $-I \le i \le I'$, $-J \le j \le J'$, $l \in \mathbb{N}$,

$$\rho_{ijl} \in \{0, \rho^*\},$$

$a_{ijl}, b_{ijl} \in \mathbb{R}$, $u_{ijl}, v_{ijl}, p_{ijl}, q_{ijl} \in [0, +\infty[$, *such that $0 \le a_{ijl} \le \Delta x$ and $0 \le b_{ijl} \le \Delta y$.*

Remark 3.3 It looks like the standard numerical discretization, taking a function piecewise constant on a square grid. But the density takes as only values 0 and ρ^* and we have to take into account the time evolution.

Remark 3.4 To simplify the presentation, we can assume that $I = J = 0$ which is just a translation of indices and $I' = J'$ by adding zero terms to have the same number of terms (adding some $\rho_{ijl}(1, u_{ijl}, v_{ijl}, p_{ijl}, q_{ijl})\mathbb{1}_{(x,y)\in P_{ij}}$ with $\rho_{ij} = 0$). In the following, we may sometimes use this change of notations by setting $N := I' + 1 = J' + 1$.

Remark 3.5 Notice that the definition of initial block and discrete blocks are consistent together because for $t = 0$, the relation (3.53) has only the term for $l = 0$ remaining and we get

$$\begin{aligned} & \rho(0,x,y)(1,u(0,x,y),v(0,x,y),p(0,x,y),q(0,x,y)) \\ = & \sum_{i=-I}^{I'}\sum_{j=-J}^{J'} \rho_{ij0}(1,u_{ij0},v_{ij0},p_{ij0},q_{ij0})\mathbb{1}_{(0,x,y)\in P_{ij0}}, \end{aligned}$$

with

$$\mathbb{1}_{(0,x,y)\in P_{ij0}} = \mathbb{1}_{i\Delta x\le x<i\Delta x+a_{ij0}}\mathbb{1}_{j\Delta y\le y<+j\Delta y+b_{ij0}}.$$

Remark 3.6 Alos, from now on, we will consider special choice of $\Delta t, \Delta x, \Delta y$, that is to say $\Delta x = \Delta y = 1/(kN)$ and $\Delta t = 1/N$ with $k, N \in N^*$.

For the case of free dynamics, we now approximate the solution of Proposition 2.1. We use the following approximation by discrete blocks.

Definition 3.4 *According to our initial block data of Proposition 3.2, the case we are going to use is the particular choice of $a_0, b_0 \in \frac{1}{k}\mathbb{Z}$ for $k \in \mathbb{N}$ and $0 \le c, d < 1/k$. We set $a_0 = r/k$ and $b_0 = s/k$ with $r, s \in \mathbb{Z}$.*

Let's set

$$\Delta x = \Delta y = 1/(kN) \text{ and } \Delta t = 1/N.$$

For any integer $N > 1/k$, there exists integers $0 \le l_N, m_N \le N$ such that $l_N\Delta x \le c < (l_N + 1)\Delta x$ and $m_N\Delta x \le d < (m_N + 1)\Delta y$. We consider the approximations given by the following sum of blocks:

$$(\rho_N, \rho_N u_N, \rho_N v_N)(t,x,y) = \sum_{i=0}^{l_N}\sum_{j=0}^{m_N}\sum_{l=0}^{+\infty} \rho^*(1,u,v)\mathbb{1}_{(t,x,y)\in P_{ijl}} \tag{3.55}$$

where

$$\mathbb{1}_{(t,x,y)\in P_{ijl}} = \mathbb{1}_{l\Delta t\le t<(l+1)\Delta t}\mathbb{1}_{a_l+i\Delta x\le x<a_l+(i+1)\Delta x}\mathbb{1}_{b_l+j\Delta y\le y<b_l+(j+1)\Delta y} \tag{3.56}$$

for $0 \le i \le l_N - 1$ and $0 \le j \le m_N - 1$ and

$$\mathbb{1}_{(t,x,y)\in P_{l_N jl}} = \mathbb{1}_{l\Delta t\le t<(l+1)\Delta t}\mathbb{1}_{a_l+l_N\Delta x\le x<a_l+c}\mathbb{1}_{b_l+j\Delta y\Delta y\le y<b_l+(j+1)\Delta y}$$

for $0 \le j \le m_N - 1$

$$\mathbb{1}_{(t,x,y)\in P_{l_N m_N l}} = \mathbb{1}_{l\Delta t\le t<(l+1)\Delta t}\mathbb{1}_{a_l+i\Delta x\le x<a_l+(i+1)\Delta x}\mathbb{1}_{b_l+m_N\Delta y\le y<b_l+d}$$

for $0 \le i \le l_N - 1$, *and*

$$\mathbb{1}_{(t,x,y)\in P_{l_N m_N l}} = \mathbb{1}_{l\Delta t \le t < (l+1)\Delta t} \mathbb{1}_{a_l + l_N \Delta x \le x < a_l + c} \mathbb{1}_{b_l + m_N \Delta y \le y < b_l + d}$$

with the sequences $(a_n)_n$ *and* $(b_n)_n$ *defined as follows. Starting from* a_0 *and* b_0*, we build the sequences as*

$$a_{2p+1} = a_0 + \left[\frac{2(p+1)u\Delta t}{\Delta x}\right]\Delta x, \qquad b_{2p+1} = b_{2p},$$

and

$$b_{2p+2} = b_0 + \left[\frac{2(p+1)v\Delta t}{\Delta y}\right]\Delta y, \qquad a_{2p+2} = a_{2p+1},$$

where the big square brackets denote the integer part.

Remark 3.7 *Notice that we can also easily add terms* $\rho_N p_N, \rho_N q_N$ *being this dynamics because they are just translated in the free dynamics evolution. And then if we have initialy* $(\rho^0 - \rho^*)p^0 \rightharpoonup 0$*, then we get* $(\rho_N - \rho^*)p_N \rightharpoonup 0$*. We choose not to note them here in order to keep the presentation simple.*

Remark 3.8 *At time* $t = (2k+1)\Delta t$*, we have a jump for the block in the* x *direction, and at time* $t = (2k+2)\Delta t$*, we have a jump for the block in the* y *direction, staying on the fixed grid at level* N *and taking an approximation of the movement.*

We have the following result which is an adaptation of a similar result of [4]. The main improvement is a more precise definition of blocks. It gives discrete blocks which approximate the solution of Proposition 2.1.

Proposition 3.5 *Let* $u, v, a_0, b_0 \in \mathbb{R}$ *such that* $a_0, b_0 \in \frac{1}{k}\mathbb{Z}$ *with* $k \in \mathbb{N}$ *and* $0 \le c, d < 1/k$*. Then there exists discrete blocks* $(\rho_N, \rho_N u_N, \rho_N v_N)$ *with initial data*

$$\mathbb{1}_{a_0 \le x \le a_0 + c} \mathbb{1}_{b_0 \le y \le b_0 + d}(1, u, v, 0, 0)$$

such that

$$\begin{cases} \partial_t \rho_N + \partial_x(\rho_N u_N) + \partial_y(\rho_N v_N) \rightharpoonup 0, \\ \partial_t(\rho_N u_N) + \partial_x(\rho_N u_N^2) + \partial_y(\rho_N u_N v_N) \rightharpoonup 0, \\ \partial_t(\rho_N v_N) + \partial_x(\rho_N u_N v_N) + \partial_y(\rho_N v_N^2) \rightharpoonup 0, \end{cases} \tag{3.57}$$

when $N \to +\infty$*, in the distributional sense.*

Proof. We set $a(t) = a_0 + ut$, $b(t) = b_0 + vt$,

$$a_\Delta(t) = \sum_{l=0}^{+\infty} a_l \mathbb{1}_{l\Delta t \le t < (l+1)\Delta t} \quad \text{and} \quad b_\Delta(t) = \sum_{l=0}^{+\infty} b_l \mathbb{1}_{l\Delta t \le t < (l+1)\Delta t},$$

where $(a_n)_n$ and $(b_n)_n$ are the sequences defined in Definition 3.4. It is proved in [4] that such functions satisfy

$$|a(t) - a_\Delta(t)| \le |u|\Delta t + \Delta x, \quad |b(t) - b_\Delta(t)| \le |v|\Delta t + \Delta y.$$

The important limit of this proof can be written in the following Lemma.

Lemma 3.6 *Setting, for any test function* $\varphi \in C_c^\infty([0,+\infty[,\mathbb{R}^2)$,

$$A(\varphi) = \int_0^{+\infty} \iint_{\mathbb{R}^2} \rho(t,x,y)\varphi(t,x,y)\,dy\,dx\,dt$$

and

$$A_N(\varphi) = \int_0^{+\infty} \iint_{\mathbb{R}^2} \rho_N(t,x,y)\varphi(t,x,y)\,dy\,dx\,dt.$$

Then we have $A_N(\varphi) \to A(\varphi)$ *when* $N \to +\infty$.

