Suppose V is a vector bundle over X = smooth projective curve with $\nabla : V \to V \otimes \Omega^1_X$ integrable connection.

If V is semistable we get a point in the moduli space $\mathcal{U}(r,X)$ of bundles. Let $M_{DR}(X,r)$ = the moduli space of (V,∇). This way we get

$$\begin{array}{c} (\text{open subset}) \subset M_{DR} \\ \downarrow \\ \mathcal{U}(r,X) \end{array}$$

This fibration is a twisted form of $T^*\mathcal{U}(r,X) \to \mathcal{U}(r,X)$.

What if V is not semistable?

(2)

Let $H \subset V$ be the maximal destabilizing subsheaf. Then $\nabla : H \to (V/H) \otimes \Omega^1_X$ is an algebraic map.

Put $E^1 := H$, $E^0 := V/H$, $\Theta = \nabla : E^1 \to E^0 \otimes \Omega^1_X$

$(E = \oplus E^p, \Theta)$ is a Higgs bundle of Hodge type (fixed by the $\mathbb{C}^*$-action)

This might be a semistable Higgs bundle.

If not, continue...

Now let $H \subseteq (E, \Theta)$ be the maximal destabilizing sub-Higgs-bundle.

We have $H = \oplus H^p$, $\Theta(H) \subset H \otimes \Omega^1_X$.

Use the new H to twist the filtration defined by the old one.

3

General Setup: X = curve

(V, ∇) a vector bundle with integrable connection

$F^{\cdot} = F^0 \supset F^1 \supset \cdots$

decreasing filtration.

It satisfies Griffiths Transversality

if $\nabla : F^p \to F^{p-1} \otimes \Omega^1_X$.

In this case, put $E := Gr_F(V) = \bigoplus E^p$

$E^p = F^p / F^{p+1}$, and θ is given

by ∇, $\theta : E^p \to E^{p-1} \otimes \Omega^1_X$.

(E, θ) is a Higgs bundle of Hodge type, i.e. fixed point of $\mathbb{C}^*$.

We say that $F^{\cdot}$ is gr-semistable if (E, θ) is a semistable Higgs bundle. If not, the maximal destabilizing subsheaf (pres. by θ) $H \subseteq E$ has $H = \bigoplus H^p$.

<u>Example</u>: "Opers"

An <u>oper</u> is $(V, \nabla, F^\cdot)$ such that $F^\cdot$ is a full flag ($Gr^p_F(V)$ rank 1) $\overset{\shortparallel}{E^p}$

and $\theta : E^p \xrightarrow{\cong} E^{p-1} \otimes \Omega^1_X$

If $g \geq 1$ this is gr. semistable.

The notion of oper is popular for geometric Langlands theory.

<u>Definition</u> a <u>partial oper</u> is $(V, \nabla, F^\cdot)$ such that $F^\cdot$ satisfies Griffiths transversality and is gr-semistable.

<u>Theorem</u>: For any (V, ∇) there exists a partial oper structure $(V, \nabla, F^\cdot)$.

- in general $F^\cdot$ is not unique
- however, the Higgs bundle (E, θ) is unique up to S-equivalence.

The grading is not necessarily unique though.

5

Construction: Suppose given $F^\cdot$ such that $(Gr_F(V), \theta)$ is not a semistable Higgs bundle. Choose $H :=$ the maximal dest. subobj. Define a new filtration by

$$G^p := \ker\left(V \to \frac{(V/F^pV)}{H^{p-1}}\right).$$

The condition $\theta(H) \subset H \otimes \Omega^1_X$ means G^p is again Griffiths-transverse and

$$0 \to Gr^p_F(V)/H^p \to Gr^p_G \to H^{p-1} \to 0$$

i.e.

$$(*) \quad 0 \to Gr_F(V)/H \to Gr_G(V) \to \underbrace{H^{[1]}}_{\text{Shift of Hodge index.}} \to 0$$

We can define a decreasing bounded invariant $\Rightarrow$ the process stops at a gr-semistable filtration.

$\beta(E)$ = slope of max dest. subobj.
$\rho(E)$ = rank of max dest. subobj.

"center of gravity"

$$g(E) := \frac{\sum p\, \mathrm{rk}(E^p)}{\mathrm{rk}(E)} \qquad g(E^{[1]}) = g(E)+1$$

$\gamma(E) := g(E/H) - g(H)$ where
H = max. dest. subobj.
So γ is shift-invariant (as are β, ρ).

