

A RELATION BETWEEN THE PARABOLIC CHERN CHARACTERS OF THE DE RHAM BUNDLES

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ABSTRACT. In this paper, we consider the weight i de Rham–Gauss–Manin bundles on a smooth variety arising from a smooth projective morphism $f : X_U \rightarrow U$ for $i \geq 0$. We associate to each weight i de Rham bundle, a certain parabolic bundle on S and consider their parabolic Chern characters in the rational Chow groups, for a good compactification S of U . We show the triviality of the alternating sum of these parabolic bundles in the (positive degree) rational Chow groups. This removes the hypothesis of semistable reduction in the original result of this kind due to Esnault and Viehweg.

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1. INTRODUCTION

Suppose X and S are irreducible projective varieties defined over the complex numbers and $\pi : X \rightarrow S$ is a morphism such that the restriction $X_U \rightarrow U$ over a nonsingular dense open set is smooth of relative dimension n . The following bundle on U , for $i \geq 0$,

$$\mathcal{H}^i := R^i \pi_* \Omega_{X_U/U}^\bullet$$

is equipped with a flat connection ∇ , called as the Gauss–Manin connection. We call the pair (\mathcal{H}^i, ∇) the de Rham bundle or the Gauss–Manin bundle of weight i .

Suppose S is a nonsingular compactification of U such that $D := S - U$ is a normal crossing divisor and the associated local system $\ker \nabla$ has unipotent monodromies along the components of D . The bundle $\overline{\mathcal{H}}^i$ is the *canonical extension* of \mathcal{H}^i ([De1]) and is

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equipped with a logarithmic flat connection $\overline{\nabla}$. It is characterised by the property that it has nilpotent residues.

By the Chern–Weil theory, the de Rham Chern classes

$$c_i^{dR}(\mathcal{H}^k) \in H_{dR}^{2i}(U)$$

vanish, and by a computation shown in [Es-Vi1, Appendix B], the de Rham classes

$$c_i^{dR}(\overline{\mathcal{H}}^k) \in H_{dR}^{2i}(S)$$

vanish too. The essential fact used is that the residues of $\overline{\nabla}$ are nilpotent.

The algebraic Chern–Simons theory initiated by S. Bloch and H. Esnault ([BE]) studies the Chern classes (denoted by c_i^{Ch}) of flat bundles in the rational Chow groups of U and S . It is conjectured by H. Esnault that the classes $c_i^{Ch}(\mathcal{H}^k)$ and $c_i^{Ch}(\overline{\mathcal{H}}^k)$ are trivial for all $i > 0$ and k ([Es1, p. 187–188], [Es2]).

The cases where it is known to be true are as follows. In [Mu], Mumford proved this for any family of stable curves. In [vdG], van der Geer proved that $c_i^{Ch}(\mathcal{H}^1)$ is trivial when $X \rightarrow S$ is a family of abelian varieties. Further, for any family of abelian varieties of dimension g , the rational Chow group elements $c_i^{Ch}(\overline{\mathcal{H}}^1)$, $i \geq 1$, were proved to be trivial by Iyer under the assumption that $g \leq 5$ ([Iy]) and by Esnault and Viehweg and for all $g > 0$ ([Es-Vi2]). Further, for some families of moduli spaces, Biswas and Iyer ([Bi-Iy]) have checked the triviality of the classes in the rational Chow groups.

In this paper we consider parabolic bundles associated to logarithmic connections (section 3) instead of canonical extensions. By Steenbrink’s theorem [St, Proposition 2.20] (see also [Kz2]), the monodromy of the VHS associated to families is quasi-unipotent and the residues have rational eigenvalues. It is natural to consider the parabolic bundles associated to such local systems. Further, these parabolic bundles are compatible with pullback morphisms (Lemma 2.5), unlike canonical extensions. One can also define the Chern character of parabolic bundles in the rational Chow groups (section 2.3). All this is possible by using a correspondence of these special parabolic bundles, termed as locally abelian parabolic bundles, with vector bundles on a particular DM-stack (Lemma 2.3).

In this framework, we show

Theorem 1.1. *Suppose $\pi : X_U \rightarrow U$ is a smooth projective morphism of relative dimension n between nonsingular varieties. Consider a nonsingular compactification $U \subset S$ such that $S - U$ is a normal crossing divisor. Then the Chern character of the alternating sum of the parabolic bundles $\overline{\mathcal{H}}^i(X_U/U)$ in each degree,*

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(X_U/U))$$

lies in $CH^0(S)_{\mathbb{Q}}$ or equivalently the pieces in all of the positive-codimension Chow groups with rational coefficients vanish.

Here $\overline{\mathcal{H}}^i(X_U/U)$ denotes the parabolic bundle associated to the weight i de Rham bundle on U .

In fact we will prove the same thing when the morphism is not generically smooth, where U is the open set over which the map is topologically a fibration, see Theorem 5.1.

If $X_U \rightarrow U$ has a semi-stable extension $X \rightarrow S$ (or a compactified family satisfying certain conditions) then the triviality of the Chern character of the alternating sum of de Rham bundles in the (positive degree) rational Chow groups is proved by Esnault and Viehweg in [Es-Vi2, Theorem 4.1]. This is termed as a logarithmic Grothendieck Riemann-Roch theorem (GRR) since GRR is applied to the logarithmic relative de Rham sheaves to obtain the relations. It might be possible to generalize the calculation of [Es-Vi2] to the case of a weak toroidal semistable reduction which always exists by [AK]. However, this seems like it would be difficult to set up.

We generalise the Esnault-Viehweg result and a good compactified family is not required over S . In particular, we do not use calculations with the Grothendieck Riemann-Roch formula, although we do use the general existence of such a formula. Further, Theorem 1.1 shows that the singularities in the fibres of extended families do not play any role, in higher dimensions. Instead, our inductive argument calls upon de Jong's semistable reduction for curves [dJ] at each inductive step.

As an application, we show

Corollary 1.2. *Suppose $X_U \rightarrow U$ is any family of projective surfaces and S is a good compactification of U . Then the parabolic Chern character satisfies*

$$ch(\overline{\mathcal{H}}^i(X_U/U)) \in CH^0(S)_{\mathbb{Q}}$$

for each $i \geq 0$.

This is proved in §5.5 Proposition 5.13. On a non-compact base S supporting a smooth family of surfaces, this was observed in [BE, Example 7.3]. We use the weight filtration on the cohomology of the singular surface, a resolution of singularities, the triviality of the classes of $\overline{\mathcal{H}}^1$ ([Es-Vi2]), and using Theorem 1.1 we deduce the proof.

The proof of Theorem 1.1 is by induction principle and using the Lefschetz theory (such an approach was used earlier in [BE2] for a similar question). We induct on the relative dimension n . By the Lefschetz theory, the cohomology of an n -dimensional nonsingular projective variety is expressed in terms of the cohomology of a nonsingular hyperplane section and the cohomology of the (extension of) *variable local system* on \mathbb{P}^1 . This helps us to apply induction and conclude the relations between the Chern classes of the de Rham bundles in the rational Chow groups (Theorem 1.1). Our proof requires a certain amount of machinery such as the notion of parabolic bundle. The reason for this is that the local monodromy transformations of a Lefschetz pencil whose fiber dimension is even, are reflections of order two rather than unipotent transvections. In spite of this machinery

we feel that the proof is basically pretty elementary, and in particular it doesn't require us to follow any complicated calculations with GRR.

A result of independent interest is Theorem 6.1 in §6, Appendix, which was written by the second author but was occasioned by the talk in Nice given by the first author. This is an analogue of Steenbrink's theorem [St, Theorem 2.18]. In this case, the relative dimension is one and the relative (logarithmic) de Rham complex has coefficients in an extension of a unipotent local system. It is proved that the associated cohomology sheaves are locally free and having a Gauss–Manin connection. Furthermore, the logarithmic connection has nilpotent residues along the divisor components (where it has poles).

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2. PARABOLIC BUNDLES

We treat some preliminaries on the notion of parabolic bundle [Se]. This takes a certain amount of space, and we are leaving without proof many details of the argument. The purpose of this discussion in our proof of the main theorem is to be able to treat the case of Lefschetz pencils of even fiber dimension, in which case the monodromy transformations are reflections of order two (rather than the more classical unipotent transformations in the case of odd fiber dimension). Thus we will at the end be considering parabolic structures with weights $0, \frac{1}{2}$ and the piece of weight $\frac{1}{2}$ will have rank one. Furthermore we will assume by semistable reduction that the components of the divisor of singularities don't touch each other. Nonetheless, it seems better to give a sketch of the general theory so that the argument can be fit into a proper context.

Suppose X is a smooth variety and D is a normal crossings divisor. Write $D = \bigcup_{i=1}^k D_i$ as a union of irreducible components, and we assume that the D_i are themselves smooth meeting transversally.

We will define the notion of *locally abelian parabolic bundle on (X, D)* . We claim that this is the right definition of this notion; however intermediate definitions may or may not be useful (e.g. our notion of “parabolic sheaf” might not be the right one). The notion which we use here appeared for example in Mochizuki [Mo], and is slightly different from the one used by Maruyama and Yokogawa [Ma-Yo] in that we consider different filtrations for all the different components of the divisor.

Also we shall only consider parabolic structures with rational weights.

A *parabolic sheaf* on (X, D) is a collection of torsion-free sheaves F_α indexed by multi-indices $\alpha = (\alpha_1, \dots, \alpha_k)$ with $\alpha_i \in \mathbb{Q}$, together with inclusions of sheaves of \mathcal{O}_X -modules

$$F_\alpha \hookrightarrow F_\beta$$

whenever $\alpha_i \leq \beta_i$ (a condition which we write as $\alpha \leq \beta$ in what follows), subject to the following hypotheses:

- (normalization/support) let δ^i denote the multiindex $\delta_i^i = 1, \delta_j^i = 0, i \neq j$, then $F_{\alpha+\delta^i} = F_\alpha(D_i)$ (compatibly with the inclusion); and
- (semicontinuity) for any given α there exists $c > 0$ such that for any multiindex ε with $0 \leq \varepsilon_i < c$ we have $F_{\alpha+\varepsilon} = F_\alpha$.

It follows from the normalization/support condition that the quotient sheaves F_α/F_β for $\beta \leq \alpha$ are supported in a schematic neighborhood of the divisor D , and indeed if $\beta \leq \alpha \leq \beta + \sum n_i \delta^i$ then F_α/F_β is supported over the scheme $\sum_{i=1}^k n_i D_i$. Let $\delta := \sum_{i=1}^k \delta^i$. Then

$$F_{\alpha-\delta} = F_\alpha(-D)$$

and $F_\alpha/F_{\alpha-\delta} = F_\alpha|_D$.

The semicontinuity condition means that the structure is determined by the sheaves F_α for a finite collection of indices α with $0 \leq \alpha_i < 1$, the *weights*.

For each component D_i of the divisor D , we have

$$F_\alpha|_{D_i} = F_\alpha/F_{\alpha-\delta^i}.$$

Thus for $\alpha = 0$ we have

$$F_0|_{D_i} = F_0/F_{-\delta^i}.$$

This sheaf over D_i has a filtration by subsheaves which are the images of the $F_{\beta_i \delta^i}$ for $-1 < \beta_i \leq 0$. The filtration stops at $\beta_i = -1$ where by definition the subsheaf is zero. Call the image subsheaf F_{D_i, β_i} and for $\beta_i > -1$ put

$$Gr_{D_i, \beta_i} := F_{D_i, \beta_i}/F_{D_i, \beta_i - \varepsilon}$$

with ε small. There are finitely many values of β_i such that the Gr is nonzero. These values are the *weights* of F along the component D_i . Any \mathbb{Z} -translate of one of these weights will also be called a weight. The global parabolic structure is determined by the sheaves F_α for multiindices α such that each α_i is a weight along the corresponding component D_i .

2.1. The locally abelian condition. A *parabolic line bundle* is a parabolic sheaf F such that all the F_α are line bundles. An important class of examples is obtained as follows: if α is a multiindex then we can define a parabolic line bundle denoted

$$F := \mathcal{O}_X \left(\sum_{i=1}^k \alpha_i D_i \right)$$

by setting

$$F_\beta := \mathcal{O}_X \left(\sum_{i=1}^k a_i D_i \right)$$

where each a_i is the largest integer such that $a_i \leq \alpha_i + \beta_i$.

If F is a parabolic sheaf, set F_∞ equal to the extension $j_*(j^*F_\alpha)$ for any α , where $j : X - D \hookrightarrow X$ is the inclusion. It is the sheaf of sections of F which are meromorphic along D , and it doesn't depend on α . Note that the F_α may all be considered as subsheaves of F_∞ .

We can define a tensor product of torsion-free parabolic sheaves: set

$$(F \otimes G)_\alpha$$

to be the subsheaf of $F_\infty \otimes_{\mathcal{O}_X} G_\infty$ generated by the $F_{\beta'} \otimes G_{\beta''}$ for $\beta' + \beta'' \leq \alpha$.

On the other hand, if E is a torsion-free sheaf on X then it may be considered as a parabolic sheaf (we say ‘‘with trivial parabolic structure’’) by setting E_α to be $E(\sum a_i D_i)$ for a_i the greatest integer $\leq \alpha_i$.