Proof. We have

$$A(\varphi) = \rho^* \sum_{l=0}^{+\infty} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a(t)}^{a(t)+c} \int_{b(t)}^{b(t)+d} \varphi(t,x,y)\,dy\,dx\,dt \tag{3.58}$$

and

$$\begin{aligned}
A_N(\varphi) \quad = \quad & \rho^* \sum_{i=0}^{l_N-1} \sum_{j=0}^{m_N-1} \sum_{l=0}^{+\infty} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a_l+i\Delta x}^{a_l+(i+1)\Delta x} \int_{b_l+j\Delta y}^{b_l+(j+1)\Delta y} \varphi(t,x,y)\,dy\,dx\,dt \\
& +\rho^* \sum_{j=0}^{m_N-1} \sum_{l=0}^{+\infty} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a_l+l_N\Delta x}^{a_l+c} \int_{b_l+j\Delta y}^{b_l+(j+1)\Delta y} \varphi(t,x,y)\,dy\,dx\,dt \\
& +\rho^* \sum_{i=0}^{l_N-1} \sum_{l=0}^{+\infty} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a_l+i\Delta x}^{a_l+(i+1)\Delta x} \int_{b_l+m_N\Delta y}^{b_l+d} \varphi(t,x,y)\,dy\,dx\,dt \\
& +\rho^* \sum_{l=0}^{+\infty} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a_l+l_N\Delta x}^{a_l+c} \int_{b_l+m_N\Delta y}^{b_l+d} \varphi(t,x,y)\,dy\,dx\,dt.
\end{aligned}$$

Let us denote by T a real such that the support in time of φ is in $[0,T]$. Denote by L_N an integer such that $(L_N-1)\Delta t \le T \le L_N\Delta t$. We have

$$\begin{aligned}
A_N(\varphi) - A(\varphi) \quad = \quad & \rho^* \sum_{i=0}^{l_N-1} \sum_{j=0}^{m_N-1} \sum_{l=0}^{+\infty} \int_{l\Delta t}^{(l+1)\Delta t} \left(\int_{a_l+i\Delta x}^{a_l+(i+1)\Delta x} \int_{b_l+j\Delta y}^{b_l+(j+1)\Delta y} \varphi(t,x,y)\,dy\,dx \right. \\
& \left. - \int_{a(t)+i\Delta x}^{a(t)+(i+1)\Delta x} \int_{b(t)+j\Delta y}^{b(t)+(j+1)\Delta y} \varphi(t,x,y)\,dy\,dx \right) dt \\
& +\rho^* \sum_{j=0}^{m_N-1} \sum_{l=0}^{+\infty} \int_{l\Delta t}^{(l+1)\Delta t} \left(\int_{a_l+l_N\Delta x}^{a_l+c} \int_{b_l+j\Delta y}^{b_l+(j+1)\Delta y} \varphi(t,x,y)\,dy\,dx \right. \\
& \left. - \int_{a(t)+l_N\Delta x}^{a(t)+c} \int_{b(t)+j\Delta y}^{b(t)+(j+1)\Delta y} \varphi(t,x,y)\,dy\,dx \right) dt \\
& +\rho^* \sum_{i=0}^{l_N-1} \sum_{l=0}^{+\infty} \int_{l\Delta t}^{(l+1)\Delta t} \left(\int_{a_l+i\Delta x}^{a_l+(i+1)\Delta x} \int_{b_l+m_N\Delta y}^{b_l+d} \varphi(t,x,y)\,dy\,dx \right.
\end{aligned}$$

$$
\begin{aligned}
& \quad - \int_{a(t)+i\Delta x}^{a(t)+(i+1)\Delta x} \int_{b(t)+m_N\Delta y}^{b(t)+d} \varphi(t,x,y)\,dy\,dx \Big)\,dt \\
& + \rho^* \sum_{l=0}^{+\infty} \int_{l\Delta t}^{(l+1)\Delta t} \Big(\int_{a_l+l_N\Delta x}^{a_l+c} \int_{b_l+m_N\Delta y}^{b_l+d} \varphi(t,x,y)\,dy\,dx \\
& \quad - \int_{a(t)+l_N\Delta x}^{a(t)+c} \int_{b(t)+m_N\Delta y}^{b(t)+d} \varphi(t,x,y)\,dy\,dx \Big)\,dt \\
= \; & \rho^* \sum_{i=0}^{l_N-1} \sum_{j=0}^{m_N-1} \sum_{l=0}^{L_N} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a_l+i\Delta x}^{a_l+(i+1)\Delta x} \int_{b_l+j\Delta y}^{b_l+(j+1)\Delta y} \\
& \quad - \Big(\varphi(t,x,y) - \varphi(t,x+a(t)-a_l, y+b(t)-b_l)\Big)\,dy\,dx\,dt \\
& + \rho^* \sum_{j=0}^{m_N-1} \sum_{l=0}^{L_N} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a_l+l_N\Delta x}^{a_l+c} \int_{b_l+j\Delta y}^{b_l+(j+1)\Delta y} \\
& \quad - \Big(\varphi(t,x,y) - \varphi(t,x+a(t)-a_l, y+b(t)-b_l)\Big)\,dy\,dx\,dt \\
& + \rho^* \sum_{i=0}^{l_N-1} \sum_{l=0}^{L_N} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a_l+i\Delta x}^{a_l+(i+1)\Delta x} \int_{b_l+m_N\Delta y}^{b_l+d} \\
& \quad - \Big(\varphi(t,x,y) - \varphi(t,x+a(t)-a_l, y+b(t)-b_l)\Big)\,dy\,dx\,dt \\
& + \rho^* \sum_{l=0}^{L_N} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a_l+l_N\Delta x}^{a_l+c} \int_{b_l+m_n\Delta y}^{b_l+d} \\
& \quad - \Big(\varphi(t,x,y) - \varphi(t,x+a(t)-a_l, y+b(t)-b_l)\Big)\,dy\,dx\,dt
\end{aligned}
$$

Let $\varepsilon > 0$. Since φ is continuous and has a compact support, there exists $\eta > 0$ such that for any (t, x_1, y_1) and (t, x_2, y_2) in the support of φ, if $|x_1 - x_2| \leq \eta$ and $|y_1 - y_2| \leq \eta$, then $|\varphi(t, x_1, y_1) - \varphi(t, x_2, y_2)| \leq \varepsilon$.

Let $N_0 \in \mathbb{N}^*$ be such that N_0 is greater than $(|u| + 1/k)/\eta$ and $(|v| + 1/k)/\eta$. Let $N \in N^*$ be greater than N_0. Now

$$|a(t) - a_\Delta(t)| \leq |u|\Delta t + \Delta x = |u|\frac{1}{N} + \frac{1}{kN} \leq \eta$$

and $|b(t) - b_\Delta(t)| \leq |v|\dfrac{1}{N} + \dfrac{1}{kN} \leq \eta$, therefore

$$
\begin{aligned}
|A_N(\varphi) - A(\varphi)| \;\leq\; & \rho^* \sum_{i=0}^{l_N-1} \sum_{j=0}^{m_N-1} \sum_{l=0}^{L_N} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a_l+i\Delta x}^{a_l+(i+1)\Delta x} \int_{b_l+j\Delta y}^{b_l+(j+1)\Delta y} \varepsilon\,dy\,dx\,dt \\
& + \rho^* \sum_{j=0}^{m_N-1} \sum_{l=0}^{L_N} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a_l+l_N\Delta x}^{a_l+c} \int_{b_l+j\Delta y}^{b_l+(j+1)\Delta y} \varepsilon\,dy\,dx\,dt \\
& + \rho^* \sum_{i=0}^{l_N-1} \sum_{l=0}^{L_N} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a_l+i\Delta x}^{a_l+(i+1)\Delta x} \int_{b_l+m_N\Delta y}^{b_l+d} \varepsilon\,dy\,dx\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+\rho^* \sum_{l=0}^{L_N} \int_{l\Delta t}^{(l+1)\Delta t} \int_{a_l+l_N\Delta x}^{a_l+c} \int_{b_l+m_n\Delta y}^{b_l+d} \varepsilon \, dy\, dx\, dt \\
\leq\; & \rho^* \sum_{i=0}^{l_N-1} \sum_{j=0}^{m_N-1} \sum_{l=0}^{L_N} \Delta t \Delta x \Delta y \varepsilon + \rho^* \sum_{j=0}^{m_N-1} \sum_{l=0}^{L_N} \Delta t \Delta x \Delta y \varepsilon \\
&+\rho^* \sum_{i=0}^{l_N-1} \sum_{l=0}^{L_N} \Delta t \Delta x \Delta y \varepsilon + \rho^* \sum_{l=0}^{L_N} \Delta t \Delta x \Delta y \varepsilon \\
\leq\; & \rho^*(l_N\Delta x)(m_N\Delta y)(L_N\Delta t)\varepsilon + \rho^*\Delta x(m_N\Delta y)(L_N\Delta t)\varepsilon \\
&+\rho^*(l_N\Delta x)\Delta y(L_N\Delta t)\varepsilon + \rho^*\Delta x\Delta y(L_N\Delta t)\varepsilon \\
\leq\; & 4\rho^* cdT\, \varepsilon.
\end{aligned} \tag{3.59}
$$

It gives that $A_N(\varphi) \to A(\varphi)$ when $N \to +\infty$. □

End of Proof of Proposition 3.5. We use the previous Lemma to conclude to our Proposition. Let $\varphi \in C_c^\infty([0,+\infty[,\mathbb{R}^2)$. The solution $(\tilde\rho, \tilde\rho\tilde u, \tilde\rho\tilde u)$ of Proposition 2.1 satisfies

$$
\begin{aligned}
0 &= \int_0^{+\infty} \iint\limits_{\mathbb{R}^2} \left(\tilde\rho\partial_t\varphi + \tilde\rho\tilde u\partial_x\varphi + \tilde\rho\tilde v\partial_y\varphi\right) dy\, dx\, dt \\
&= A(\partial_t\varphi) + uA(\partial_x\varphi) + vA(\partial_y\varphi).
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\int_0^{+\infty} \iint\limits_{\mathbb{R}^2} \left(\rho_N\partial_t\varphi + \rho_N u_N\partial_x\varphi + \rho_N v_N\partial_y\varphi\right) dy\, dx\, dt \\
=\; & A_N(\partial_t\varphi) + uA_N(\partial_x\varphi) + vA_N(\partial_y\varphi).
\end{aligned}
$$

Since $A(\partial_t\varphi) + uA(\partial_x\varphi) + vA(\partial_y\varphi) = 0$, then we get that

$$
A_N(\partial_t\varphi) + uA_N(\partial_x\varphi) + vA_N(\partial_y\varphi) \underset{N\to+\infty}{\to} 0
$$

applying the previous Lemma to $\partial_t\varphi$, $\partial_x\varphi$ and $\partial_y\varphi$. That is to say

$$
\int_0^{+\infty} \iint\limits_{\mathbb{R}^2} \left(\rho_N\partial_t\varphi + \rho_N u_N\partial_x\varphi + \rho_N v_N\partial_y\varphi\right) dy\, dx\, dt \underset{Nt\to+\infty}{\to} 0
$$

for any test function φ.
Since the speeds u and v are constant, they can be factorized on each term, thus we get that

$$
\int_0^{+\infty} \iint\limits_{\mathbb{R}^2} \left(\rho_N u_N\partial_t\varphi + \rho_N u_N^2\partial_x\varphi + \rho_N v_N u_N\partial_y\varphi\right) dy\, dx\, dt \underset{N\to+\infty}{\to} 0
$$

and

$$
\int_0^{+\infty} \iint\limits_{\mathbb{R}^2} \left(\rho_N v_N\partial_t\varphi + \rho_N u_N v_N\partial_x\varphi + \rho_N v_N^2\partial_y\varphi\right) dy\, dx\, dt \underset{N\to+\infty}{\to} 0
$$

for any test function φ. □

Remark 3.9 We can also easily add terms $\rho_N p_N, \rho_N q_N$ in this dynamics because they are unchanged in the free dynamics evolution.

We define now the dynamics in the case of a collision, to approximate the solution of proposition 2.3 by discrete blocks. We first consider that the shock happens during the x direction movement in the splitting.

Remark 3.10 As for the free dynamics case, blocks are initialy considered in subdivision with path $1/k$ that we refined by approximations by $1/(kN)$.

Definition 3.7 *Let $u_l, p_l, v_l, q_l, u_r, p_r, v_r, q_r \geq 0$ and $\rho^* > 0$. Assume that $u_l > u_r$. We take, for $k, N \in \mathbb{N}^*$,*

$$\Delta x = \Delta y = 1/(kN) \text{ and } \Delta t = 1/N.$$

We start at $t = L\Delta t$ from a situation with two disjoint blocks:

$$\begin{aligned}(\rho, \rho u, \rho v, \rho p, \rho q)(t, x, y) &= \rho^*(1, u_l, v_l, p_l, q_l)\mathbb{1}_{x\in[\alpha^l - P\Delta x, \alpha^l[}\mathbb{1}_{y\in[b,b+\mu[}\\ &\quad +\rho^*(1, u_r, v_r, p_r, q_r)\mathbb{1}_{x\in[\alpha^r, \alpha^r+Q\Delta x[}\mathbb{1}_{y\in[b,b+\mu[},\end{aligned}$$

with $P, Q \in \mathbb{N}$, $\alpha^l, \alpha^r \in \frac{1}{kN}\mathbb{Z}$ and such that $0 < \dfrac{\alpha^r - \alpha^l}{u_l - u_r} \leq \Delta t$. Then a collision has to happen in time $t^* = L\Delta t + \dfrac{\alpha^r - \alpha^l}{u_l - u_r}$ between $L\Delta t$ and $(L+1)\Delta t$. Notice that L changes with Δt, that is to say with N. Notice that the values on a discret block are unchanged on $[L\Delta t, (L+1)\Delta t[$. In order to have the conservation of the mass and a good approximation of the velocity and of the available speed, we take for time $t = (L+1)\Delta t$ the dynamics:

$$\begin{aligned}(\rho, \rho u, \rho v, \rho p, \rho q)(t, x, y) &= \rho^*(1, u_r, v_l, p_l + u_l - u_r, q_l)\mathbb{1}_{x\in[\alpha^f - P\Delta x, \alpha^f[}\mathbb{1}_{y\in[b,b+\mu[}\\ &\quad +\rho^*(1, u_r, v_r, p_r, q_r)\mathbb{1}_{x\in[\alpha^f, \alpha^f+Q\Delta x[}\mathbb{1}_{y\in[b,b+\mu[}\end{aligned}$$

where

$$\alpha^f = \left[\frac{u_r t^* + \alpha^r + u_r(\Delta t - t^*)}{\Delta x}\right]\Delta x = \left[\frac{\alpha^r + u_r\Delta t}{\Delta x}\right]\Delta x,$$

with the big square brackets denoting the integer part.

Remark 3.11 We have similar formulas for a shock in the y direction substituting Δx by Δy and u by v.

Remark 3.12 Notice that to simplify the presentation, we consider two blocks sharing the same length in the y direction, which is the fixed direction during this step of the movement. Indeed, in areas where they have different lengths, only the common area will intervene, otherwise the block moves at constant speed. It would be the opposite in the x direction for the next step of the motion.

We prove now that the discrete blocks defined previously are approximations of the solution of proposition 2.3.

Proposition 3.8 *We denote by* $(\rho_N, u_N, v_N, p_N, q_N)$ *the discrete blocks constructed in definition 3.7. Then the functions* $(\rho_N, \rho_N u_N, \rho_N v_N, \rho_N p_N, \rho_N q_N)$ *tend in the distributional sense, when* $N \to +\infty$*, to the function* $(\rho, \rho u, \rho v, \rho p, \rho q)$ *of proposition 2.3. Furthermore the functions satisfy*

$$0 \le u_N \le esssup\ u^0, \quad 0 \le v_N \le esssup\ v^0,$$

$$0 \le p_N \le esssup\ u^0 + esssup\ p^0, \quad 0 \le q_N \le esssup\ v^0 + esssup\ q^0.$$

Proof. We consider the case of a shock in the x direction with the previous notations. Let's denote by L (keep in mind that L changes with Δt, that is to say with N) the integer such that $t^* \in [L\Delta t, (L+1)\Delta t[$. We fix ε small enough in order that $t^* > L\Delta t + \varepsilon$ and assume that there was no shock betweeen $(L+1)\Delta t$ and $(L+1)\Delta t + \varepsilon$. Thus before $L\Delta t + \varepsilon$ and after $(L+1)\Delta t + \varepsilon$, the movement is without constraint, which we have studied already. Notice also that after the shock, the positions of the blocks move as in the case without constraints starting with the new defined positions at the instant of shock.
On $[L\Delta t + \varepsilon, (L+1)\Delta t + \varepsilon[$, the part of the functions located near the collision can be written as

$$\begin{aligned}
& (\rho_N, \rho_N u_N, \rho_N v_N, \rho_N p_N, \rho_N q_N)(t,x,y) \\
= \ & \rho^*(1, u_l, v, p_l, q) \mathbb{1}_{L\Delta t+\varepsilon \le t < (L+1)\Delta t} \mathbb{1}_{\alpha^l - P\Delta x \le x < \alpha^l} \mathbb{1}_{b \le y < b+\mu} \\
& + \rho^*(1, u_r, v, p_r, q) \mathbb{1}_{L\Delta t+\varepsilon \le t < (L+1)\Delta t} \mathbb{1}_{\alpha^r \le x < \alpha^r + Q\Delta x} \mathbb{1}_{b \le y < b+\mu} \\
& + \rho^*(1, u_r, v, p_l + u_l - u_r, q) \mathbb{1}_{(L+1)\Delta t \le t < (L+1)\Delta t + \varepsilon} \mathbb{1}_{\alpha^f - P\Delta x \le x < \alpha^f} \mathbb{1}_{b \le y < b+\mu} \\
& + \rho^*(1, u_r, v, p_r, q) \mathbb{1}_{(L+1)\Delta t \le t < (L+1)\Delta t + \varepsilon} \mathbb{1}_{\alpha^f \le x < \alpha^f + Q\Delta x} \mathbb{1}_{b \le y < b+\mu}.
\end{aligned}$$