With this notation we may define for any vector bundle E on X the parabolic bundle

$$E \left(\sum_{i=1}^k \alpha_i D_i \right) := E \otimes_{\mathcal{O}_X} \left(\sum_{i=1}^k \alpha_i D_i \right).$$

Lemma 2.1. *Any parabolic line bundle has the form $L(\sum_{i=1}^k \alpha_i D_i)$ for L a line bundle on X . This may be viewed as $L(B)$ where B is a rational divisor on X (supported on D).*

□

Definition 2.2. *A parabolic sheaf F is a locally abelian parabolic bundle if, in a Zariski neighborhood of any point $x \in X$ there is an isomorphism between F and a direct sum of parabolic line bundles.*

Most questions about locally abelian parabolic bundles can be treated by reducing to a local question then looking at the case of line bundles and using the structure result of Lemma 2.1.

2.2. Relationship with bundles on DM-stacks. This material is the subject of a number of recent papers and preprints by Biswas [Bi2], Cadman [Cad], Matsuki and Olsson [Ma-Ol] and specially Niels Borne [Bo].¹

Suppose $f : (X', D') \rightarrow (X, D)$ is a morphism of smooth varieties with normal crossings divisors, such that $f^{-1}(D) \subset D'$. Then we would like to define the *pullback* f^*F of a locally abelian parabolic bundle F on (X, D) , as a locally abelian parabolic bundle on (X', D') . This is not entirely trivial to do. We propose two approaches. We first will pass through the notion of bundles on Deligne-Mumford stacks in order to give a precise definition. On

¹We thank A. Chiodo for useful discussions about this and for informing us of these references.

the other hand, we can use the locally abelian structure to reduce to the case of parabolic line bundles. In this case we require that

$$f^*(L(B)) = (f^*L)(f^*B)$$

using the pullback $f^*(B)$ for a rational divisor B (note that if B is supported on D then the pullback will be supported on D').

Any construction using a local choice of frame as in Definition 2.2 or something else such as the resolution we will discuss later, leaves open the problem of seeing that the construction is independent of the choices which were made. In order to get around this kind of problem, we use the relationship between parabolic bundles and bundles on certain Deligne-Mumford stacks.

We define a DM-stack denoted $Z := X[\frac{D_1}{n_1}, \dots, \frac{D_k}{n_k}]$. Localizing in the etale topology over X we may assume that the D_i are defined by equations z_i . Only some of the components will appear in any local chart; renumber things so that these are $z_i = 0$ for $i = 1, \dots, k'$. Then define the local chart for Z to be given with coordinates u_i by the equations $z_i = u_i^{n_i}$ for $i = 1, \dots, k'$ and $z_i = u_i$ for the other i . Without repeating the general theory of DM-stacks, this defines a smooth DM-stack Z whose coarse moduli space is X and whose stabilizer groups along the components D_i are the groups $\mathbb{Z}/n_i\mathbb{Z}$.

This DM-stack may alternatively be considered as a ‘‘Q-variety’’ in the sense of Mumford [Mu, section 2].

On Z we have divisors which we may write as $\frac{D_i}{n_i}$, given in the local coordinate patch by $u_i = 0$. In particular, if B is a rational divisor supported along D such that $B = \sum b_i D_i$ and if $n_i b_i \in \mathbb{Z}$ (that is, the denominator of b_i divides n_i) then B becomes an actual divisor on Z by writing

$$B = \sum (n_i b_i) \frac{D_i}{n_i}.$$

Let $p : Z \rightarrow X$ denote the projection. If E is a vector bundle on Z then p_*E is a torsion-free sheaf on X , and in fact it is a bundle. We may define the *associated parabolic bundle* $\mathbf{a}(E)$ on (X, D) by the formula

$$\mathbf{a}(E)_\alpha := p_*(E(\sum_i \alpha'_i D_i))$$

where α'_i is the greatest rational number $\leq \alpha_i$ with denominator dividing n_i .

Lemma 2.3. *The above construction establishes an equivalence of categories between vector bundles on Z , and locally abelian parabolic bundles on (X, D) whose weights have denominators dividing the n_i .*

Proof. See [Bo, Theorem 5]. The setup there is slightly different in that all of the components D_i are combined together, and Borne considers more generally torsion-free coherent sheaves. To adapt this to our situation, we will just comment on why the parabolic bundle on X obtained from a bundle on Z must satisfy the locally abelian condition. Suppose

E is a bundle on the DM stack Z . It suffices to see the following claim: that there is a Zariski open covering $X = \bigcup_i X_i$ which induces a covering $Z = \bigcup_i Z_i$ with $Z_i := Z \times_X X_i$, such that every $E|_{Z_i}$ splits as a direct sum of line bundles on the DM-stacks Z_i . Fix a point $x \in X$ on a crossing point of components which we may assume are numbered D_1, \dots, D_b . In a Zariski neighborhood X_i of x , we have by construction

$$Z_i = Y_i/G_i$$

where Y_i is a smooth scheme, $G_i = \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_b$,

$$Y_i \rightarrow X_i$$

is a Galois ramified covering with group G_i and X_i is the categorical quotient, and $y \in Y_i$ is the unique point lying over x and $G_i \cdot y = y$. The bundle $E|_{Z_i}$ is given by the data of a G_i -equivariant bundle E_{Y_i} on Y_i . In particular G_i acts on the fiber $E_{Y_i, y}$ and since G_i is abelian, there is a direct sum decomposition

$$E_{Y_i, y} = \bigoplus_{j=1}^r \mathbb{C}(\xi_j)$$

where ξ_j are characters of G_i . Define the G_i -equivariant direct sum of line bundles

$$F := \bigoplus_{j=1}^r \mathcal{O}_{Y_i}(\xi_j) := \bigoplus_{j=1}^r \mathcal{O}_{Y_i} \otimes_{\mathbb{C}} \mathbb{C}(\xi_j).$$

Let $u_0 : E_{Y_i} \rightarrow F$ be any morphism which restricts over y to the given G_i -equivariant isomorphism

$$E_{Y_i, y} \cong F_y,$$

and put $u := \sum_{g \in G_i} g^{-1} u_0 \circ g$. This is now G_i -equivariant, and is still an isomorphism over y . Let Y'_i be the Zariski open set where u is an isomorphism. It is G_i -equivariant so it comes from an open neighborhood X'_i of x . Over the corresponding stack Z'_i our morphism u descends to an isomorphism between E and a direct sum of line bundles. Thus the parabolic bundle on X will be a direct sum of parabolic line bundles locally in the Zariski topology. We refer to [Bo] for the remainder of the proof. \square

One expects more general parabolic bundles to correspond to certain saturated sheaves on these DM stacks but that goes beyond our present requirements.

The inverse functor in Lemma 2.3 will be taken as the definition of the pullback p^* from certain locally abelian parabolic bundles on (X, D) (that is, those with appropriate denominators) to bundles considered as having their trivial parabolic structure on the stack Z via the map $p : Z \rightarrow X$.

Lemma 2.4. *The inverse functor p^* is compatible with morphisms of DM-stacks of the form*

$$Z_1 := X\left[\frac{D_1}{n_1}, \dots, \frac{D_k}{n_k}\right] \xrightarrow{q} Z_2 := X\left[\frac{D_1}{m_1}, \dots, \frac{D_k}{m_k}\right]$$

whenever m_i divides n_i . In other words, if F is a parabolic bundle whose denominators divide the m_i and if $E = p^*F$ is the associated bundle on Z_2 , then q^*E is the associated bundle on Z_1 (by a natural isomorphism satisfying a cocycle condition).

□

Lemma 2.5. *Suppose $f : (X', D') \rightarrow (X, D)$ is a morphism of smooth varieties with normal crossings divisors, such that $f^{-1}(D) \subset D'$. Then the pullback f^*F of a locally abelian parabolic bundle F on (X, D) , is defined as a locally abelian parabolic bundle on (X', D') .*

Proof. Using the equivalence of Lemma 2.3, we can define the pullback of a locally abelian parabolic bundle. Indeed, given a morphism $f : (X', D') \rightarrow (X, D)$ and any positive integers n_1, \dots, n_k for the components D_i of D , there exist integers $n'_1, \dots, n'_{k'}$ for the components D'_i of D' such that f extends to a morphism

$$f_{DM} : X' \left[\frac{D'_1}{n'_1}, \dots, \frac{D'_{k'}}{n'_{k'}} \right] \rightarrow X \left[\frac{D_1}{n_1}, \dots, \frac{D_k}{n_k} \right]$$

If F is a locally abelian parabolic bundle on X we can choose integers n_i divisible by the denominators of the weights of F , then choose n'_j as above. Thus F corresponds to a bundle E on the DM-stack $X \left[\frac{D_1}{n_1}, \dots, \frac{D_k}{n_k} \right]$. The pullback $f_{DM}^*(E)$ is a bundle on the DM-stack $X' \left[\frac{D'_1}{n'_1}, \dots, \frac{D'_{k'}}{n'_{k'}} \right]$. Define $f^*(F)$ to be the locally abelian parabolic bundle on (X', D') corresponding (again by Lemma 2.3) to the bundle $f_{DM}^*(E)$.

Using Lemma 2.4, the parabolic bundle $f^*(F)$ is independent of the choice of n_i and n'_j . □

This definition can be extended, using local charts which are schemes, to the pullback to any DM-stack, and it is compatible with the previous notation in the sense that if $p : Z \rightarrow X$ is the projection from the DM-stack used in Lemma 2.3 to the original variety, then the bundle $E = p^*(F)$ which corresponds to F is indeed the pullback as defined two paragraphs ago (which would be a parabolic bundle on the DM-stack Z but which is actually a regular bundle with trivial parabolic structure).

2.3. Chern character of parabolic bundles. Mumford, Gillet, Vistoli ([Mu],[Gi],[Vi]) have defined Chow groups for DM-stacks. Starting with (X, D) and choosing denominators n_i we obtain the stack $Z := X \left[\frac{D_1}{n_1}, \dots, \frac{D_k}{n_k} \right]$. The coarse moduli space of Z is the original X , so from [Gi, Theorem 6.8], the pullback and pushforward maps establish an isomorphism of rational Chow groups

$$CH^*(Z)_{\mathbb{Q}} \cong CH^*(X)_{\mathbb{Q}}.$$

We also have a Chern character for bundles on a DM-stack [Gi, section 8]. If F is a locally abelian parabolic bundle on (X, D) , choose n_i divisible by the denominators of the

weights of F and define the Chern character of F by

$$\mathrm{ch}^{\mathrm{par}}(F) := p_*(\mathrm{ch}(p^*F)) \in CH^*(X)_{\mathbb{Q}}.$$

This doesn't depend on the choice of n_i , by Lemma 2.4, and the compatibility of the Chern character with pullbacks on DM-stacks.

Using the fact that Chern character commutes with pullback for bundles on DM-stacks, we obtain:

Lemma 2.6. *Suppose $f : (X', D') \rightarrow (X, D)$ is a morphism of smooth varieties with normal-crossings divisors, such that $f^{-1}(D)$ is supported on D' . If F is a locally abelian parabolic bundle on (X, D) then the pullback f^*F is a locally abelian parabolic bundle on (X', D') and this construction commutes with Chern character:*

$$\mathrm{ch}^{\mathrm{par}}(f^*F) = f^*(\mathrm{ch}^{\mathrm{par}}(F)) \in CH^*(X')_{\mathbb{Q}}.$$

□

If F is a usual vector bundle considered with its trivial parabolic structure, then $\mathrm{ch}(F)$ is the usual Chern character of F (this is the case $n_i = 1$ in the definition). Also, if f is a morphism then the pullback $f^*(F)$ is again a usual vector bundle with trivial parabolic structure.

We can easily describe the pullback and Chern characters for parabolic line bundles. If $B = \sum b_i D_i$ is a rational divisor on X , supported on D , and if $f : (X', D') \rightarrow (X, D)$ is a morphism as before, we have the formula

$$f^*(\mathcal{O}_X(B)) = \mathcal{O}_{X'}(B')$$

where $B' = \sum b_i (f^*D_i)$ with $f^*(D_i)$ the usual pullback of Cartier divisors. Similarly, if E is a vector bundle on X then we have

$$f^*(E(B)) = (f^*E)(B').$$

For the Chern character, recall that $\mathrm{ch}(\mathcal{O}_X(B)) = e^B$ for a divisor with integer coefficients B . This formula extends to the case of a rational divisor, to give the formula for the Chern character of a parabolic line bundle in the rational Chow group. More generally for a vector bundle twisted by a rational divisor we have

$$\mathrm{ch}^{\mathrm{par}}(E(B)) = \mathrm{ch}(E) \cdot e^B.$$

The Chern character is additive on the K_0 -group of vector bundles on a DM-stack, so (modulo saying something good about choosing appropriate denominators whenever we want to apply the formula) we get the same statement for the Chern character on the K_0 -group of locally abelian parabolic bundles. One should say what is an *exact sequence* of locally abelian parabolic bundles. Of course a morphism of parabolic sheaves $F \rightarrow G$ is just a collection of morphisms of sheaves $F_\alpha \rightarrow G_\alpha$ and the kernel and cokernel are still parabolic sheaves. A short sequence

$$0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$$

of parabolic sheaves is *exact* if the resulting sequences

$$0 \rightarrow F_\alpha \rightarrow G_\alpha \rightarrow H_\alpha \rightarrow 0$$

are exact for all α . This notion is preserved by the equivalence of Lemma 2.3 and indeed more generally the pullback functor f^* preserves exactness. All objects in these sequences are bundles so there are no higher *Ext* sheaves and in particular there is no requirement of flatness when we say that f^* preserves exactness, although the p^* of Lemma 2.3 is flat.