We consider a test function $\varphi \in C_c^\infty([0, +\infty[, \mathbb{R}^2)$ and $S : \mathbb{R} \to \mathbb{R}$ a continuous function. We have

$$\begin{aligned}
& \int_0^{+\infty} \iint_{\mathbb{R}^2} (\rho_N \partial_t \varphi + \rho_N u_N \partial_x \varphi + \rho_N v_N \partial_y \varphi)\, S(u_N + p_N)\, dy\, dx\, dt \qquad (3.60) \\
= \ & \int_{L\Delta t+\varepsilon}^{(L+1)\Delta t} \int_{\alpha^l - P\Delta x}^{\alpha^l} \int_b^{b+\mu} \rho^* \left(\partial_t \varphi + u_l \partial_x \varphi + v \partial_y \varphi\right) S(u_l + p_l)\, dy\, dx\, dt \\
& + \int_{L\Delta t+\varepsilon}^{(L+1)\Delta t} \int_{\alpha^r}^{\alpha^r + Q\Delta x} \int_b^{b+\mu} \rho^* \left(\partial_t \varphi + u_r \partial_x \varphi + v \partial_y \varphi\right) S(u_r + p_r)\, dy\, dx\, dt \\
& + \int_{(L+1)\Delta t}^{(L+1)\Delta t+\varepsilon} \int_{\alpha^f - P\Delta x}^{\alpha^f} \int_b^{b+\mu} \rho^* \left(\partial_t \varphi + u_r \partial_x \varphi + v \partial_y \varphi\right) S(u_l + p_l)\, dy\, dx\, dt \\
& + \int_{(L+1)\Delta t}^{(L+1)\Delta t+\varepsilon} \int_{\alpha^f}^{\alpha^f + Q\Delta x} \int_b^{b+\mu} \rho^* \left(\partial_t \varphi + u_r \partial_x \varphi + v \partial_y \varphi\right) S(u_r + p_r)\, dy\, dx\, dt \\
& + R_N(\varphi),
\end{aligned}$$

where $R_N(\varphi) \to 0$ corresponding to the part of ρ_N which follows a movement without constraints and has already been studied. We denote by $(\rho, \rho u, \rho\ v, \rho p, \rho\ q)$ the solution defined in Proposition 2.4 with $c = P\Delta x$ and $d = Q\Delta y$. We have

$$\int_0^{+\infty} \iint_{\mathbb{R}^2} (\rho \partial_t \varphi + \rho u \partial_x \varphi + \rho v \partial_y \varphi)\, S(u + p)\, dy\, dx\, dt \qquad (3.61)$$

$$
\begin{aligned}
&= \int_{L\Delta t+\varepsilon}^{t^*} \int_{a_l(t)-c}^{a_l(t)} \int_{b(t)}^{b(t)+\mu} \rho^* \left(\partial_t\varphi + u_l\partial_x\varphi + v_l\partial_y\varphi\right) S(u_l+p_l)\, dy\, dx\, dt \\
&\quad + \int_{L\Delta t+\varepsilon}^{t^*} \int_{a_r(t)}^{a_r(t)+d} \int_{b(t)}^{b(t)+\mu} \rho^* \left(\partial_t\varphi + u_r\partial_x\varphi + v_r\partial_y\varphi\right) S(u_r+p_r)\, dy\, dx\, dt \\
&\quad + \int_{t^*}^{(L+1)\Delta t+\varepsilon} \int_{a_r(t)-c}^{a_r(t)} \int_{b(t)}^{b(t)+\mu} \rho^* \left(\partial_t\varphi + u_r\partial_x\varphi + v_l\partial_y\varphi\right) S(u_l+p_l)\, dy\, dx\, dt \\
&\quad + \int_{t^*}^{(L+1)\Delta t+\varepsilon} \int_{a_r(t)}^{a_r(t)+d} \int_{b(t)}^{b(t)+\mu} \rho^* \left(\partial_t\varphi + u_r\partial_x\varphi + v_r\partial_y\varphi\right) S(u_r+p_r)\, dy\, dx\, dt \\
&= \int_{L\Delta t+\varepsilon}^{t^*} \int_{\alpha^f-P\Delta x}^{\alpha^f} \int_{b}^{b+\mu} \rho^* \left(\partial_t\varphi + u_l\partial_x\varphi + v_l\partial_y\varphi\right)(t, x-\alpha^f+a_l(t), y-b+b(t)) S(u_l+p_l)\, dy\, dx\, dt \\
&\quad + \int_{L\Delta t+\varepsilon}^{t^*} \int_{\alpha^f}^{\alpha^f+Q\Delta y} \int_{b}^{b+\mu} \rho^* \left(\partial_t\varphi + u_r\partial_x\varphi + v_r\partial_y\varphi\right)(t, x-\alpha^f+a_r(t), y-b+b(t)) S(u_r+p_r)\, dy\, dx\, dt \\
&\quad + \int_{t^*}^{(L+1)\Delta t+\varepsilon} \int_{\alpha^f-P\Delta x}^{\alpha^f} \int_{b}^{b+\mu} \rho^* \left(\partial_t\varphi + u_r\partial_x\varphi + v_l\partial_y\varphi\right)(t, x-\alpha^f+a_r(t), y-b+b(t)) S(u_l+p_l)\, dy\, dx\, dt \\
&\quad + \int_{t^*}^{(L+1)\Delta t+\varepsilon} \int_{\alpha^f}^{\alpha^f+Q\Delta y} \int_{b}^{b+\mu} \rho^* \left(\partial_t\varphi + u_r\partial_x\varphi + v_r\partial_y\varphi\right)(t, x-\alpha^f+a_r(t), y-b+b(t)) S(u_r+p_r)\, dy\, dx\, dt
\end{aligned}
$$

For $S(z)=1$ and $S(z)=z$, the term (3.61) equals to zero. We study the difference between (3.60) and (3.61).

We have then to consider the quantity

$$
\begin{aligned}
B_N(\varphi) &= \int_{L\Delta t+\varepsilon}^{t^*} \int_{\alpha^f-c}^{\alpha^f} \int_{b}^{b+\mu} \Big(\varphi(t,x,y) - \varphi(t, x-\alpha^f+a_l(t), y)\Big)\, dy\, dx\, dt \\
&\quad + \int_{L\Delta t+\varepsilon}^{t^*} \int_{\alpha^f}^{\alpha^f+d} \int_{b}^{b+\mu} \Big(\varphi(t,x,y) - \varphi(t, x-\alpha^f+a_r(t), y)\Big)\, dy\, dx\, dt \\
&\quad + \int_{t^*}^{(L+1)\Delta t+\varepsilon} \int_{\alpha^f-c}^{\alpha^f+d} \int_{b}^{b+\mu} \Big(\varphi(t,x,y) - \varphi(t, x-\alpha^f+a^f(t), y)\Big)\, dy\, dx\, dt.
\end{aligned}
$$

We have $a_l(t) = \alpha^l + u_l(t - L\Delta)$, $a_r(t) = \alpha^r + u_r(t - L\Delta t)$ and $x^* = \alpha^l + u_l(t^* - L\Delta t)$, then for $t \in [L\Delta t, (L+1)\Delta t[$,

$$
|\alpha^f - a^f(t)| \le |u_f(t - L\Delta t)| + \Delta x \le |u_f|\Delta t + \Delta x,
$$

$$
|\alpha^f - a_l(t)| \le |(u_l - u_f)(t^* - L\Delta t)| + |u_f(\Delta t - t)| + \Delta x \le (|u_l - u_f| + |u_f|)\Delta t + \Delta x,
$$

and

$$
|\alpha^f - a_r(t)| \le |(u_r - u_f)(t^* - L\Delta t)| + |u_f(\Delta t - t)| + \Delta x \le (|u_r - u_f| + |u_f|)\Delta t + \Delta x.
$$

Then we do as in the case without constraints (for the terms $A_N(\varphi)$) to get that $B_N(\varphi) \to 0$ when $N \to +\infty$. The L^∞ bounds are obvious because of the definition of the dynamics. □

We obtain a similar result to build discrete blocks which are approximations of the solution of Proposition 2.4.

3.3 General case of approximations of solutions and BV estimates

We want now to get approximations of solutions for any block initial data of the form of our approximation processus, that is to say with the following form (3.50)-(3.51). These blocks have the form

$$\begin{aligned}(\rho^0(x,y), \quad & \rho^0(x,y)u^0(x,y), \rho^0(x,y)v^0(x,y), \rho^0(x,y)p^0(x,y), \rho^0(x,y)q^0(x,y)) \\ &= \sum_{i=-I}^{I'} \sum_{j=-J}^{J'} \rho^*(1, u_{ij}, v_{ij}, p_{ij}, q_{ij}) \mathbb{1}_{a_{ij}\leq x\leq b_{ij}} \mathbb{1}_{c_{ij}\leq y\leq d_{ij}} \end{aligned} \tag{3.62}$$

which is a linear sum of terms as the ones considered in previous subsections. Then we have the following merging result.