Thus we can define the K_0 of the category of locally abelian parabolic bundles, and the Chern character is additive here. Also the Chern character is multiplicative on tensor products (this might need some further proof which we don't supply here). And the pullback f^* is defined on the K_0 group.

Caution: The Chern character doesn't provide an isomorphism between K_0 and the Chow group, on a DM-stack. For this one must consider an extended Chern character containing further information over the stacky locus, see [To] [Ch-Ru] [AGV]. Thus, the same holds for parabolic bundles: two parabolic bundles with the same Chern character are not necessarily the same in the K_0 -group of parabolic bundles.

We don't give a formula for the Chern character of a parabolic bundle F in terms of the Chern characters of the constituent bundles F_α .² The formula we had claimed in the first version of the present paper was wrong. We will treat this question elsewhere. For our present purposes, what we need to know is contained in the following lemma and its corollary.

Lemma 2.7. *Suppose F is a parabolic bundle. Then the difference between F and any one of its F_α in the rational Chow ring of X or equivalently the rational K -theory of X , is an element which is concentrated over D (i.e. it may be represented by a rational combination of sheaves concentrated on D).*

Proof. There is a localization sequence in rational K -theory of X (see [Sr, Proposition 5.15]) :

$$K_0(D) \otimes \mathbb{Q} \rightarrow K_0(X) \otimes \mathbb{Q} \rightarrow K_0(X - D) \otimes \mathbb{Q} \rightarrow 0.$$

The restrictions of F and F_α to $U = X - D$ are isomorphic vector bundles so $\text{ch}(F) - \text{ch}(F_\alpha)$ maps to zero in $K_0(X - D) \otimes \mathbb{Q}$. Thus it comes from $K_0(D) \otimes \mathbb{Q}$. \square

Corollary 2.8. *If $\text{ch}^{\text{par}}(F) \in CH^0(X)_\mathbb{Q}$ for a parabolic bundle, then any F_α is equivalent to an element coming from $CH^*(D)$.*

²This point seems to be somewhat delicate, for example D. Panov in his thesis [Pa] gives corrected versions of some of the formulae obtained by Biswas [Bi].

3. THE PARABOLIC BUNDLE ASSOCIATED TO A LOGARITHMIC CONNECTION

Suppose as above that X is a smooth projective variety with normal crossings divisor D , and let $U := X - D$. Write as before $D = D_1 \cup \cdots \cup D_k$ the decomposition of D into smooth irreducible components.

Suppose (E, ∇) is a vector bundle with logarithmic connection ∇ on X such that the singularities of ∇ are concentrated over D . Fix a component D_i of the divisor. Define the *residue* of (E, ∇) along D_i to be the pair $(E|_{D_i}, \eta_i)$ where $E|_{D_i}$ is the restriction of E to D_i . The residual transformation η_i is the action of the vector field $z \frac{\partial}{\partial z}$ where z is a local coordinate of D_i . This is independent of the choice of local coordinate. The connection induces an operator

$$\nabla_{D_i} : E|_{D_i} \rightarrow (E|_{D_i}) \otimes_{\mathcal{O}_{D_i}} (\Omega_X^1(\log D))|_{D_i},$$

whose projection by the residue map is η_i . This will not in general lift to a connection on $E|_{D_i}$ (an error in our first version pointed out by H. Esnault). Such a lift exists locally in the étale topology where we can write X as a product of D_i and the affine line, and in this case η_i is an automorphism of the bundle with connection $E|_{D_i}$, in particular the eigenvalues of η_i are locally constant functions along D_i . The latter holds without étale localization, and the eigenvalues are actually constant because we assume D_i irreducible.

Assume the following condition:

Definition 3.1. *We say that (E, ∇) has rational residues if the eigenvalues of the residual transformations of ∇ along components of D (that is, the η_i above) are rational numbers.*

In this case we will construct a parabolic vector bundle $F = \{F_\alpha\}$ on (X, D) together with isomorphisms

$$F_\alpha|_U \cong E|_U.$$

We require that ∇ extend to a logarithmic connection on each F_α . Finally, we require that the residue of ∇ on the piece $F_\alpha/F_{\alpha-\varepsilon\delta^i}$ which is concentrated over D_i , be an operator with eigenvalue $-\alpha_i$.

Definition 3.2. *A parabolic bundle with these data and properties is called a parabolic bundle associated to (E, ∇) .*

The notion of a parabolic bundle associated to a connection was discussed in [In] and [IIS]. In those places, no restriction is placed on the residues, and parabolic structures with arbitrary complex numbers for weights are considered. They use full flags and treat the case of curves. It isn't immediately clear how the full flags would generalize for normal crossings divisors in higher dimensions.

Lemma 3.3. *For any vector bundle with logarithmic connection having rational residues, there exists a unique associated parabolic bundle, and furthermore it is locally abelian.*

Proof. This construction is basically the one discussed by Deligne in [De1] Proposition 5.4 and attributed to Manin [Man]. We give three constructions.

(I) Suppose E is a vector bundle with a logarithmic connection on (X, D) . Applying the discussion of [De1] we may assume the eigenvalues are contained in $(-1, 0]$. Consider the residue $\eta_i \in \text{End}(E|_{D_i})$ whose eigenvalues $-\alpha_i^j$ are rational numbers and satisfying $\alpha_i^1 < \alpha_i^2 < \dots < \alpha_i^{n_i}$.

The eigenspaces of the fibres of $E|_{D_i}$ given by the endomorphism η_i associated to an eigenvalue $-\alpha_i^j$ (which is a constant), defines a subsheaf $A_{\alpha_i^j}$ of $E|_{D_i}$, called the *generalized eigenspaces*. Let $F_i^{\alpha_i^j} = \sum_{l=1}^j A_{\alpha_i^l}$. Define the subsheaves $\overline{F_i^{\alpha_i^j}}$ of E by the exact sequence:

$$0 \longrightarrow \overline{F_i^{\alpha_i^j}} \longrightarrow E \longrightarrow E|_{D_i}/F_i^{\alpha_i^j} \longrightarrow 0.$$

Given a set of rational weights $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, i.e, α_i is a weight along D_i , we can associate a subsheaf

$$F_\alpha = \bigcap_{i=1}^k \overline{F_i^{\alpha_i}}$$

of E . This defines a parabolic structure on (X, D) . Moreover, each F_α restricts on U to $E|_U$.

Since the residue endomorphism η_i preserves the generalized eigenspaces, it preserves the sheaves $F_i^{\alpha_i^j}$ and hence ∇ induces a logarithmic connection on each $F_i^{\alpha_i^j}$ and thus on F_α .

For the locally abelian condition, note that the same construction may be done over the analytic topology, and since the local monodromy groups are abelian on punctured neighborhoods of the crossing points of D which are products of punctured discs, we get a decomposition into a direct sum of parabolic line bundles, locally in the analytic topology. Artin approximation gives it locally in the etale topology, and the argument mentioned in Lemma 2.3 gives the locally abelian condition in the Zariski topology.

(II) In [De1, Proposition 5.4] Deligne constructs a bundle associated to any choice of lifting function $\tau : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}$. In fact the same construction will work if we choose different liftings τ_i for each component of the divisor D_i . Given α , choose liftings τ_i which send \mathbb{R}/\mathbb{Z} to the intervals $[-\alpha_i, 1 - \alpha_i)$ respectively. Applying the extension construction gives the bundle F_α and these organize into our parabolic bundle on X , locally abelian by the same reasoning as in the previous paragraph.

(III) Choose n_i such that the residues of ∇ along D_i are integer multiples of $\frac{1}{n_i}$. Then the parabolic bundle associated to (E, ∇) corresponds to the Deligne canonical extension of (E, ∇) over the Deligne-Mumford stack $Z := X[\frac{D_1}{n_1}, \dots, \frac{D_k}{n_k}]$, see §2.2. Note that the pullback of (E, ∇) to a logarithmic connection over this Deligne-Mumford stack, has integer residues so exactly the construction of [De1] can be carried out here. Uniqueness of the construction implies descent from an etale covering of the stack to the stack Z

itself. As noted in the proof of Lemma 2.3, a bundle on Z corresponds to a locally abelian parabolic structure on X . The fact that the bundle has a logarithmic connection over Z translates into the statement that the F_α are preserved by the logarithmic connection which is generically defined from U . \square

Lemma 3.4. *Let F be the parabolic bundle associated to a logarithmic connection. On the bundle F_α the eigenvalues of the residue of ∇ along D_i are contained in the interval $[-\alpha_i, 1 - \alpha_i]$.*

Proof. The restriction to D_i may be expressed as

$$F_\alpha|_{D_i} = F_\alpha / F_{\alpha - \delta^i},$$

so the condition in the definition of associated parabolic bundle fixes the eigenvalues on the graded pieces here as being in the required interval. \square

The proof of 3.3 also shows that

Lemma 3.5. *Given an exact sequence of logarithmic connections with rational residues along D :*

$$0 \longrightarrow E'' \longrightarrow E \longrightarrow E' \longrightarrow 0$$

there is an exact sequence of parabolic bundles

$$0 \longrightarrow F_{E''} \longrightarrow F_E \longrightarrow F_{E'} \longrightarrow 0$$

such that $F_{E''}, F_E, F_{E'}$ are the parabolic bundles associated to E'', E and E' respectively. Similarly the construction is compatible with direct sums.

Lemma 3.6. *Suppose $Y \subset X$ is a subvariety intersecting D transversally. Let F be the parabolic bundle associated to a vector bundle with logarithmic connection (E, ∇) over X . Then $F|_Y$ is the parabolic bundle associated to the restriction of (E, ∇) to Y .*

Proof. Since the intersection of Y and D is transversal, D restricts to a normal crossing divisor D_Y on Y . The restriction of (E, ∇) on Y has rational residues $\{\alpha_i\}$ which is a subset of the rational residues of (E, ∇) and corresponds to those sheaves F_{α_i} which restricted to Y are nonzero. In other words, using the construction of parabolic structure in Lemma 3.3, it follows that $F|_Y$ is the parabolic bundle associated to the restriction of (E, ∇) to Y . \square

3.1. Computing cohomology using the associated parabolic bundle. Suppose X is a smooth projective variety with normal crossings divisor $D \subset X$ and let $U := X - D$. Let $j : U \rightarrow X$ denote the inclusion.

Suppose (E, ∇) is a vector bundle on X with logarithmic connection and rational residues along D , with associated parabolic bundle F . Let $DR(E, \nabla)$ denote the de Rham complex

$$E \rightarrow E \otimes_{\mathcal{O}_X} \Omega_X^1(\log D) \rightarrow E \otimes_{\mathcal{O}_X} \Omega_X^2(\log D) \rightarrow \dots$$

with d_∇ as differential.

Define the *logarithmic de Rham cohomology*

$$\mathcal{H}_{DR}(X; E, \nabla) := \mathbb{H}(X, DR(E, \nabla)).$$

In this notation the fact that it is logarithmic along D is encoded in the fact that ∇ is a logarithmic connection along D . On the other hand, let E_U^∇ denote the local system on U of flat sections for the connection ∇ .

Theorem 3.7. (Deligne) *Suppose that the residues of ∇ have no strictly positive integers as eigenvalues. Then*

$$\mathcal{H}_{DR}^i(X; E, \nabla) \cong H^i(U, E_U^\nabla).$$

Proof. This is [De1], Proposition 3.13 and Corollaire 3.14. \square

Corollary 3.8. *Suppose (E, ∇) is a logarithmic connection. Let F denote the associated parabolic bundle, so that for any α we obtain again a logarithmic connection (F_α, ∇) . If $\alpha = (\alpha_1, \dots, \alpha_k)$ with all $\alpha_i > 0$ then*

$$\mathcal{H}_{DR}^i(X; F_\alpha, \nabla) \cong H^i(U, E_U^\nabla).$$

Proof. By Lemma 3.4 the eigenvalues of the residue of ∇ on F_α are in $[-\alpha_i, 1 - \alpha_i]$. If $\alpha_i > 0$ this is guaranteed not to contain any positive integers. \square

3.2. The relative logarithmic de Rham complex. Suppose (E, ∇) is a vector bundle with logarithmic connection relative to a pair (X, D) where X is smooth and D is a normal crossings divisor. In practice, E will be of the form F_α as per Theorem 3.8.

Suppose we have a map $f : X \rightarrow Y$ such that components of D are either over the singular divisor J in Y , or have relative normal crossings over Y [De1]. We have the logarithmic complex $\Omega_X(\log D)$ and the subcomplex $f^*\Omega_Y(\log J)$. Put

$$\Omega_{X/Y}(\log D) := \Omega_X(\log D) / f^*\Omega_Y(\log J).$$

We can form a differential d_∇ using ∇ on $E \otimes \Omega_{X/Y}(\log D)$. Call the resulting logarithmic de Rham complex $DR(X/Y, E, \nabla)$. In this notation the divisors D and J are implicit.