Proposition 3.9 *Let $\rho^* > 0$. Let $\rho^0 \in L^1(\mathbb{R}^2)$, $u^0, v^0, p^0, q^0 \in L^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$ such that $0 \leq \rho^0 \leq \rho^*$. We consider the sequence of block initial data $(\rho_k^0, u_k^0, v_k^0, p_k^0, q_k^0)_{k\geq 1}$ defined by (3.50)-(3.51). Then, for any $k \in \mathbb{N}^*$, there exists $(\rho_{kl}, u_{kl}, v_{kl}, p_{kl}, q_{kl})_l$ discrete blocks associated to the initial data $(\rho_k^0, u_k^0, v_k^0, p_k^0, q_k^0)$ such that*

$$\begin{cases} \partial_t \rho_{kl} + \partial_x(\rho_{kl}u_{kl}) + \partial_y(\rho_{kl}v_{kl}) = R_{kl} \rightharpoonup 0, \\ \partial_t(\rho_{kl}u_{kl}) + \partial_x(\rho_{kl}u_{kl}(u_{kl}+p_{kl})) + \partial_y(\rho_{kl}(u_{kl}+p_{kl})v_{kl}) = S_{kl} \rightharpoonup 0, \\ \partial_t(\rho_{kl}v_{kl}) + \partial_x(\rho_{kl}u_{kl}(v_{kl}+q_{kl})) + \partial_y(\rho_{kl}v_{kl}(v_{kl}+q_{kl})) = T_{kl} \rightharpoonup 0, \end{cases} \tag{3.63}$$

and $(\rho_{kl} - \rho^*)p_{kl} \rightharpoonup 0$, $(\rho_{kl} - \rho^*)q_{kl} \rightharpoonup 0$ when $l \to +\infty$, in the distributional sense.

Proof. As long as there is no collision, each block moves freely and then locally proposition 3.5 gives approximations of the solution by discrete blocks. Until the first collision between two blocks, $(\rho_{kl}, \rho_{kl}u_{kl}, \rho_{kl}v_{kl}, \rho_{kl}p_{kl}, \rho_{kl}q_{kl})$ is thus defined locally by the sum of functions like defined in definition 3.4. Every time a collision between two blocks happens, let us say during a movement in direction x (it is similar in the y direction), proposition 3.8 gives an approximation of the solution by discrete blocks, thus at this time, the corresponding part of $(\rho_{kl}, \rho_{kl}u_{kl}, \rho_{kl}v_{kl}, \rho_{kl}p_{kl}, \rho_{kl}q_{kl})$ is modified according to definition 3.7. Then, it moves freely as in proposition 3.5 until the next collision. This way, it defined approximations of solutions as expected. □

We move on now to the proof of L^∞ and BV estimates for these functions.

Proposition 3.10 *The blocks of proposition 3.9 satisfy, for any $t \geq 0$,*

$$0 \leq \rho_{kl} \leq \rho^*, \tag{3.64}$$

$$0 \leq u_{kl} \leq esssup\ u^0, \quad 0 \leq v_{kl} \leq esssup\ v^0, \tag{3.65}$$

$$0 \leq p_{kl} \leq esssup\ u^0 + esssup\ p^0, \quad 0 \leq q_{kl} \leq esssup\ v^0 + esssup\ q^0, \tag{3.66}$$

$$\iint_{\mathbb{R}\times[-a,a]} |\partial_x w_{kl}(t,x,y)| \leq \iint_{\mathbb{R}\times[-a_t,a_t]} |\partial_x w_k^0(x,y)|, \tag{3.67}$$

$$\iint_{[-a,a]\times\mathbb{R}} |\partial_y w_{kl}(t,x,y)| \leq \iint_{[-a_t,a_t]\times\mathbb{R}} |\partial_y w_k^0(x,y)|, \tag{3.68}$$

for $w = u$ and v,

$$\iint\limits_{\mathbb{R}\times[-a,a]} |\partial_x p_{kl}(t,x,y)| \le \iint\limits_{\mathbb{R}\times[-a_t,a_t]} |\partial_x u_k^0(x,y)| + \iint\limits_{\mathbb{R}\times[-a_t,a_t]} |\partial_x p_k^0(x,y)|, \tag{3.69}$$

$$\iint\limits_{[-a,a]\times\mathbb{R}} |\partial_y p_{kl}(t,x,y)| \le \iint\limits_{[-a_t,a_t]\times\mathbb{R}} |\partial_y u_k^0(x,y)| + \iint\limits_{[-a_t,a_t]\times\mathbb{R}} |\partial_y p_k^0(x,y)| \tag{3.70}$$

$$\iint\limits_{\mathbb{R}\times[-a,a]} |\partial_x q_{kl}(t,x,y)| \le \iint\limits_{\mathbb{R}\times[-a_t,a_t]} |\partial_x v_k^0(x,y)| + \iint\limits_{\mathbb{R}\times[-a_t,a_t]} |\partial_x q_k^0(x,y)|, \tag{3.71}$$

$$\iint\limits_{[-a,a]\times\mathbb{R}} |\partial_y q_{kl}(t,x,y)| \le \iint\limits_{[-a_t,a_t]\times\mathbb{R}} |\partial_y v_k^0(x,y)| + \iint\limits_{[-a_t,a_t]\times\mathbb{R}} |\partial_y q_k^0(x,y)| \tag{3.72}$$

for any $a > 0$ and setting

$$a_t = a + t\max(\|u^0\|_\infty + \|p^0\|_\infty, \|v^0\|_\infty + \|q^0\|_\infty).$$

Proof. The L^∞ bounds are obvious by construction. For simplicity, we skip the indice k and l and denote by u a function u_{kl}. We have a relation like (3.52). It allows to consider the evolution across shocks of quantities

$$\sum_{i=2}^{n^0} |u_i^0 - u_{i-1}^0| \quad \text{and} \quad \sum_{i=2}^{n^0} |p_i^0 - p_{i-1}^0|$$

with u_i^0 the velocities of the successive blocks and i is the indice in the x or y direction if we are located during an evolution in x or in y. If a collision happens at time t^* between blocks k and $k+1$ during this evolution, then blocks k and $k+1$ merge with velocity u_{k+1}, then the variations of u become

$$|u_2^0 - u_i^0| + |u_3^0 - u_2^0| + \ldots + |u_{k+2}^0 - u_k^0| + \ldots + |u_n^0 - u_{n-1}^0|.$$

Shock after shock on x direction or y direction, we get a non-increasing of the quantity $\iint\limits_{\mathbb{R}\times[-a,a]} |\partial_x u_{kl}(t,x,y)|$ and $\iint\limits_{\mathbb{R}\times[-a,a]} |\partial_y u_{kl}(t,x,y)|$ and we get (3.67) and (3.68) for $w = u$ taking into account the localisation of the blocks. On an other side, on a collision at time t^* between blocks k and $k+1$ during this evolution, then blocks k and $k+1$ merge with velocity u_{k+1} but on the term p, we have an increase from $|p_{k+2} - p_{k+1}| + |p_{k+2} - p_{k+1}|$ to $|p_{k+2} - p_{k+1}| + |u_{k+1} - u_k| + |p_{k+2} - p_{k+1}|$. Thus we get (3.69) and (3.70). We obtain similarly (3.67) and (3.68) for $w = v$ and (3.71)-(3.72). □

Remark 3.13 The combination of the bounds (3.44)-(3.45) and (3.67)-(3.72) gives

$$\iint\limits_{\mathbb{R}\times[-a,a]} |\partial_x u_{kl}(t,x,y)| \le \iint\limits_{\mathbb{R}\times[-a_t,a_t]} \left|\partial_x u_k^0(x,y)\right| \le (1+a_t^2) \iint\limits_{\mathbb{R}^2} \left|\partial_x u^0(x,y)\right|, \tag{3.73}$$

$$\iint\limits_{[-a,a]\times\mathbb{R}} |\partial_y u_{kl}(t,x,y)| \leq \iint\limits_{[-a_t,a_t]\times\mathbb{R}} \left|\partial_y u_k^0(x,y)\right| \leq (1+a_t^2) \iint\limits_{\mathbb{R}^2} \left|\partial_y u^0(x,y)\right|, \tag{3.74}$$

$$\iint\limits_{\mathbb{R}\times[-a,a]} |\partial_x v_{kl}(t,x,y)| \leq \iint\limits_{\mathbb{R}\times[-a_t,a_t]} \left|\partial_x v_k^0(x,y)\right| \leq (1+a_t^2) \iint\limits_{\mathbb{R}^2} \left|\partial_x v^0(x,y)\right|, \tag{3.75}$$

$$\iint\limits_{[-a,a]\times\mathbb{R}} |\partial_y v_{kl}(t,x,y)| \leq \iint\limits_{[-a_t,a_t]\times\mathbb{R}} \left|\partial_y v_k^0(x,y)\right| \leq (1+a_t^2) \iint\limits_{\mathbb{R}^2} \left|\partial_y v^0(x,y)\right|, \tag{3.76}$$

$$\begin{aligned} \iint\limits_{\mathbb{R}\times[-a,a]} |\partial_x p_{kl}(t,x,y)| &\leq \iint\limits_{\mathbb{R}\times[-a_t,a_t]} \left|\partial_x u_k^0(x,y)\right| + \iint\limits_{\mathbb{R}\times[-a_t,a_t]} \left|\partial_x p_k^0(x,y)\right| \\ &\leq (1+a_t^2)\left(\iint\limits_{\mathbb{R}^2} \left|\partial_x u^0(x,y)\right| + \iint\limits_{\mathbb{R}^2} \left|\partial_x p^0(x,y)\right|\right), \end{aligned} \tag{3.77}$$