Define a complex of quasicoherent sheaves on Y as follows:

$$\mathcal{H}_{DR}(X/Y; E, \nabla) := \mathbb{R}f_*DR(X/Y, E, \nabla).$$

To define this precisely, choose an open covering of X by affine open sets X_β such that the map f is affine on the multiple intersections. Let $\check{C}DR(X/Y, E, \nabla)$ denote the simple complex associated to the double complex obtained by applying the Čech complex in each degree. The components are acyclic for f_* so we can set

$$\mathbb{R}f_*DR(X/Y, E, \nabla) := f_*\check{C}DR(X/Y, E, \nabla).$$

The component sheaves of the Čech complex on X are quasicoherent but the differentials are not \mathcal{O}_X -linear. However, the differentials are $f^{-1}\mathcal{O}_Y$ -linear, so the direct image complex consists of quasicoherent sheaves on Y .

The complex $DR(X/Y, E, \nabla)$ is filtered by the Hodge filtration which is the “filtration bête”. This presents $DR(X/Y, E, \nabla)$ as a successive extension of complexes $E \otimes \Omega_{X/Y}^m(\log D)[-m]$. The higher derived direct image becomes a successive extension of the higher derived direct images of these component complexes, for example this is exactly true if we use the Čech construction above.

The standard argument of Mumford [Mu] shows that each

$$\mathbb{R}f_*E \otimes \Omega_{X/Y}^m(\log D)[-m]$$

is a perfect complex over Y and formation of this commutes with base changes $b : Y' \rightarrow Y$. Therefore the same is true of the successive extensions. Thus formation of $\mathcal{H}_{DR}(X/Y; E, \nabla)$ commutes with base change.

At this point we are imprecise about the conditions on Y' and b . All objects should be defined after base change. For example if $X' := Y' \times_Y X$ is smooth and the pullbacks of D to X' and J to Y' are normal crossings, then everything is still defined. In §6 we will want to consider a more general situation in which the logarithmic de Rham complex is still defined. To be completely general, refer to Illusie [Il] for example for an intrinsic definition, but we could take as definition

$$DR(X'/Y', E', \nabla') := pr_2^*DR(X/Y, E, \nabla)$$

for $pr_2 : X' \rightarrow X$ the projection, and with that the base-change formula holds for any $b : Y' \rightarrow Y$.

This gives the following lemma which is well known.

Lemma 3.9. *Suppose (E, ∇) is a vector bundle with logarithmic connection on (X, D) . The relative logarithmic de Rham cohomology $\mathcal{H}_{DR}(X/Y; E, \nabla)$ is a perfect complex on Y and if $b : Y' \rightarrow Y$ is a morphism then letting $X' := Y' \times_Y X$ and (E', ∇') be the pullbacks, we have a quasiisomorphism*

$$\mathcal{H}_{DR}(X'/Y'; E', \nabla') \stackrel{qi}{\simeq} b^*\mathcal{H}_{DR}(X/Y; E, \nabla).$$

The complex $\mathcal{H}_{DR}(X/Y; E, \nabla)$ has a logarithmic Gauss-Manin connection which induces the usual Gauss-Manin connection on the cohomology sheaves. This is well known from the theory of \mathcal{D} -modules, see also [Kz2]. We discuss it in §6.

As Deligne says in [De1] 3.16, the calculation of cohomology also works in the relative case. Restrict over the open set U which is the complement of J . Then $X_U \rightarrow U$ is a smooth map and the divisor $D_U := D \cap X_U$ is a union of components all of whose multiple intersections are smooth over Y . Let $W := X_U - D_U$. Let $f_W : W \rightarrow U$ be the map restricted to our open set. With the current hypotheses, topologically it is a fibration. The relative version of Lemma 3.7 (see also [Kz2]) says the following.

Lemma 3.10. *In the above situation, suppose that the eigenvalues of the residue of ∇ along horizontal components, that is components of D_U , are never positive integers. Then the vector bundle with Gauss-Manin connection $\mathcal{H}_{DR}^i(X_U/U; E_U, \nabla)$ on U is the*

unique bundle with connection and regular singularities corresponding to the local system $R^i f_{W,*} \mathbb{C}_W$ on U .

3.3. Chern characters for higher direct images. We now introduce some notation for calculations in Chow groups. Suppose $f : X \rightarrow Y$ is a flat morphism. If E is a vector bundle on X then its higher direct image complex $\mathbb{R} f_*(E)$ is a perfect complex on Y and we can define its Chern character by

$$\mathrm{ch}(\mathbb{R} f_*(E)) := \sum_i (-1)^i \mathrm{ch}(\mathbb{R}^i f_*(E)).$$

The Grothendieck-Riemann-Roch theorem implies the following observation which says that the Chern character of the higher direct image complex is a function of the Chern character of E .

Proposition 3.11. *Suppose $f : X \rightarrow Y$ is a proper morphism between smooth varieties. Then there is a map on rational Chow groups*

$$\chi_{X/Y} : CH(X)_{\mathbb{Q}} \rightarrow CH(Y)_{\mathbb{Q}}$$

which represents the Euler characteristic for higher direct images in the sense that

$$\mathrm{ch}(\mathbb{R} f_*(E)) = \chi_{X/Y}(\mathrm{ch}(E))$$

for any vector bundle or coherent sheaf E on X .

Proof. The GRR formula gives an explicit formula for $\chi_{X/Y}$ [Fu, Theorem 15.2]. One also notes that the Chow groups calculate the rational K_0 and it is clear that $\mathrm{ch} \circ \mathbb{R} f_*$ is additive on exact sequences so it passes to a function on K_0 . \square

Esnault-Viehweg in [Es-Vi2] use the explicit formula for $\chi_{X/Y}$. We use less of GRR here in the sense that we use only existence of the function rather than calculating with the formula.

3.4. Modification of the base. Consider the situation of a family $f : X \rightarrow S$. Let \mathcal{O}_X denote the trivial bundle with its standard connection on X . We assume that S is irreducible. Let $U \subset S$ denote a nonempty open set over which f is smooth, and let $X_U := X \times_S U$.

The $\mathcal{H}_{DR}^i(X_U/U)$ are vector bundles on U . If the complementary divisor $J := S - U$ has normal crossings then the Gauss-Manin connections on the cohomology vector bundles have logarithmic singularities with rational residues (this last statement is the content of the well-known ‘‘monodromy theorem’’). Let

$$\overline{\mathcal{H}}_{DR}^i(X_U/U)$$

denote the parabolic bundle on S associated to this logarithmic connection. Our goal is to investigate the alternating sum of the Chern characters of these bundles on S . In view of this goal, we can make the following reduction.

Lemma 3.12. *In the above situation, suppose $p : S' \rightarrow S$ is a generically finite surjective map from an irreducible variety to S , and if $U' \subset S'$ is a nonempty open set mapping to U , let $X' := X \times_S S'$. Assume that $J' := S' - U'$ again has normal crossings. Then the parabolic bundle $\overline{\mathcal{H}}_{DR}^i(X_{U'}/U')^{\text{par}}$, which is defined as the one which corresponds to the Gauss-Manin connection on $\mathcal{H}_{DR}^i(X_{U'}/U')$, is isomorphic to the pullback from S to S' of the parabolic bundle $\overline{\mathcal{H}}_{DR}^i(X_U/U)^{\text{par}}$ on S .*

Proof. Notice that over U' the statement is true for the cohomology bundle, since it is a base-change theorem for smooth morphisms. Now by Lemma 2.5, we deduce the statement for the parabolic bundle as well. \square

Corollary 3.13. *Let $p : S' \rightarrow S$ and U' be as in the previous lemma. Then*

$$\sum_i (-1)^i \text{ch}^{\text{par}}(\overline{\mathcal{H}}_{DR}^i(X_{U'}/U')) = p^* \sum_i (-1)^i \text{ch}^{\text{par}}(\overline{\mathcal{H}}_{DR}^i(X_U/U)).$$

In particular, $\sum_i (-1)^i \text{ch}^{\text{par}}(\overline{\mathcal{H}}_{DR}^i(X_{U'}/U'))$ is in the degree zero piece of the rational Chow group of S' , if and only if $\sum_i (-1)^i \text{ch}(\overline{\mathcal{H}}_{DR}^i(X_U/U)^{\text{par}})$ is in the degree zero piece of the rational Chow group of S .

\square

Since our goal is to prove that the alternating sum in question at the end of the previous corollary is in the degree zero piece of the rational Chow group, we can safely replace $X \rightarrow S$ and the open set U by the base-change $X' \rightarrow S'$ whenever $S' \rightarrow S$ is a generically finite rational map, with any nonempty open subset $U' \subset S'$. In what follows we will often make this type of reduction. There are several different flavours, for example we could simply decrease the size of the open set U ; we could do any kind of birational transformation on S ; or we could take a finite covering of an open set of S and complete to a smooth variety.

In particular by using finite coverings we may assume that the monodromy transformations at infinity are unipotent rather than just quasi-unipotent. In this case, the associated parabolic extension is just a regular bundle and it is the same as the Deligne canonical extension.

In order to avoid overly heavy notation in what follows, we will keep the same letters when we replace S, X, U and speak of this as “making a generically finite modification of S ,” or some similar such phrase.

4. LEFSCHETZ FIBRATIONS

4.1. Lefschetz pencil [Kz]. Suppose $X \subset \mathbb{P}^N$ is a nonsingular projective variety of dimension n , defined over the complex numbers. Consider its dual variety $\hat{X} \subset \mathbb{P}^{\hat{N}}$. A Lefschetz pencil is a projective line $\mathbb{P}^1 \subset \mathbb{P}^{\hat{N}}$ such that its intersection with \hat{X} is a reduced

zero-dimensional subscheme. Furthermore, these points correspond to hyperplane sections of X with only one ordinary double point as its singularity.

Then there is a Lefschetz fibration

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & X \\ \downarrow \varphi & & \\ \mathbb{P}^1 & & \end{array}$$

such that Z is a blow-up of X along the base locus B defined by the vanishing of any two hyperplane sections $H_s, H_t \in \mathbb{P}^1$.

Remark 4.1. *We remark that if n is odd, then the monodromy transformations of the Lefschetz fibration in the middle dimensional cohomology, are reflections (i.e., they have order 2 with one eigenvalue as -1 and the rest as $+1$). In particular, they are not unipotent transformations.*

4.2. Semistable reduction for families of rational curves. We need the genus 0 version of semistable reduction as proven by de Jong [dJ].

Lemma 4.2. (de Jong) *Suppose $P \rightarrow S$ is a morphism whose general fibers are projective lines, with a divisor $K \subset P$. Then there is a finite covering and modification of the base $S' \rightarrow S$ and a family $P' \rightarrow S'$ with divisor $K' \subset P'$ such that $(P', K') \rightarrow S'$ is a semistable family of marked rational curves, and such that the open set $U' \subset S'$ over which the family is a smooth family of marked projective lines, has complement $D' := S' - U'$ which is a divisor with normal crossings. Further, the variety P' can be assumed to be smooth.*

Proof. This is [dJ, Theorem 4.1, Proposition 3.6]. □

4.3. A Lefschetz fibration for families. Suppose S is a smooth projective variety and $J \subset S$ is a normal crossings divisor with smooth components. Let $U := S - J$. Suppose $f : X_U \rightarrow U$ is a smooth projective morphism over U , where X_U is smooth. Let n denote the relative dimension of X_U over U .

In the following discussion we allow modification of the base S as described in Lemma 3.12 and Corollary 3.13.

Suppose we are given an embedding $X_U \rightarrow S \times \mathbb{P}^N$. Choose a general line $\mathbb{P}^1 \subset \mathbb{P}^N$. After possibly restricting to a smaller open set we may assume that this line defines a Lefschetz pencil for the subvariety X_u for every $u \in U$.

Let $P_0 := U \times \mathbb{P}^1$, with projection

$$q : P_0 \longrightarrow S$$

so that we have a family of Lefschetz pencils

$$X_U \xleftarrow{\alpha} Z_0 \xrightarrow{\varphi} P_0$$

where $\alpha : Z_0 \rightarrow X_U$ is blowing up the smooth family of base loci $B_0 \rightarrow U = S - J$, and the map $Z_0 \rightarrow P_0$ is a Lefschetz pencil of the fiber X_s for $s \in S - J$.

Let $K_0 \subset P_0$ denote the family of points over which the Lefschetz pencils have singular fibers. This is a divisor which is smooth over the base open set U : the different double points stay distinct as $u \in U$ moves by the hypothesis that we have a Lefschetz pencil for every $u \in U$.

By going to a finite covering of U we may suppose that each irreducible component of the divisor $K_0 \subset P_0$ maps isomorphically onto U . Thus we can consider (P_0, K_0) as a family of marked rational curves.

Apply the semistable reduction result Lemma 4.2, recalled in the previous subsection. After further modification of the base, then completing the families X_U , Z_0 and B_0 we obtain the following situation.

Keep the notations that $J \subset S$ is a divisor with normal crossings and $U := S - J$ an open subset. We have a semistable marked rational curve $q : (P, K) \rightarrow S$ with divisor $D := K \cup q^{-1}(J) \subset P$. We have a diagram

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & P \\ \downarrow \alpha & & \downarrow q \\ X & \xrightarrow{f} & S. \end{array}$$

We also have a subvariety $B \subset X$ corresponding to the base loci.

If we denote in general by a subscript U the restriction of objects over the open set U , then $P_U \cong U \times \mathbb{P}^1$ and

$$X_U \xleftarrow{\alpha_U} Z_U \xrightarrow{\varphi_U} P_U \cong U \times \mathbb{P}^1$$

is a family of Lefschetz pencils indexed by U . The divisor $K_U \subset P_U$ which is a union of components mapping isomorphically to U , is the divisor of singularities of φ_U . The subvariety $B_U \subset X_U$ is the family of base loci for the Lefschetz pencils.

Lemma 4.3. *There is a decomposition of local systems over U ,*

$$R^i \alpha_* \mathbb{Q} = R^0 q_*(R^i \varphi_* \mathbb{Q}) \oplus R^1 q_*(R^{i-1} \varphi_* \mathbb{Q}) \oplus R^2 q_*(R^{i-2} \varphi_* \mathbb{Q}).$$

Proof. See [De2, p.112]. □

We need a de Rham version of this statement. It would be good to have a direct analogue, however the singularities in the Lefschetz fibration are not normal crossings singularities. Thus, we consider an open subset W obtained by removing the singular fibers. In the start of the next section we will give a series of reductions which basically say that it suffices to look at this open set.

It should be possible to obtain a direct de Rham version of Lemma 4.3 by applying the general theory of \mathcal{D} -modules, see [CK].

Consider \mathcal{O}_{Z_U} the trivial vector bundle with standard connection d . This corresponds to the constant local system.

Taking the relative de Rham cohomology we obtain a vector bundle together with its Gauss-Manin connection

$$\mathcal{H}_{DR}^i(Z_U/U, (\mathcal{O}_{Z_U}, d)) := \mathbb{R}^i(f \circ \alpha)_* DR(Z_U/U, \mathcal{O}_{Z_U}, d)$$

on U . Let $W := P_U - K_U$ be the complement of the locus of singularities of the family of Lefschetz pencils. Let $Z_W := \varphi^{-1}(W) \subset Z$. The map φ is smooth over W .

Similarly using the morphism φ , define (E_W^i, ∇_W) to be the vector bundle with Gauss-Manin connection

$$E_W^i := \mathcal{H}_{DR}^i(Z_W/W, (\mathcal{O}_{Z_W}, d)) := \mathbb{R}^i \varphi_* DR(Z_W/W, \mathcal{O}_{Z_W}, d)$$

As usual this extends over P to a logarithmic connection with rational residues [St]. Let (E^i, ∇) denote the associated parabolic bundle on P , and let (E_U^i, ∇_U) denote the restriction to P_U which is also the associated parabolic bundle on P_U . Note that (E^i, ∇) has singularities along the divisor $D := K \cup q^{-1}(J)$ so (E_U^i, ∇_U) has singularities along K_U .

For any multi-index α for the divisor D we obtain a vector bundle with logarithmic connection (E_α^i, ∇) on P with singularities along D . Let $(E_{\alpha,U}^i, \nabla_U)$ be its restriction to P_U . This is the same as the corresponding bundle associated to the parabolic bundle E_U^i for multi-index α_U which contains only the indices for the components of the divisor K_U .

In what follows we will also consider the morphism $Z_W \rightarrow U$ which doesn't have projective fibers. We can still define the relative de Rham cohomology $\mathcal{H}_{DR}^m(Z_W/U, (\mathcal{O}_Z, d))$ which is again a vector bundle with integrable Gauss-Manin connection on U . This has regular singularities and corresponds to the local system $R^m(\varphi_W \circ q_W)_* \mathbb{C}_{Z_W}$ on U . For our present purposes this can be considered as a matter of notation: the $\mathcal{H}_{DR}^m(Z_W/U, (\mathcal{O}_Z, d))$ can be defined as the unique bundle with regular singular integrable connection on U corresponding to the local system $R^m(\varphi_W \circ q_W)_* \mathbb{C}_{Z_W}$, so we don't need to explain in detail how to construct it which would involve resolving the double point singularities in the Lefschetz fibration and then taking relative logarithmic de Rham cohomology.

Lemma 4.4. *Assume that $\alpha_i > 0$ for indices corresponding to components of the divisor K . The Leray spectral sequence for $Z_W \rightarrow W \rightarrow U$ is*

$$E_2^{i,j} = \mathcal{H}_{DR}^i(P_U/U, E_{\alpha,U}^j, \nabla) \Rightarrow \mathcal{H}_{DR}^{i+j}(Z_W/U, (\mathcal{O}_Z, d)).$$

This is a spectral sequence in the category of vector bundles over U with integrable connection having regular singularities at the complementary divisor J .

Proof. The category of vector bundles over U with integrable connection having regular singularities at the complementary divisor J is equivalent to the category of local systems over U , by [De1]. Therefore it suffices to have the spectral sequence for the associated local systems.

The Leray spectral sequence for $Z_W \xrightarrow{\varphi_W} W \xrightarrow{q_W} U$ is

$$E_2^{i,j} = R^i q_{W,*}(R^j \varphi_{W,*} \mathbb{C}_W) \Rightarrow R^{i+j}(q_W \circ \varphi_W)_* \mathbb{C}_W.$$

By Corollary 3.8 in each fiber, we have that

$$\mathcal{H}_{DR}^i(P_U/U, E_{\alpha,U}^k, \nabla)$$

is the vector bundle with regular singular integrable connection corresponding to the local system

$$R^i q_{W,*}(R^j \varphi_{W,*} \mathbb{C}_W).$$

Note that the Gauss-Manin connection on $\mathcal{H}_{DR}^i(P_U/U, E_{\alpha,U}^j, \nabla)$ is known to have regular singularities, in fact we shall use its extension as a logarithmic connection given by the relative logarithmic de Rham cohomology over S .

On the other hand,

$$\mathcal{H}_{DR}^{i+j}(Z_W/U, (\mathcal{O}_Z, d))$$

is the vector bundle with integrable connection corresponding to the local system $R^{i+j}(q_W \circ \varphi_W)_* \mathbb{C}_W$. If we use the paragraph before the statement of the present lemma then this is by definition, even.

In view of the equivalence of categories, we get the desired spectral sequence. \square

Let

$$\overline{\mathcal{H}}_{DR}^i(P_U/U, E_{\alpha,U}^j, \nabla)$$

and

$$\overline{\mathcal{H}}_{DR}^{i+j}(Z_W/U, (\mathcal{O}_Z, d))$$

denote the parabolic bundles on S associated to these vector bundles with logarithmic connection on U .

Corollary 4.5. *Assume that $\alpha_i > 0$ for indices corresponding to components of the divisor K . In the Chow group of S tensored with \mathbb{Q} we have*

$$\sum_m (-1)^m \text{ch}^{\text{par}} \overline{\mathcal{H}}_{DR}^m(Z_W/U, (\mathcal{O}_Z, d)) = \sum_{i,j} (-1)^{i+j} \text{ch}^{\text{par}} \overline{\mathcal{H}}_{DR}^i(P_U/U, E_{\alpha,U}^j, \nabla).$$

Proof. Formation of the associated parabolic bundle commutes with exact sequences by Lemma 3.5. The spectral sequence of Lemma 4.4 thus gives the result. \square

5. MAIN THEOREM

5.1. The main statement. We now give the statement of our main theorem in the form suitable for our inductive argument.

Suppose S is a smooth projective variety and $A \subset S$ is a normal crossings divisor with smooth components. Let $U := S - A$. Suppose $f : X \rightarrow S$ is a projective morphism which is smooth over U , and let $X_U := X \times_S U$. Let

$$\mathcal{H}_{DR}^i(X_U/U) := \mathbb{R}^i f_*(\Omega_{X_U/U}, d)$$

denote the bundle of i -th relative de Rham cohomology bundle on U . It has a Gauss-Manin connection ∇ which is well-known to have regular singularities and rational residues ([St, Proposition 2.20]). Let $\overline{\mathcal{H}}_{DR}^i(X_U/U)$ denote the parabolic bundle associated to $(\mathcal{H}_{DR}^i(X_U/U), \nabla)$ on U .

We extend this notation to the case when $f : X \rightarrow S$ is not generically smooth. There is still an open set $U \subset S$ over which the map is topologically a fibration for the usual topology, and we assume that the complement has normal crossings. In this case, define $\mathcal{H}_{DR}^i(X_U/U)$ to be the unique vector bundle with regular singular connection still denoted ∇ which corresponds to the local system $R^i f_{U,*} \mathbb{C}_{X_U}$ over U . Again let $\overline{\mathcal{H}}_{DR}^i(X_U/U)$ denote the associated parabolic bundle on S .

We state a general form of our main theorem. It will be useful to have this generality in the inductive proof.

Theorem 5.1. *Fix n . Suppose X, A, U, Y, f are as above, including the case where f is projective but not generically smooth. Suppose that $f : X \rightarrow S$ has relative dimension $\leq n$. Then the alternating sum of Chern characters of the parabolic extensions to S*

$$\sum_{i=0}^{2n} (-1)^i \text{ch}^{\text{par}}(\overline{\mathcal{H}}^i(X_U/U))$$

lies in $CH^0(S)_{\mathbb{Q}}$ or equivalently the pieces in all of the positive-codimension Chow groups with rational coefficients vanish.

The proof will be by induction on n and using Lefschetz pencils. ³

5.2. Preliminary reductions. We start with some reductions. The first one was discussed in §3.5, Lemma 3.12 and Corollary 3.13.

Lemma 5.2. *In proving the theorem for $f : X \rightarrow S$, we can modify S by a generically finite morphism $S' \rightarrow S$ and it suffices to prove the theorem for $X' \rightarrow S'$.*

Lemma 5.3. *In order to prove the theorem for n and for morphisms $X \rightarrow S$ which may not be smooth, it suffices to prove the theorem for any $Y_U \rightarrow U$ a smooth projective family of relative dimension $\leq n$.*

Proof. We prove the present lemma also by induction on n . Therefore, we may consider a family $f : X \rightarrow S$ of relative dimension n and assume that the theorem is known for arbitrary (not necessarily smooth) projective families of relative dimension $< n$.

After possibly making a modification of the base S as in Lemma 5.2, specially to decrease the size of the open set U and resolve singularities of the complementary divisor, we can assume that we have an open set $W \subset X$ such that $f : W \rightarrow U$ is smooth and

³Bloch and Esnault [BE2] used a similar approach earlier to prove a Riemann–Roch statement for Chern–Simons class which essentially required to consider the value at the generic point of the base S .

topologically a fibration. Let $T := X - W$ and assume that $(X_U, T_U) \rightarrow U$ is topologically a fibration.

Choose another compactification $W \subset Y$ with a morphism $g : Y \rightarrow U$ such that g is smooth and projective. Let $V := Y - W$ be the complementary divisor which we assume to have relative normal crossings over U .

Fix a point $u \in U$. We have pairs (X_u, T_u) and (Y_u, V_u) . In both of these, the smooth complementary open set is W_u . The cohomology of either of these pairs is equal to the compactly supported cohomology of W_u . Poincaré duality says

$$H^i((X_u, T_u), \mathbb{C}) \cong H^{2n-i}(W_u, \mathbb{C}) \cong H^i((Y_u, V_u), \mathbb{C}).$$

On the other hand, we have a long exact sequence relating $H^*(X_u, \mathbb{C})$, $H^*(T_u, \mathbb{C})$, and $H^i((X_u, T_u), \mathbb{C})$. Similarly we have a long exact sequence relating $H^*(Y_u, \mathbb{C})$, $H^*(V_u, \mathbb{C})$, and $H^i((Y_u, V_u), \mathbb{C})$. As u varies in U , the Poincaré duality isomorphisms are isomorphisms of local systems over U , and the long exact sequences are long exact sequences in the category of local systems over U . In view of the fact that the associated parabolic bundle commutes with isomorphisms and with taking exact sequences, we obtain a long exact sequence relating

$$\overline{\mathcal{H}}_{DR}(X_U/U), \quad \overline{\mathcal{H}}_{DR}(T_U/U), \quad \text{and} \quad \overline{\mathcal{H}}_{DR}^{2n-\cdot}(W/U),$$

and a long exact sequence relating

$$\overline{\mathcal{H}}_{DR}(Y_U/U), \quad \overline{\mathcal{H}}_{DR}(V_U/U), \quad \text{and} \quad \overline{\mathcal{H}}_{DR}^{2n-\cdot}(W/U).$$

The shift of indices induced by Poincaré duality is $2n$, an even number, so it doesn't affect the sign of the alternating sum. Thus we obtain

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(X_U/U)) - \sum_{i=0}^{2n-2} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(T_U/U)) = \sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(W/U))$$

and

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(Y_U/U)) - \sum_{i=0}^{2n-2} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(V_U/U)) = \sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(W/U)).$$

In particular, by the induction hypothesis we know the statement of the theorem for $T \rightarrow S$ and $V \rightarrow S$ because these have relative dimension $\leq n-1$. On the other hand we also know the statement of the theorem for $Y \rightarrow S$, because this is the hypothesis of our present reduction lemma. Putting these together we obtain the statement of the theorem for $X \rightarrow S$. \square

Lemma 5.4. *The statement of the theorem is true in relative dimension $n = 0$.*

Proof. Indeed, in this case by making a generically finite modification of the base, we can reduce to the case when the morphism $X_U \rightarrow U$ is a disjoint union of sections isomorphic to U . Thus the relative cohomology sheaf, nontrivial only in degree zero, is a trivial bundle. \square

Remark: It would be good to have the statement of Theorem 5.1 for families which are not necessarily projective. Our method of proof gives the statement for smooth quasiprojective families. The reduction from the general case to these cases would seem to require a more detailed utilisation of Grothendieck-Verdier duality theory than we are prepared to do here.