$$\begin{aligned} \iint\limits_{[-a,a]\times\mathbb{R}} |\partial_y p_{kl}(t,x,y)| &\leq \iint\limits_{[-a_t,a_t]\times\mathbb{R}} \left|\partial_y u_k^0(x,y)\right| + \iint\limits_{[-a_t,a_t]\times\mathbb{R}} \left|\partial_y p_k^0(x,y)\right| \\ &\leq (1+a_t^2)\left(\iint\limits_{\mathbb{R}^2} \left|\partial_y u^0(x,y)\right| + \iint\limits_{\mathbb{R}^2} \left|\partial_y p^0(x,y)\right|\right), \end{aligned} \tag{3.78}$$

$$\begin{aligned} \iint\limits_{\mathbb{R}\times[-a,a]} |\partial_x q_{kl}(t,x,y)| &\leq \iint\limits_{\mathbb{R}\times[-a_t,a_t]} \left|\partial_x v_k^0(x,y)\right| + \iint\limits_{\mathbb{R}\times[-a_t,a_t]} \left|\partial_x q_k^0(x,y)\right| \\ &\leq (1+a_t^2)\left(\iint\limits_{\mathbb{R}^2} \left|\partial_x v^0(x,y)\right| + \iint\limits_{\mathbb{R}^2} \left|\partial_x q^0(x,y)\right|\right), \end{aligned} \tag{3.79}$$

$$\begin{aligned} \iint\limits_{[-a,a]\times\mathbb{R}} |\partial_y q_{kl}(t,x,y)| &\leq \iint\limits_{[-a_t,a_t]\times\mathbb{R}} \left|\partial_y v_k^0(x,y)\right| + \iint\limits_{[-a_t,a_t]\times\mathbb{R}} \left|\partial_y q_k^0(x,y)\right| \\ &\leq (1+a_t^2)\left(\iint\limits_{\mathbb{R}^2} \left|\partial_y v^0(x,y)\right| + \iint\limits_{\mathbb{R}^2} \left|\partial_y q^0(x,y)\right|\right), \end{aligned} \tag{3.80}$$

for any $a > 0$ *and with* $a_t = a + t\max(\|u^0\|_\infty + \|p^0\|_\infty, \|v^0\|_\infty + \|q^0\|_\infty)$.

We have now to get stability results in order to get solution for the system with constraint for a large class of initial data.

4 Existence results

4.1 Limit of approximations of solutions

In dimension one, we have directly obtained explicit solutions for any block initial data. In the current two-dimension case, at this stage, we only have approximations of solutions for general block initial data. We need first to pass to the limit of the third arrow of figure 1, that is to say in the case where we only have

$$\begin{cases} \partial_t \rho_l + \partial_x(\rho_l u_l) + \partial_y(\rho_l v_l) = R_l \rightharpoonup 0, \\ \partial_t(\rho_l(u_l + p_l)) + \partial_x(\rho_l u_l(u_l + p_l)) + \partial_y(\rho_l(u_l + p_l)v_l) = S_l \rightharpoonup 0, \\ \partial_t(\rho_l(v_l + q_l)) + \partial_x(\rho_l u_l(v_l + q_l)) + \partial_y(\rho_l v_l(v_l + q_l)) = T_l \rightharpoonup 0 \end{cases} \tag{4.81}$$

when $l \to +\infty$, with a limit in the distribution sense, instead of having $R_l = S_l = T_l = 0$. We prove now that in this situation, we can extract a subsequence whose limit is a solution.

Theorem 4.1 (Limit of approximations) *Let $\rho^* > 0$. Consider initial data $(\rho^0, u^0, v^0, p^0, q^0)$ with regularities (1.14) such that (1.20)-(1.21). We consider the sequence of blocks initial data $(\rho_k^0, u_k^0, v_k^0, p_k^0, q_k^0)_{k\geq 1}$ defined by (3.50)-(3.51). For any k, this initial data allows to consider the sequence $(\rho_{kl}, u_{kl}, v_{kl}, p_{kl}, q_{kl})_l$ defined by proposition 3.9. Then, extracting a subsequence if necessary, as $l \to +\infty$, we have*

$$(\rho_{kl}, u_{kl}, v_{kl}, p_{kl}, q_{kl}) \rightharpoonup (\rho_k, u_k, v_k, p_k, q_k)$$

in the distributional sense, where $(\rho_k, u_k, v_k, p_k, q_k)$ with regularities (1.15)-(1.16), are weak solutions of (1.10), that is to say (1.17)-(1.19), with constraints (1.11)-(1.12) and initial data $(\rho_k^0, u_k^0, v_k^0, p_k^0, q_k^0)$. and satisfy the bounds

$$0 \leq \rho_k \leq 1, \quad \iint_{\mathbb{R}^2} \rho_k(t,x,y)\,dx\,dy \leq \iint_{\mathbb{R}^2} \rho^0(x,y)\,dx\,dy, \tag{4.82}$$

$$0 \leq u_k \leq esssup\ u^0, \quad 0 \leq v_k \leq esssup\ v^0, \tag{4.83}$$

$$0 \leq p_k \leq esssup\ u^0 + esssup\ p^0, \quad 0 \leq q_k \leq esssup\ u^0 + esssup\ q^0, \tag{4.84}$$

$$\iint_{\mathbb{R}\times[-a,a]} |\partial_x w_k(t,x,y)| \leq (1 + a_t^2) \iint_{\mathbb{R}^2} \left|\partial_x w^0(x,y)\right|, \tag{4.85}$$

$$\iint_{[-a,a]\times\mathbb{R}} |\partial_y w_k(t,x,y)| \leq (1 + a_t^2) \iint_{\mathbb{R}^2} \left|\partial_y w^0(x,y)\right|, \tag{4.86}$$

for $w = u$ and v and

$$\iint_{\mathbb{R}\times[-a,a]} |\partial_x p_k(t,x,y)| \leq (1 + a_t^2) \left(\iint_{\mathbb{R}^2} \left|\partial_x u^0(x,y)\right| + \iint_{\mathbb{R}^2} \left|\partial_x p^0(x,y)\right| \right), \tag{4.87}$$

$$\iint_{[-a,a]\times\mathbb{R}} |\partial_y p_k(t,x,y)| \le (1+a_t^2)\left(\iint_{\mathbb{R}^2} \left|\partial_y u^0(x,y)\right| + \iint_{\mathbb{R}^2} \left|\partial_y p^0(x,y)\right|\right), \tag{4.88}$$

$$\iint_{\mathbb{R}\times[-a,a]} |\partial_x q_k(t,x,y)| \le (1+a_t^2)\left(\iint_{\mathbb{R}^2} \left|\partial_x v^0(x,y)\right| + \iint_{\mathbb{R}^2} \left|\partial_x q^0(x,y)\right|\right), \tag{4.89}$$

$$\iint_{[-a,a]\times\mathbb{R}} |\partial_y q_k(t,x,y)| \le (1+a_t^2)\left(\iint_{\mathbb{R}^2} \left|\partial_y v^0(x,y)\right| + \iint_{\mathbb{R}^2} \left|\partial_y q^0(x,y)\right|\right), \tag{4.90}$$

for any $a>0$ and with $a_t = a + t\max(\|u^0\|_\infty + \|p^0\|_\infty, \|v^0\|_\infty + \|q^0\|_\infty)$.

Proof. Since $(\rho_k, u_k, v_k, p_k, q_k)_{k\ge 1}$ are bounded in $L^\infty(]0,+\infty[\times\mathbb{R}^2)$, then there exists a subsequence such that

$$\rho_k \rightharpoonup \rho, \quad u_k \rightharpoonup u, \quad v_k \rightharpoonup v, \quad p_k \rightharpoonup p, \quad q_k \rightharpoonup q \quad \text{in } L^\infty_{w*}(]0,+\infty[\times\mathbb{R}^2). \tag{4.91}$$

From the first equation of (4.81), the sequence $(\rho_k)_{k\ge1}$ satisfies the estimate:

$$\left|\iint_{\mathbb{R}^2} (\rho_{kl}(t,x,y) - \rho_{kl}(s,x,y))\varphi(x,y)\,dx\,dy\right| \le C_\varphi|t-s| + \left|\int_s^t \iint_{\mathbb{R}^2} R_{kl}\varphi\right|, \tag{4.92}$$

with

$$C_\varphi = \sup_{k\ge1}\|u_k^0\|_{L^\infty}\left(\iint_{\mathbb{R}^2}|\partial_x\varphi|\,dx\,dy\right) + \sup_{k\ge1}\|v_k^0\|_{L^\infty}\left(\iint_{\mathbb{R}^2}|\partial_y\varphi|\,dx\,dy\right).$$

Adapting the proof of (3.59) but on a time space of length $|t-s|$ instead of T, we similarly get a bound for $\int_s^t \iint_{\mathbb{R}^2} R_{kl}\varphi$ of the form $|t-s|\varepsilon C$ (instead of $T\varepsilon C$). Then we get again a bound of the form

$$\left|\iint_{\mathbb{R}^2} (\rho_{kl}(t,x,y) - \rho_{kl}(s,x,y))\varphi(x,y)\,dx\,dy\right| \le \tilde{C}_\varphi|t-s|. \tag{4.93}$$

Then, applying lemma 5.2, we have $\rho_{kl} \to \rho_k$ in $C([0,T], L^\infty_{w*}(\mathbb{R}^2))$ when $l\to+\infty$. Furthermore $(u_k)_{k\ge1}$ is bounded in $BV_{loc}(\mathbb{R}^2)$ uniformly in time sur $[0,T]$. We can then apply lemma 5.1, with $C_a = (1+a_T^2)K$, with $K = \max(K_1,K_2)$ and we get that $\rho_{kl}u_{kl} \rightharpoonup \rho_k u_k$ in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$. Similarly, we have $\rho_{kl}v_{kl} \rightharpoonup \rho_k v_k$, $\rho_{kl}p_{kl} \rightharpoonup \rho_k p_k$ and $\rho_{kl}q_{kl} \rightharpoonup \rho_k q_k$ in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$.