5.3. Lefschetz pencil and reduction to the case of the open set Z_W/U . We now assume given a projective family $f : X \rightarrow S$ of relative dimension n such that X_U is smooth over U , and assume the theorem is known for relative dimension $\leq n - 1$. After modification of the base as per Lemma 3.12, arrange to have a family of Lefschetz pencils as in §4.2. Keep the same notations Z, P, W from there.

We assume that we have modified S enough so that the monodromies of the local systems which we encounter on U or W are unipotent around components of J or vertical components of $f^{-1}(J)$. In this case, the extended bundles from U to S are actual bundles rather than parabolic bundles. We may use the same notation ch for the parabolic Chern character of parabolic bundles as that used for regular bundles, but where possible we indicate when a nontrivial parabolic structure might be involved by the notation ch^{par} .

By our inductive hypothesis, we know the statement of the theorem for the morphisms

$$\varphi : Z \rightarrow P, \quad q : P \rightarrow S.$$

Let (E^i, ∇) denote the parabolic bundles considered in §4.2. By the inductive hypothesis for the map φ we have

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(E^i) \in CH^0(P)_{\mathbb{Q}}.$$

We are assuming from the previous paragraph that the monodromies of the local system E^{∇} are unipotent around components of $f^{-1}(J)$, so the parabolic structure on E along these components of the divisor D is a trivial parabolic structure on a usual bundle. It is only in the case of a Lefschetz pencil of even fiber dimension (that is, when $n - 1$ is even) that E has nontrivial parabolic structure along the horizontal components K .

We show in the next two lemmas that the main step in the proof will be to treat the case Z_W/U .

Lemma 5.5. *Suppose we know the statement of the theorem for Z_W/U , in other words suppose we know that*

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(Z_W/U)) \in CH^0(S)_{\mathbb{Q}}.$$

Then we can conclude the statement of the theorem for Z_U/U .

Proof. Let $Z_{K,U} := Z_U - Z_W$. It is equal to the preimage $\varphi_U^{-1}(K_U)$, so it is a closed subset of Z_U .

Proceed as in the proof of Lemma 5.3. The pair $(Z_U, Z_{K,U})$ is a fibration over U . For each $u \in U$ the relative cohomology $H^\cdot((Z_u, Z_{K,u}), \mathbb{C})$ fits into a long exact sequence with $H^\cdot(Z_u, \mathbb{C})$ and $H^\cdot(Z_{K,u}, \mathbb{C})$. On the other hand, $H^\cdot((Z_u, Z_{K,u}), \mathbb{C})$ is isomorphic to the compactly supported cohomology of W_u which is Poincaré dual to the cohomology of W_u with $i \mapsto 2n - i$. This shift doesn't affect the signs in our alternating sums. As before, we obtain long exact sequences of local systems on U and then of associated parabolic bundles on S . Thus, the hypothesis of the present lemma implies that

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(Z_U/U)) - \sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(Z_{K,U}/U)) \in CH^0(S)_{\mathbb{Q}}.$$

On the other hand the morphism $Z_{K,U} \rightarrow U$ is projective of relative dimension $n - 1$. It is not smooth, but we have taken care to include the non-smooth case in our inductive statement. Thus, by the inductive assumption we know that

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(Z_{K,U}/U)) \in CH^0(S)_{\mathbb{Q}}.$$

We obtain the same conclusion for Z_U/U . □

Lemma 5.6. *Suppose we know the statement of the theorem for Z_U/U , in other words suppose we know that*

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(Z_U/U)) \in CH^0(S)_{\mathbb{Q}}.$$

Then we can conclude the statement of the theorem for X_U/U .

Proof. Recall that $B \subset X$ is the family of base loci for the Lefschetz pencil, and Z is obtained from X by blowing up along B , at least in the part lying over U . Let \tilde{B} denote the inverse image of B in Z . The relative dimension of B_U/U is $n - 2$ and the relative dimension of \tilde{B}_U/U is $n - 1$. Thus we can assume that we know the statement of the theorem for these families. Furthermore $Z_U - \tilde{B}_U \cong X_U - B_U$ and this is a smooth quasiprojective variety with topological fibration to U . Use the same argument as in Lemma 5.3 applying the long exact sequences for pairs and Poincaré duality on $(X_U - B_U)/U$. By this argument applied once to each side, if we know the statement of the theorem for Z_U/U then we get it for $(Z_U - \tilde{B}_U)/U = (X_U - B_U)/U$ and hence for X_U/U . □

Corollary 5.7. *Suppose we know the statement of the theorem for Z_W/U , then we can conclude the statement of the theorem for X_U/U .*

Proof. Put together the previous two lemmas. □

5.4. **The proof for Z_W/U .** Up to now we have reduced to the problem of proving the statement of the theorem for the family of open smooth varieties Z_W/U . Here is where we use the Lefschetz pencil $Z_W \xrightarrow{\varphi} W \xrightarrow{q} U$ as in §4.

Recall that we defined

$$E_W^i := \mathcal{H}_{DR}^i(Z_W/W, (\mathcal{O}_{Z_W}, d)) := \mathbb{R}^i \varphi_* DR(Z_W/W, \mathcal{O}_{Z_W}, d)$$

with its Gauss-Manin connection ∇_W on W . Then (E^i, ∇) was the associated parabolic bundle on P , with (E_U^i, ∇_U) the restriction to P_U . For any multi-index α for the divisor D we have (E_α^i, ∇) which is a vector bundle on P with logarithmic connection having singularities along D , and $(E_{\alpha,U}^i, \nabla_U)$ is the restriction to P_U .

We assume that $\alpha_i > 0$ for components D_i appearing in K , that is components which subject to S , whereas we assume from now on that $\alpha_i = 0$ for vertical components D_i , that is the components which map to components of J .

By Corollary 4.5 we have

$$\sum_m (-1)^m \text{ch} \overline{\mathcal{H}}_{DR}^m(Z_W/U) = \sum_{i,j} (-1)^{i+j} \text{ch} \overline{\mathcal{H}}_{DR}^i(P_U/U, E_{\alpha,U}^j, \nabla).$$

On the left in this formula is the quantity which we are trying to show lies in $CH^0(S)_{\mathbb{Q}}$.

Recall that the quantity on the right is defined to be the parabolic bundle on S associated to the vector bundle $\mathcal{H}_{DR}^i(P_U/U, E_{\alpha,U}^j, \nabla)$ with regular singular Gauss-Manin connection over U .

Lemma 5.8. *The extended parabolic bundle $\overline{\mathcal{H}}_{DR}^i(P_U/U, E_{\alpha,U}^j, \nabla)$ is a vector bundle with trivial parabolic structure, and it corresponds to the vector bundle $\mathcal{H}_{DR}^i(P/S, E_\alpha^j, \nabla)$ obtained by taking the relative logarithmic de Rham cohomology over S .*

Proof. This is the contents of our analogue of Steenbrink's theorem, Theorem 6.1 (see §6). Note that the fact that we have chosen $\alpha_i = 0$ for vertical components of the divisor D_i means that the logarithmic connection ∇ on E_α^i has nilpotent residues along the vertical components D_i . Recall for this that we are assuming that we have made sufficient modification $S' \rightarrow S$ so that the monodromy transformations of (E, ∇) around vertical components of the divisor D are unipotent. \square

Corollary 5.9. *For the proof of Theorem 5.1 it suffices to show that*

$$\sum_{i,j} (-1)^{i+j} \text{ch} \mathcal{H}_{DR}^i(P/S, E_\alpha^j, \nabla) \in CH^0(S)_{\mathbb{Q}}.$$

\square

We now proceed with the proof of the formula to which we have reduced in the previous corollary.

Use the inductive statement for the family $\varphi : Z \rightarrow P$. Note that this has relative dimension $n - 1$. The parabolic bundle on P associated to the higher direct image local

system on the open set W is E^i . Therefore our inductive hypothesis for Z/P says that

$$\sum_i (-1)^i \text{ch}^{\text{par}}(E^i) \in CH^0(P)_{\mathbb{Q}}.$$

Next observe that by Lemma 2.7, the difference between the parabolic Chern character of the parabolic bundle, and the Chern character of any component bundle, comes from the divisor D . In fact, due to our assumption about the monodromy, E^i is just a usual bundle along the vertical components of the divisor. Thus, for multi-indices α such that $\alpha_i = 0$ along vertical components, the difference comes from the divisor K :

$$\text{ch}^{\text{par}}(E^i) - \text{ch}(E_{\alpha}^i) \in \text{Image}(CH^{\cdot}(K)_{\mathbb{Q}} \longrightarrow CH^{\cdot}(P)_{\mathbb{Q}}).$$

To be more precise this means that the difference is in the subspace of $CH^{\cdot}(P)_{\mathbb{Q}}$ spanned by Chern characters of sheaves concentrated on K . Thus we get

$$\sum_i (-1)^i \text{ch}(E_{\alpha}^i) \in CH^{\cdot}(K)_{\mathbb{Q}} + CH^0(P)_{\mathbb{Q}}.$$

Write

$$\sum_i (-1)^i \text{ch}(E_{\alpha}^i) = \mathbf{k} + \mathbf{r}$$

with $\mathbf{k} \in CH^{\cdot}(K)_{\mathbb{Q}}$ and $\mathbf{r} \in CH^0(P)_{\mathbb{Q}}$. Note that \mathbf{r} is just an integer, the alternating sum of the ranks of the bundles.

Now look at the relative logarithmic de Rham complex for E_{α}^i :

$$E_{\alpha}^i \rightarrow E_{\alpha}^i \otimes_{\mathcal{O}_P} \Omega_{P/S}^1(\log D).$$

The Hodge to de Rham spectral sequence and multiplicativity for tensor products and additivity for exact sequences of the Chern character, gives the formula

$$\begin{aligned} \sum_{i,j} (-1)^{i+j} \text{ch} \mathcal{H}_{DR}^i(P/S, E_{\alpha}^j, \nabla) = \\ \chi_{P/S} \left(\left[\sum_i (-1)^i \text{ch}(E_{\alpha}^i) \right] \cdot \left[\sum_{j=0,1} (-1)^j \text{ch} \Omega_{P/S}^j(\log D) \right] \right). \end{aligned}$$

Recall here that $\chi_{P/S}$ represents the function (see Proposition 3.11), entering into the GRR formula for P/S calculating the relative Euler characteristic in $CH^{\cdot}(S)_{\mathbb{Q}}$ for an element of $CH^{\cdot}(P)_{\mathbb{Q}}$.

This formula decomposes into two pieces according to the decomposition $\mathbf{k} + \mathbf{r}$ above:

$$\begin{aligned} \sum_{i,j} (-1)^{i+j} \text{ch} \mathcal{H}_{DR}^i(P/S, E_{\alpha}^j, \nabla) = & \chi_{P/S} \left(\mathbf{k} \cdot \left[\sum_{j=0,1} (-1)^j \text{ch} \Omega_{P/S}^j(\log D) \right] \right) \\ & + \chi_{P/S} \left(\mathbf{r} \cdot \left[\sum_{j=0,1} (-1)^j \text{ch} \Omega_{P/S}^j(\log D) \right] \right). \end{aligned}$$

Lemma 5.10. *In the above decomposition, the first piece vanishes:*

$$\chi_{P/S} \left(\mathbf{k} \cdot \left[\sum_{j=0,1} (-1)^j \text{ch } \Omega_{P/S}^j(\log D) \right] \right) = 0$$

Proof. We claim that the argument inside the function $\chi_{P/S}$ is already zero. Indeed, \mathbf{k} is a sum of Chern characters of sheaves supported on K , but on the other hand the divisor K is a union of components which appear in D and only intersect single vertical components of the divisor D , because $(P, K) \rightarrow S$ is a semistable family of pointed curves. Thus we have a residue isomorphism

$$\Omega_{P/S}^1(\log D)|_K \cong \mathcal{O}_K = \Omega_{P/S}^0(\log D)|_K,$$

so the difference $\sum_{j=0,1} (-1)^j \text{ch } \Omega_{P/S}^j(\log D)$ restricts to zero on K . When we multiply it by \mathbf{k} we get zero. \square

After this we are left only with the second piece:

$$\chi_{P/S} \left(\mathbf{r} \cdot \left[\sum_{j=0,1} (-1)^j \text{ch } \Omega_{P/S}^j(\log D) \right] \right).$$

The class \mathbf{r} is just an integer considered in $CH^0(P)_{\mathbb{Q}}$. Thus, the following lemma will complete the proof.

Lemma 5.11. *In our situation,*

$$\chi_{P/S} \left(\sum_{j=0,1} (-1)^j \text{ch } \Omega_{P/S}^j(\log D) \right) \in CH^0(S)_{\mathbb{Q}}.$$

Proof. This quantity is equal to

$$\sum_i (-1)^{i+j} \text{ch } \mathcal{H}_{DR}^i(P/S, (\mathcal{O}_P, d)).$$

By the same argument as above using Steenbrink's theorem for the fibration $q : P \rightarrow S$, the terms $\mathcal{H}_{DR}^i(P/S, (\mathcal{O}_P, d))$ are the vector bundles over S associated by the Deligne canonical extension to the local systems $R^i q_{U,*} \mathbb{C}_{P_U}$ defined on U . Since P/S is semistable the monodromy here is unipotent. However in this case $P_U \rightarrow U$ is a family of smooth rational curves. Thus the direct image local systems are \mathbb{C}_U for $i = 0, 2$ and 0 for $i = 1$. Their canonical extensions are trivial bundles on S so

$$\sum_i (-1)^{i+j} \text{ch } \mathcal{H}_{DR}^i(P/S, (\mathcal{O}_P, d)) \in CH^0(S)_{\mathbb{Q}}.$$

\square

Multiplied by the integer \mathbf{r} , the result of this lemma shows that the second term in the previous decomposition lies in $CH^0(S)_{\mathbb{Q}}$. As we saw in Lemma 5.10 that the other piece vanishes, we obtain:

Corollary 5.12.

$$\sum_{i,j} (-1)^{i+j} \text{ch} \mathcal{H}_{DR}^i(P/S, E_\alpha^j, \nabla) \in CH^0(S)_\mathbb{Q}.$$

□

In view of Corollary 5.9, this completes the proof of Theorem 5.1. □

5.5. Family of projective surfaces. Consider a family of projective surfaces

$$\pi : X \longrightarrow S$$

where S is smooth. Then, with notations as in Theorem 5.1, we have

Proposition 5.13. *The parabolic Chern character of the parabolic bundles associated to weight i de Rham bundles satisfies*

$$\text{ch}(\overline{\mathcal{H}}^i(X/S)) \in CH^0(S)_\mathbb{Q}$$

for each $i \geq 0$.

We first consider the following case:

Lemma 5.14. *Suppose $\pi : X \longrightarrow S$ is a generically smooth morphism of relative dimension 2. Then*

$$\text{ch}(\overline{\mathcal{H}}^i(X/S)) \in CH^0(S)_\mathbb{Q}$$

for each $i \geq 0$.

Proof. Firstly, the Hard Lefschetz theorem and [Es-Vi2, Theorem 1.1] imply that

$$\text{ch}(\overline{\mathcal{H}}^1(X/S)), \text{ch}(\overline{\mathcal{H}}^3(X/S)) \in CH^0(S)_\mathbb{Q}.$$

Hence by Theorem 5.1, we conclude that

$$\text{ch}(\overline{\mathcal{H}}^2(X/S)) \in CH^0(S)_\mathbb{Q}.$$

□

Now suppose $\pi : X \longrightarrow S$ be any projective morphism. Let $\text{Spec } K$ be the generic point of S and $X_K := X \times_{\mathbb{C}} \text{Spec } K$. Since the field K is of characteristic zero, by Hironaka's theorem [Hi], there is a resolution of singularities

$$f : X'_K \longrightarrow X_K$$

defined over the field K such that f is a sequence of blow-ups and X'_K is nonsingular.

This implies that there is a non-empty open subset $U \subset S$ and a commutative diagram:

$$\begin{array}{ccc} X'_U & \xrightarrow{f} & X_U \\ \downarrow \pi' & & \downarrow \pi \\ U & = & U \end{array}$$

such that π' is a smooth morphism and the variety $X'_U \rightarrow U$ is a fibrewise resolution of singularities of $X_U \rightarrow U$. By Lemma 3.12, we assume that $D := S - U$ is a normal crossing divisor.

Since a fibre X_s of π is a singular surface, the cohomology of X_s carries a mixed Hodge structure. In particular, over the open subset of S where π is a topological fibration (which again by Lemma 3.12 can be assumed to be U), we obtain a filtration of local systems with associated-graded

$$(1) \quad Gr.R^i\pi_*\mathbb{Q} = \bigoplus_{k=0}^i Gr_k R^i\pi_*\mathbb{Q}.$$

Here $Gr_k R^i\pi_*\mathbb{Q}$ is the local system corresponding to the weight k graded piece of the weight filtration $\{W_k\}$ on the cohomology $H^i(X_s, \mathbb{Q})$. More precisely, $H^i(X_s, \mathbb{Q})$ carries a weight filtration

$$(0) = W_{-1} \subset W_0 \subset \dots \subset W_i = H^i(X_s, \mathbb{Q})$$

such that $Gr_k H^i(X_s, \mathbb{Q})$ carries a polarised pure Hodge structure of weight k . In other words, $Gr_k R^i\pi_*\mathbb{Q}$ is the local system associated to a family of polarised pure Hodge structures of weight k . We refer to [De3] for the details.

Also, notice that the morphism $f_s^* : W_i = H^i(X_s, \mathbb{Q}) \rightarrow H^i(X'_s, \mathbb{Q})$ is a morphism of mixed Hodge structures whose kernel is W_{i-1} . In particular, the i -th graded piece $Gr_i H^i(X_s, \mathbb{Q})$ is a polarised pure sub-Hodge structure of weight i of $H^i(X'_s, \mathbb{Q})$. When $i = 2$, the complementary sub-Hodge structure is generated by the algebraic classes since $X'_s \rightarrow X_s$ is a sequence of blow-ups and normalizations for $s \in U$.

These statements put together over U corresponds to saying that

Lemma 5.15. *There is a decomposition of local systems*

$$R^2\pi'_*\mathbb{Q} = Gr_2 R^2\pi_*\mathbb{Q} \oplus T$$

where T is a trivial local system.

Proof. The kernel of the map on H_2 of a surface corresponding either to blowing up a point, or to normalization, is generated by algebraic cycles. Therefore the kernel of $H_2(X'_s) \rightarrow H_2(X_s)$ is generated by algebraic cycles. This gives an exact sequence

$$0 \rightarrow Gr_2 R^2\pi_*\mathbb{Q} \rightarrow R^2\pi'_*\mathbb{Q} \rightarrow T \rightarrow 0$$

where T corresponds to a local system of finite monodromy. After going to a finite cover of U we may assume T is trivial. The cup product on $R^2\pi'_*\mathbb{Q}$ is a nondegenerate form which is nondegenerate on $Gr_2 R^2\pi_*\mathbb{Q}$ so we get an orthogonal splitting of the exact sequence. \square

Consider the parabolic bundles $\overline{\mathcal{G}}_k(\mathcal{H}^i)$ associated to the graded pieces $Gr_k R^i\pi_*\mathbb{Q}$, for each i, k , on S .

Then we have

Lemma 5.16. *The parabolic Chern character of the above parabolic bundles satisfies*

$$ch(\overline{\mathcal{G}}_k(\mathcal{H}^i)) \in CH^0(S)_{\mathbb{Q}}$$

for $i, k \leq 2$.

Proof. By Lemma 5.14, we have $ch(\overline{\mathcal{H}}^2(X'/S)) \in CH^0(S)_{\mathbb{Q}}$. Hence by Lemma 5.15, we conclude that

$$ch(\overline{\mathcal{G}}_2(\mathcal{H}^2)) \in CH^0(S)_{\mathbb{Q}}.$$

Since the parabolic bundle $\overline{\mathcal{G}}_0(\mathcal{H}^i)$ for any i , is a trivial bundle and $\overline{\mathcal{G}}_1(\mathcal{H}^i)$ corresponds to a family of polarised pure Hodge structures of weight one, by [Es-Vi2, Theorem 1.1], we conclude that

$$ch(\overline{\mathcal{G}}_1(\mathcal{H}^2)) \in CH^0(S)_{\mathbb{Q}}.$$

□

Using (1) and Lemma 3.5, we conclude that $\overline{\mathcal{H}}^2(X/S)$ (resp. $\overline{\mathcal{H}}^1(X/S)$) has a filtration whose associated-graded is $\bigoplus_{k=0}^2 \overline{\mathcal{G}}_k(\mathcal{H}^2)$ (resp. $\bigoplus_{k=0}^1 \overline{\mathcal{G}}_k(\mathcal{H}^1)$).

By Lemma 5.16, we obtain

$$ch(\overline{\mathcal{H}}^i(X/S)) \in CH^0(S)_{\mathbb{Q}}$$

for $i = 0, 1, 2$. Now by Theorem 5.1, we obtain

$$ch(\overline{\mathcal{H}}^3(X/S)) \in CH^0(S)_{\mathbb{Q}}.$$

This completes the proof of Proposition 5.13.

6. APPENDIX: AN ANALOGUE OF STEENBRINK'S THEOREM

Suppose $f : P \rightarrow S$ is a semistable morphism of relative dimension one, with respect to a divisor J on S and its pullback $f^{-1}J$ on P , both of which have normal crossings. Assume for simplicity that the total space of P is smooth (which is possible by [dJ, Proposition 3.6]).

Semistability implies that the components of $f^{-1}J$ occur with multiplicity one, and that f is flat. Let $U := S - J$ and $V := P - f^{-1}J = f^{-1}U$ be open sets in S and P respectively. Suppose $K \subset P$ is another divisor, isomorphic to a disjoint union of copies of S on which f is the identity. Suppose that the components of K meet $f^{-1}J$ transversally at smooth points of the latter, so that $(P, K) \rightarrow S$ is a semistable family of pointed curves. Let $D := K \cup f^{-1}J$.

Suppose that (E, ∇) is a logarithmic connection on P with singularities along D with rational residues. Suppose that the residual transformations of ∇ along components of $f^{-1}J$ are nilpotent. Assume also that for every component K_i of the horizontal divisor K we have a weight α_i such that the eigenvalues of the residues of ∇ along K_i are contained in $[-\alpha_i, 1 - \alpha_i)$.

Let $E \otimes \Omega_{P/S}(\log D)$ be the relative logarithmic de Rham complex (including K in the logarithmic part). Set

$$F := \mathcal{H}_{DR}(P/S, E, \nabla) = \mathbb{R}f_*(E \otimes \Omega_{P/S}(\log D)).$$

This is a complex of \mathcal{O}_S -modules, quasi-isomorphic to a complex of vector bundles, in other words it is a *perfect complex* on S . We can now state Steenbrink's theorem in this setting:

Theorem 6.1. *Keep the above hypothesis of a semistable family and logarithmic connection (E, ∇) such that the residues of ∇ along components of $f^{-1}J$ are nilpotent and the residues along K_i have eigenvalues in $[-\alpha_i, 1 - \alpha_i)$. Then the higher direct image complex F splits locally over S into a direct sum of its cohomology sheaves $H^i(F)$. Furthermore, these cohomology sheaves are locally free on S . They have Gauss-Manin connections which are logarithmic along J , and the residues of their Gauss-Manin connections along J are nilpotent. In particular, the cohomology sheaves $H^i(F)$ are the canonical extensions of the Gauss-Manin connections over the open set U (denoted $\overline{\mathcal{H}}^i$) and*

$$\mathrm{ch}(F) = \sum_i (-1)^i \mathrm{ch}(\overline{\mathcal{H}}^i).$$

Steenbrink proved this in the context of a geometric family [St, Theorem 2.18]. A weaker version of this theorem was proved earlier by N. Katz [Kz2]. Since then there have been a large number of generalizations of his theorem, see [Fa] [Fa2] [Sa] [II] [KN] [KMN] [IKN] [Ko] [Ts] [Ts2] [El] [Og] [Cai] among others. The argument we sketch below is probably contained in some of these references in some way.⁴

We will indicate a proof which we hope will be enlightening. The first thing to note is that it suffices to prove that the dimensions of the cohomology groups of the fibers $F_x := F \otimes_{\mathcal{O}_S} \mathbf{k}(x)$ remain constant. Therefore it suffices to look at a general curve going into each point, and since the semistability hypothesis is preserved under base change to a general curve in the base ([Ab-dJ]) we may assume that S is a curve. Note also that the Gauss-Manin connection is globally defined before we restrict to a curve, but it suffices to look over a curve in order to prove that the residues are nilpotent, so for this part also it suffices to assume that S is a curve.

The idea is that one can define the Gauss-Manin connection on the level of the complex F , Lemma 6.2 below [Kz2]. We then observe in Proposition 6.4 that these standard facts

⁴For example the argument given by Illusie in [II] uses the condition that the higher direct image is a perfect complex. Illusie doesn't treat the case of local coefficient systems, necessary for our inductive argument. He refers to Faltings [Fa], [Fa2] for treating some more general cases. The techniques we use here certainly come from this theory (the second author went to Faltings' course on p -adic Hodge theory). Also the note of Cailotto [Cai] seems to use an argument similar to the one we give here. Illusie, Kato and Nakayama touch on this result in §6 and Theorem 7.1 of [IKN], and refer for the case of a system of local coefficients to Kato-Matsubara-Nakayama [KMN]. There, this result is treated for local coefficients in a variation of Hodge structure, which would be sufficient for our purposes since (E, ∇) which intervenes in our argument is a variation of Hodge structure coming from the Lefschetz pencil.

imply local freeness of the cohomology sheaves by a direct calculation: any cohomology sheaves not locally free would lead to eigenvalues of the residues differing by nonzero integers which contradicts nilpotency. This is basically the same as the main lemma in [Cai].