Now the second equation of (4.81) gives that

$$\frac{d}{dt}\iint_{\mathbb{R}^2} \rho_{kl}(u_{kl}+p_{kl})(t,x,y)\varphi(x,y)\,dx\,dy$$
$$= \iint_{\mathbb{R}^2} \rho_{kl}u_{kl}(u_{kl}+p_{kl})(t,x,y)\partial_x\varphi(x,y)\,dx\,dy + \iint_{\mathbb{R}^2} \rho_k(u_{kl}+p_{kl})v_{kl}(t,x,y)\partial_y\varphi(x,y)\,dx\,dy + \iint_{\mathbb{R}^2} S_l\varphi,$$

thus the sequence $\displaystyle\iint_{\mathbb{R}^2} \rho_{kl}(u_{kl}+p_{kl})(t,x,y)\varphi(x,y)\,dx\,dy$ is bounded in BV_t. Therefore, in the same pattern as the proof of lemma 5.2 (see also [7]), we can extract a subsequence such that

$$\iint_{\mathbb{R}^2} \rho_{kl}(u_{kl}+p_{kl})(t,x,y)\varphi(x,y)\,dx\,dy \underset{l\to+\infty}{\to} \iint_{\mathbb{R}^2} \rho_k(u_k+p_k)(t,x,y)\varphi(x,y)\,dx\,dy \text{ in } L^1(]0,T[),$$

for all $\varphi \in C_c^\infty(\mathbb{R}^2)$. We can then apply lemma 5.1 with $\gamma_{kl} = \rho_{kl}(u_{kl}+p_{kl})$ this time and $\omega_{kl} = u_{kl}$ (and also with v_{kl}) and we get that $\rho_{kl}(u_{kl}+p_{kl})u_{kl} \rightharpoonup \rho_k(u_k+p_k)u_k$ and $\rho_{kl}(u_{kl}+p_{kl})v_{kl} \rightharpoonup \rho_k(u_k+p_k)v$ in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$. Similarly, we also have $\rho_{kl}u_{kl}(v_{kl}+q_{kl}) \rightharpoonup \rho_k u_k(v_k+q_k)$ and $\rho_k v_k(v_k+q_{kl}) \rightharpoonup \rho_k v_k(v_k+q_k)$ in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$. We can now pass to the limit in the weak formulation to get (1.17)-(1.19) with the initial data (ρ^0, u^0, v^0). Now $\rho_{kl}p_{kl} \rightharpoonup \rho_k p_k$ and $p_{kl} \rightharpoonup p_k$, thus $p_{kl}(\rho_{kl}-\rho^*) \rightharpoonup p_k(\rho_k-\rho^*)$. But we also have $p_{kl}(\rho_{kl}-\rho^*) \rightharpoonup 0$. Then by unicity of the limit, we get $p(\rho-\rho^*)=0$. Similarly, we have $q(\rho-\rho^*)=0$. □

The first consequence of this result is that we will obtain solutions for any block initial data (not explicit in every cases here contrary to the one-dimensional case). Then we will get existence of solutions for any initial data.

4.2 Existence result

We prove here the existence theorem 1.1.

Proof of theorem 1.1. Let ρ_k^0, u_k^0, v_k^0, p_k^0, q_k^0 ($k \in \mathbb{N}^*$) be the block initial data defined by (3.50)-(3.51) associated to ρ^0, u^0, v^0, p^0, q^0 provided by proposition 3.2. Proposition 3.9 gives, for any k, a sequence $(\rho_{kl}, u_{kl}, v_{kl}, p_{kl}, q_{kl})_l$ such that

$$\begin{cases} \partial_t\rho_{kl} + \partial_x(\rho_{kl}u_{kl}) + \partial_y(\rho_{kl}v_{kl}) = R_{kl} \rightharpoonup 0, \\ \partial_t(\rho_{kl}(u_{kl}+p_{kl})) + \partial_x(\rho_{kl}u_{kl}(u_{kl}+p_{kl})) + \partial_y(\rho_{kl}(u_{kl}+p_{kl})v_{kl}) = S_{kl} \rightharpoonup 0, \\ \partial_t(\rho_{kl}(v_{kl}+q_{kl})) + \partial_x(\rho_{kl}u_{kl}(v_{kl}+q_{kl})) + \partial_y(\rho_{kl}v_{kl}(v_{kl}+q_{kl})) = T_{kl} \rightharpoonup 0 \end{cases}$$

in the distributional sense. At k fixed, these functions satisfy the bounds of theorem 4.1 and we can apply it to get that, up to subsequence, and making a diagonal Cantor process, the convergence $(\rho_{kl}, u_{kl}, v_{kl}, p_{kl}, q_{kl}) \underset{l\to+\infty}{\rightharpoonup} (\rho_k, u_k, v_k, p_k, q_k)$ holds in the distributional sense for any k. The previous obtained limits $(\rho_k, u_k, v_k, p_k, q_k)$, with regularities (1.15)-(1.16), are weak solutions of (1.10), with constraints (1.11)-(1.12) and initial data $(\rho_k^0, u_k^0, v_k^0, p_k^0, q_k^0)$, and satisfy the bounds (4.82)-(4.90). We can now adapt the proof of Theorem 4.1, it is easier here because we have 0 instead of the terms R_l, S_l and T_l, to this sequence, and get, up to a subsequence when $k\to\infty$, $(\rho_k, u_k, v_k, p_k, q_k) \rightharpoonup (\rho, u, v, p, q)$, where (ρ, u, v, p, q), are weak solutions of (1.10), with constraints (1.11)-(1.12) and initial data $(\rho^0, u^0, v^0, p^0, q^0)$, and satisfy the bounds (1.22)-(1.30). □

5 Appendix: technical results

The first result is to help us passing to the limit in the products. It is an extension in dimension two of a similar lemma in dimension one proved in [3]. The proof can be found in [4]. Notice that we also have to consider locally BV bounds.

Lemma 5.1 *Consider for any $k \in \mathbb{N}$, some functions $\gamma_k \in L^\infty(]0,T[\times\mathbb{R}^2)$, $\omega_k \in L^\infty(]0,T[, BV_{loc}(\mathbb{R}^2))$ and $\gamma \in L^\infty(]0,T[\times\mathbb{R}^2)$, $\omega \in L^\infty(]0,T[, BV_{loc}(\mathbb{R}^2))$. Let us assume that $(\gamma_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^\infty(]0,T[\times\mathbb{R}^2)$ that tends to γ in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$, and satisfies, for any $\Gamma \in C_c^\infty(\mathbb{R}^2)$,*

$$\iint\limits_{\mathbb{R}^2} (\gamma_k - \gamma)(t,x,y)\Gamma(x,y)\,dx\,dy \underset{k\to+\infty}{\to} 0, \tag{5.1}$$

either i) a.e. $t \in]0,T[$ or ii) in $L^1_t(]0,T[)$. Let us also assume that $(\omega_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^\infty(]0,T[\times\mathbb{R}^2)$ that tends to ω in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$, and assume that, for any $a > 0$, there exists $C_a > 0$ such that, for any $t \in [0,T]$,

$$\iint\limits_{[-a,a]^2} |\partial_x \omega_k(t,x,y)| \le C_a, \qquad \iint\limits_{[-a,a]^2} |\partial_y \omega_k(t,x,y)| \le C_a, \qquad \text{for any } k. \tag{5.2}$$

Then $\gamma_k\omega_k \rightharpoonup \gamma\omega$ in $L^\infty_{w*}(]0,T[\times\mathbb{R}^2)$, as $k \to +\infty$.

Remark 5.1 This is a result of compensated compactness, which uses the compactness in (x,y) for $(\omega_k)_k$ given by (5.2) and the weak compactness in t for $(\gamma_k)_k$ given by (5.1) to pass to the weak limit in the product $\gamma_k\omega_k$.

The second result gives some continuity in time. The proof is an easy adaptation in dimension two of lemma 4.4 of [7]. The main idea is to use a countable dense set in $C_c^\infty(\mathbb{R}^2)$ for the L^1-norm and Ascoli's theorem. Since there is no new difficulty, we skip the proof here.