Let \mathcal{L} denote the sheaf of differential operators of order ≤ 1 , whose symbols in degree 1 are vector fields tangent to J . Since we are assuming that S is a curve, an action of \mathcal{L} on F will contain all of the information that we need. In order to fix the ideas, note several facts about \mathcal{L} . It has left and right structures of \mathcal{O}_S -module which are different. We have an exact sequence, compatible with both the left and right module structures:

$$(2) \quad 0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{L} \rightarrow (\Omega_S^1(\log J))^* \rightarrow 0.$$

In view of the assumption that S is a curve, so J is a collection of points, we have that

$$(\Omega_S^1(\log J))^* = T_S(-J)$$

is just the sheaf of tangent vector fields vanishing at the points of J . Thus we can write the exact sequence as

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{L} \xrightarrow{\sigma} T_S(-J) \rightarrow 0.$$

The fiber $T_S(-D)_x$ over a point $x \in J$ is a complex line generated by the canonical element denoted $(t \frac{\partial}{\partial t})_x$ which is independent of the choice of coordinate t at x .

In the above exact sequence, the map σ is the *symbol* of a differential operator. Using it we can describe the difference between the left and right structures of \mathcal{O}_S -module. For $v \in \mathcal{L}$ and $a \in \mathcal{O}_S$ we have

$$v \cdot a - a \cdot v = \sigma(v)(a),$$

this answer $\sigma(v)(a)$ means the derivative of a along the vector field $\sigma(v)$, and $\sigma(v)(a)$ is considered as an element of $\mathcal{O}_S \subset \mathcal{L}$.

Suppose $x \in J$. The fiber of \mathcal{L} over x is the same when it is calculated on the left or the right. Let \mathfrak{m}_x be the maximal ideal at x and $\mathbb{C}_x := \mathcal{O}_S/\mathfrak{m}_x$. We have

$$\mathbb{C}_x \otimes_{\mathcal{O}_S} \mathcal{L} = \mathcal{L} \otimes_{\mathcal{O}_S} \mathbb{C}_x = \mathbb{C} \oplus \langle t \frac{\partial}{\partial t} \rangle_x.$$

Denote this by \mathcal{L}_x . This is special for points $x \in J$ because we have taken differential operators generated by vector fields which are tangent to J . In particular it means that an action of \mathcal{L} induces an action of \mathcal{L}_x on the fiber over x .

For general points in S the quotient of \mathcal{L} by the maximal ideal on the left and on the right are not canonically isomorphic, and we could no longer restrict to an action on the fiber in this way.

Lemma 6.2. *Let $F := \mathcal{H}_{DR}(P/S, E, \nabla)$ be the relative logarithmic de Rham cohomology, under the hypothesis of Theorem 6.1, specifically let it be the Čech complex obtained from*

an affine open covering. There is an action of \mathcal{L} on F given by

$$\mathcal{L} \otimes_{\mathcal{O}_S} F \rightarrow F$$

whose restriction to the scalars $\mathcal{O}_S \subset \mathcal{L}$ is the identity. Furthermore if $x \in J$ then the action of the vector $(t \frac{\partial}{\partial t})_x \in \mathcal{L}_x$ on the fiber F_x is nilpotent.

Proof. Katz constructs an action of the vector field $t \frac{\partial}{\partial t}$ by derivations on the full higher direct image complex in [Kz2], see also [KO]. This is the required action of \mathcal{L} , inducing the identity on the scalars. Katz shows that the indicial polynomial of the action of $(t \frac{\partial}{\partial t})_x$ divides a product of indicial polynomials of the residues of ∇ on vertical components. In our case where the residues of ∇ are supposed to be nilpotent in the hypothesis of 6.1, we get that $(t \frac{\partial}{\partial t})_x \in \mathcal{L}_x$ is nilpotent. ⁵ \square

Now note that \mathcal{L} is locally free as a right \mathcal{O}_S -module, so the tensor product is also the derived tensor product

$$\mathcal{L} \otimes_{\mathcal{O}_S} F = \mathcal{L} \otimes_{\mathcal{O}_S}^{\mathbb{L}} F.$$

In this spirit, we have a homotopy invariance of the existence of this action.

Lemma 6.3. *Locally over S , suppose F' is a perfect complex, that is a complex of vector bundles of finite length, quasiisomorphic to F . Then there is still an action*

$$\mathcal{L} \otimes_{\mathcal{O}_S} F' \rightarrow F'$$

with the property that for any $x \in J$, the action of \mathcal{L}_x on $H^i(F'_x)$ induces the identity action of the scalars $\mathbb{C} \subset \mathcal{L}_x$ and a nilpotent action of the vector $(t \frac{\partial}{\partial t})_x$.

Proof. Being quasiisomorphic means that there is a chain of quasiisomorphisms going in different directions relating F' and F . However, since F' consists of projective objects locally in the Zariski topology of S , we can represent this by an actual morphism of complexes $F' \rightarrow F$. Here we allow ourselves to replace S by a smaller neighborhood of the point $x \in J$ we are interested in. Similarly, $\mathcal{L} \otimes_{\mathcal{O}_S} F'$ is a complex of vector bundles so the map

$$\mathrm{Hom}(\mathcal{L} \otimes_{\mathcal{O}_S} F', F') \rightarrow \mathrm{Hom}(\mathcal{L} \otimes_{\mathcal{O}_S} F', Q \otimes \langle d \log t \rangle^*)$$

is a quasiisomorphism of complexes over S , and again possibly after going to an open set, our map

$$\mathcal{L} \otimes_{\mathcal{O}_S} F' \rightarrow Q \otimes \langle d \log t \rangle^*$$

lifts to a map

$$\mathcal{L} \otimes_{\mathcal{O}_S} F' \rightarrow F'$$

up to addition of a map of the form $d(\kappa)$, in other words up to a homotopy of complexes. Thus, for any choice of F' which is a complex of bundles representing F up to quasiisomorphism, we get an action of \mathcal{L} . This will induce the same map as the original

⁵We thank H. Esnault for pointing out the reference [Kz2] which replaces the lengthy discussion of this proof, using a truncated version of the theory of formal categories [Be], in our original version. One could also now invoke either the theory of \mathcal{D} -modules, or the *log-crystalline site* to get the same statement.

$\mathcal{L} \otimes F \rightarrow F$ on cohomology, and the same is true after applying a derived functor such as taking the fiber over a point $x \in J$. \square

Replace the notation F' above by the shorter notation F in what follows. See [Cai] for a similar statement with application to a Steenbrink-type theorem.

Proposition 6.4. *Suppose F is a complex of vector bundles over an affine curve or a complex disk S . Let $x \in S$ and let J be the divisor $[x]$. Define \mathcal{L} as before with respect to this divisor. Suppose that we are given an action*

$$M : \mathcal{L} \otimes_{\mathcal{O}_S} F \rightarrow F$$

where the tensor product uses the right module structure on \mathcal{L} and leaves the left module structure in the answer. Suppose that, when restricted to $\mathcal{O}_S \subset \mathcal{L}$ it gives a map

$$M_0 : F = \mathcal{O}_S \otimes_{\mathcal{O}_S} F \rightarrow F$$

which induces the identity on cohomology sheaves and on the cohomology spaces of F_x . Suppose furthermore that for our point x the morphism

$$M_x : \mathcal{L}_x \otimes_{\mathcal{O}_S} F_x \rightarrow F_x$$

induces a nilpotent action of $(t \frac{\partial}{\partial t})_x \subset \mathcal{L}_x$ on F_x . Then, locally in a neighborhood of x the complex F splits as a direct sum of its cohomology sheaves, and these cohomology sheaves are locally free.

Proof. The map M gives a map

$$H^i(M) : \mathcal{L} \otimes_{\mathcal{O}_S} \mathcal{H}^i(F) \rightarrow \mathcal{H}^i(F)$$

on cohomology sheaves. Away from D , this map is a connection, and as is well-known it implies that the cohomology sheaves are locally free. Thus our only problem is to prove that they are locally free near a point $x \in J$.

Over the disc or affine curve S the complex F is quasiisomorphic to the direct sum of its cohomology sheaves. This is because there are no Ext^i terms for $i \geq 2$ over \mathcal{O}_S since S is one-dimensional.

We can then choose a minimal resolution of each cohomology sheaf. This can be done explicitly, because by the Chinese remainder theorem we have that $H^i(F)$ is a direct sum of locally free modules and modules of the form $\mathcal{O}_S/(z^n)$. For the latter pieces if they exist (our goal is to show that they don't occur) choose as resolution

$$0 \rightarrow \mathcal{O}_S \xrightarrow{z^n} \mathcal{O}_S \rightarrow \mathcal{O}_S/(z^n) \rightarrow 0.$$

In this way we obtain a quasiisomorphism $R \rightarrow F \rightarrow R$ between F and a complex R of locally free sheaves with the ‘‘minimality’’ property that the differentials in R vanish at our singular point $x \in D$.

Now replace F by the minimal resolution R . We still get a map

$$M^R : \mathcal{L} \otimes_{\mathcal{O}_S} R \rightarrow R.$$

Its restriction M_0^R to $\mathcal{O}_S \subset \mathcal{L}$ is a map $M_0 : R \rightarrow R$ which induces the identity on cohomology sheaves. The fiber R_x is a complex whose differentials are equal to zero, in particular

$$R_x \cong \bigoplus_i H^i(F_x).$$

Therefore, $M_{0,x}^R$ is equal to the identity by our hypothesis that M_0 induces the identity on $H^i(F_x)$. In particular, $M_{0,x}^R$ is invertible, so by Nakayama's lemma, after possibly restricting the size of our disc or affine neighborhood of x , we may assume that $M_0^R : R \rightarrow R$ is invertible. Define

$$M' := (M_0^R)^{-1} \circ M^R : \mathcal{L} \otimes_{\mathcal{O}_S} R \rightarrow R.$$

We have now succeeded in normalizing so that the restriction of M' to $\mathcal{O}_S \subset \mathcal{L}$ is the identity. One can see, using (2), that this exactly defines a logarithmic connection denoted ∇ on the whole complex R . In particular each R^i now has a logarithmic connection and the differentials of R are compatible with these connections.

Note that the fiber \mathcal{L}_x is a vector space which is well-defined with respect to either the left or right \mathcal{O}_S -module structures of \mathcal{L} because the symbols $\sigma(v)$ of elements of \mathcal{L} vanish at $x \in D$. The exact sequence for \mathcal{L} splits over x to give just

$$\mathcal{L}_x \cong \mathbb{C} \oplus T_S(-x)_x \cong \mathbb{C} \oplus \mathbb{C}.$$

Thus M gives a well-defined map

$$M_x(t \frac{\partial}{\partial t})_x : F_x \rightarrow F_x.$$

The hypothesis of the present proposition is that this endomorphism of F_x induces nilpotent endomorphisms on cohomology. Composing with $R_x \rightarrow F_x \rightarrow R_x$ which are again quasiisomorphisms, gives the map

$$M_x^R(t \frac{\partial}{\partial t})_x : R_x \rightarrow R_x.$$

However, $R_x \cong H^*(F_x)$. Thus the action of $M_x^R(t \frac{\partial}{\partial t})_x$ on R_x is nilpotent.

We claim that the connection ∇ on the complex R has nilpotent residues at x . Indeed, the residue of ∇ at x is equal to $(M_{0,x}^R)^{-1} \circ M_x^R(t \frac{\partial}{\partial t})_x$, and we know from the hypothesis of our proposition that $M_{0,x}^R$ is the identity, so the residue of ∇ is nilpotent.

The claim implies, in particular, that each R^i is a Deligne canonical extension.

Lemma 6.5. *The differentials of the complex R are zero.*

Proof. Look at any one of the differentials

$$d_i : R^i \rightarrow R^{i+1}.$$

This map is a map of vector bundles with logarithmic connection. Thus d_i may be considered as a flat section of the vector bundle with logarithmic connection $\text{Hom}(R^i, R^{i+1})$. This latter is also a Deligne canonical extension. On the other hand, $d_i(x) = 0$ by the

minimality assumption for R . We are now in the following situation: we have a vector bundle with logarithmic connection which is a Deligne canonical extension (i.e. the residue is nilpotent), and we have a flat section which vanishes at x . We claim that this implies that the section vanishes everywhere. The claim is clearly true when we are dealing with sections of the canonical extension of a trivial connection. However, as we are working over a disc, our Deligne extension is a successive extension of trivial pieces, so by an induction argument, we get that $d_i = 0$. \square

Corollary 6.6. *The complex F is locally a direct sum of its cohomology sheaves, and these cohomology sheaves are locally free.*

Proof. This statement is invariant under quasiisomorphism, and it is true for R because of the previous corollary so it is true for F . \square

This corollary completes the proof of the proposition. \square

Proof of Theorem 6.1: Lemma 6.2 constructs the Gauss-Manin connection on the cohomology complex and shows that it satisfies the condition that the residual actions over points $x \in J$ are nilpotent, and in Proposition 6.4 we have seen that this nilpotence property implies that the cohomology complex F is locally a direct sum of locally free cohomology sheaves. Along the way, the nilpotence property provides the statement of the second part of Theorem 6.1. This completes the proof when S is a curve, and as stated at the beginning of the section, it is enough to treat the case when the base is a curve. The last statement about Chern characters follows because a complex and the direct sum of its cohomology are equivalent in K -theory. \square

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