Lemma 5.2 *Let $(n_k)_{k\in\mathbb{N}^*}$ be a bounded sequence in $L^\infty(]0,T[\times\mathbb{R}^2)$ which satisfies: for all $\varphi \in C_c^\infty(\mathbb{R}^2)$, the sequence $\left(\int_{\mathbb{R}} n_k(t,x,y)\varphi(x,y)\,dx\,dy\right)_k$ is uniformly Lipschitz continuous on $[0,T]$, i.e. $\exists C_\varphi > 0$, $\forall k \in \mathbb{N}^*$, $\forall s,t \in [0,T]$,*

$$\left| \iint\limits_{\mathbb{R}^2} (n_k(t,x,y) - n_k(s,x,y))\varphi(x,y)\,dx\,dy \right| \le C_\varphi |t-s|.$$

Then, up to a subsequence, it exists $n \in L^\infty(]0,T[\times\mathbb{R}^2)$ such that $n_k \to n$ in $C([0,T], L^\infty_{w*}(\mathbb{R}^2))$, i.e.

$$\forall \Gamma \in L^1(\mathbb{R}^2), \qquad \sup_{t\in[0,T]} \left| \iint\limits_{\mathbb{R}^2} (n_k(t,x,y) - n(t,x,y))\Gamma(x,y)\,dx\,dy \right| \underset{k\to+\infty}{\to} 0.$$

We conclude the Annex by the proof of Lemma 2.2.
Proof of Lemma 2.2. Notice that

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{a_1(t)}^{a_2(t)} \int_{b_1(t)}^{b_2(t)} \varphi(t,x,y)\,dy\,dx\right) \\
= \;& \int_{a_1(t)}^{a_2(t)} \int_{b_1(t)}^{b_2(t)} \partial_t \varphi(t,x,y)\,dy\,dx \\
&+ \int_{b_1(t)}^{b_2(t)} (\varphi(t,a_2(t),y)a_2'(t) - \varphi(t,a_1(t),y)a_1'(t))\,dy \\
&+ \int_{a_1(t)}^{a_2(t)} (\varphi(t,x,b_2(t))b_2'(t) - \varphi(t,x,b_1(t))b_1'(t))\,dx.
\end{aligned}$$

Integrating this relation between σ and s, we have

$$\begin{aligned}
&\left(\int_{a_1(s)}^{a_2(s)}\int_{b_1(t)}^{b_2(t)} \varphi(t,x,y)\,dy\,dx\right) - \left(\int_{a_1(\sigma)}^{a_2(\sigma)}\int_{b_1(t)}^{b_2(t)} \varphi(t,x,y)\,dy\,dx\right)\\
= \; & \int_\sigma^s \int_{a_1(t)}^{a_2(t)}\int_{b_1(t)}^{b_2(t)} \partial_t\varphi(t,x,y)\,dy\,dx\,dt\\
&+\int_\sigma^s \int_{b_1(t)}^{b_2(t)} (\varphi(t,a_2(t),y)-\varphi(t,a_1(t),y))u\,dy\,dt\\
&+\int_\sigma^s \int_{a_1(t)}^{a_2(t)} (\varphi(t,x,b_2(t))-\varphi(t,x,b_1(t)))v\,dx\,dt\\
= \; & \int_\sigma^s \int_{a_1(t)}^{a_2(t)}\int_{b_1(t)}^{b_2(t)} \partial_t\varphi(t,x,y)\,dy\,dx\,dt\\
&+\int_\sigma^s \int_{b_1(t)}^{b_2(t)}\int_{a_1(t)}^{a_2(t)} \partial_x\varphi(t,x,y)\,dx\,u\,dy\,dt\\
&+\int_\sigma^s \int_{a_1(t)}^{a_2(t)}\int_{b_1(t)}^{b_2(t)} \partial_y\varphi(t,x,y)\,v\,dx\,dt\\
= \; & \int_\sigma^s \int_{a_1(t)}^{a_2(t)}\int_{b_1(t)}^{b_2(t)} \left(\partial_t\varphi(t,x,y)+u\partial_x\varphi(t,x,y)+v\partial_y\varphi(t,x,y)\right)\,dy\,dx\,dt. \;\square
\end{aligned}$$

References

[1] A. Aw, M. Rascle, *Resurrection of second order models of traffic flow*, SIAM J. Appl. Math. 60 (2000), 916–938

[2] L. Barthélemy, *Problème d'obstacle pour une équation quasi-linéaire du premier ordre*, Ann. Fac. Sci. Toulouse Math. 9 (1988), 137–159.

[3] F. Berthelin, *Existence and weak stability for a two-phase model with unilateral constraint*, Math. Models & Methods in the Applied Sciences, 12 (2002), 249–272.

[4] F. Berthelin, *Theoretical study of a multi-dimensional pressureless model with unilateral constraint*, SIAM, Journal on Mathematical Analysis, vol. 49, no. 3, (2017), 2287–2320.

[5] F. Berthelin, *Numerical flux-splitting for a class of hyperbolic systems with unilateral constraint*, Mathematical Modelling and Numerical Analysis 37 no. 3, (2003), 479–494.

[6] F. Berthelin, F. Bouchut, *Weak solutions for a hyperbolic system with unilateral constraint and mass loss*, Annales de l'Institut Henry Poincaré, Analyse non linéaire 20 (2003), 975–997.

[7] F. Berthelin, D. Broizat, *A model for the evolution of traffic jams in multi-lane*, Kinetic and Related Models, Vol. 5, no. 4, (2012), 697–728.

[8] F. Berthelin, P. Degond, M. Delitala, M. Rascle, *Model for the formation and evolution of traffic jams*, ARMA, Archives for Rational Mechanics Analysis, 187, no. 2 (2008), 185–220.

[9] F. Berthelin, P. Degond, V. LeBlanc, S. Moutari, M. Rascle, J. Royer, *A traffic-flow model with constraint for the modeling of traffic jams*, M3AS, Mathematical Models & Methods in Applied Science, Vol. 18, no. 1 (2008), 1269–1298.

[10] F. Berthelin, P. Goatin, Particle approximation of a constrained model for traffic flow, *NoDEA, Nonlinear Differential Equations and applications*, vol. 24, no. 5 (2017), 24–55.

[11] F. Bouchut, *On zero pressure gas dynamics*, Advances in kinetic theory and computing, 171-190, Ser. Adv. Math. Appl. Sci. 22 (1994), World Sci. Publishing, River Edge, NJ.

[12] F. Bouchut, Y. Brenier, J. Cortes, J.-F. Ripoll, *A hierarchy of models for two-phase flows*, Journal of Nonlinear Science 10 (2000), 639–660.

[13] F. Bouchut, F. James, *Duality solutions for pressureless gases, monotone scalar conservation laws, and uniqueness*, Comm. Partial Differential Equation 24 (1999), 2173–2189.

[14] Y. Brenier, E. Grenier, *Sticky particles and scalar conservation laws*, SIAM J. Numer. Anal. 35 (1998) 2317–2328.

[15] D. Bresch, C. Perrin, E. Zatorska, *Singular limit of a Navier–Stokes system leading to a free/congested zones two-phase model*, Comp. Rendus Math. 352 (2014), 685–690.

[16] R. Bürger, P. Goatin, D. Inzunza and L.M. Villada, *A non-local pedestrian flow model accounting for anisotropic interactions and domain boundaries*, Math. Biosci. Eng., 17, no. 5 (2020), 5883-5906

[17] C. Chalons, P. Goatin and N. Seguin, *General constrained conservation laws. Application to pedestrian flow modeling*, Netw. Heterog. Media, 8, no. 2 (2013), 433–463.

[18] R.M. Colombo, E. Rossi *Modeling Crowd Movements in Domains with Boundaries*, IMA Journal of Applied Mathematics 84, no. 5 (2019), 833–853.

[19] P. Degond and J. Hua, *Modeling Self-organized hydrodynamics with congestion and path formation in crowds*, Journal of Computational Physics, 237 (2013), 299–319.

[20] W. E., Y.G. Rykov, Y.G. Sinai, *Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics*, Comm. Math. Phys. 177 (1996), 349–380.

[21] G. Gagneux, A.-M. Lefevere, M. Madaune-Tort, *Une approche analytique d'un modèle black oil des écoulements triphasiques compressibles en ingénierie pétrolière*, J. Méc. Théor. Appl. 6 (1987), no. 4, 547–569.

[22] E. Grenier, *Existence globale pour le système des gaz sans pression*, C. R. Acad. Sci. Paris Sér. I Math. 321 (1995), 171–174.

[23] M. Ishii, *Thermo-fluid dynamic theory of two-phase flow*, Eyrolles, Paris, 1975.

[24] L. Lévi, *Problèmes unilatéraux pour des équations non linéaires de convection-réaction*, Ann. Fac. Sci. Toulouse Math. 4 (1995), 593–631.

[25] L. Lévi, *Obstacle problems for scalar conservation laws*, ESAIM: M2AN 35 (2001), 575–593.

[26] N. Masmoudi, P.-L. Lions, *On a free boundary barotropic model*, Ann. Inst. H. Poincaré Anal. Nonlinéaire 16 (1999), 373–410.

[27] C. Perrin, E. Zatorska, *Free/congested two-phase model from weak solutions to multi-dimensional compressible Navier-Stokes equations*, Comm. Partial Differential Equations 40, no. 8 (2015), 1558–1589.

[28] J. Wells, *Bounded continuous vector-valued functions on a locally compact space*, Michigan Math. J., Volume 12, issue 1 (1965), 119–126.

[29] M. Zhang, *A non-equilibrium traffic model devoid of gas-like behavior*, Transportation Res. B, 36 (2002), 275–298.