<u>Prop.</u> $(\beta(E), \rho(E), \gamma(E))$ decreases in lexicographic order.

<u>Proof</u>: See $(*)$: $\beta_{new} \leq \beta_{old}$
if equality, $(Gr_G)_{max} \hookrightarrow H^{[1]}$
so $\rho_{new} \leq \rho_{old}$; again
if equality, let $K := (Gr_G)_{max}$
$g(K) = g(H)+1$, $g(Gr_{G/K}) = g(Gr_{F/H})$
so $\gamma_{new} = \gamma_{old} - 1$. $\square$.

<u>Proof of theorem</u>: The invariant takes finitely many values (assuming (V, ∇) irreducible). $\square$.
(($\Rightarrow$ no gaps in the Hodge numbers))

Geometric interpretation:

$$\mathcal{M}_{Hod} = \left\{ (\lambda, V, \nabla) : \begin{array}{l} \nabla : V \to V \otimes \Omega^1_X \\ \nabla(ae) = a\nabla(e) + \lambda\, da\, e \end{array} \right\}$$

$$\mathcal{M}_{Hod} \xrightarrow{\lambda} \mathbb{A}^1$$

$\lambda^{-1}(0) = \mathcal{M}_H$ = moduli space of semistable Higgs bundles (deg 0)*

$\lambda^{-1}(1) = \mathcal{M}_{DR}$ = moduli space of integrable connections.

$\mathbb{G}_m$ acts on $\mathcal{M}_{Hod}$ over its action on $\mathbb{A}^1$. All fixed points in $\mathcal{M}_H$.

$t(\lambda, V, \nabla) = (t\lambda, V, t\nabla)$

$$(\mathcal{M}_H)^{\mathbb{G}_m} = \coprod P_\alpha \quad \text{connected pieces.}$$

$\forall y \in \mathcal{M}_{Hod}$, $\lim\limits_{t \to 0} t \cdot y \in P_\alpha$ exists.

If $y \leftrightarrow (V, \nabla) \in \mathcal{M}_{DR}$, choose a gr-semistable $F^{\cdot}$

Then $\lim\limits_{t \to 0} (V, t\nabla) = (Gr_F(V), \theta)$.

A similar construction holds for points of $\mathcal{M}_H$.

Let $G_\alpha := \{ y \in M_{DR},\ \lim_{t\to 0} t\cdot y \in P_\alpha \}$ (8)

$G_\alpha \longrightarrow P_\alpha \qquad \left((V,\nabla) \longmapsto (Gr_F(V), \theta)\right)$

$\tilde{G}_\alpha = \{ y \in M_H,\ \lim_{t\to 0} t\cdot y \in P_\alpha \}$

"abelianization".

$M_{DR} = \coprod G_\alpha \qquad M_H = \coprod \tilde{G}_\alpha$

"oper stratification"

The space of opers is the stratum G_α corresponding to $P_\alpha = \{ Sym^{r-1}(K^{-1/2} \oplus K^{1/2}) \}$

(unique closed stratum?)

<u>Variations</u> of <u>Hodge</u> <u>Structure</u>:

$$\begin{tikzcd} G_\alpha^{VHS} \arrow[r, phantom, "\overset{\text{real}}{\subset}"] \arrow[dr, "\cong"'] & G_\alpha \arrow[d] \\ & P_\alpha \end{tikzcd}$$

if $(V, \nabla, F^\cdot, \langle , \rangle)$ is a polarised VHS then $F^\cdot$ is automatically gr-semistable.

Locally $G_\alpha^{VHS} = G_\alpha \cap M_B(X,r)_{\mathbb{R}}$ (at least near smooth points)

(9)

The lowest stratum G_0 corresponds to connections of the form $(V, \nabla = \nabla_U + A)$
(S.) Stable — U ← flat unitary.

$P_0 = \mathcal{U}(r, X) =$ semistable v.b. on X.

$$\begin{array}{c} \widetilde{G}_0 = T^* P_0 \\ \downarrow \\ P_0 \end{array} \qquad G_0 \text{ princ. } \widetilde{G}_0\text{-torsor}$$

To make these statements without problems, let's consider the case of projective connections of degree d with $(d, r) = 1$.

Equivalently, parabolic bundles with a single weight $(\frac{d}{r})$ at a pt. $p \in X$ or bundles on the orbifold $X[\frac{1}{r}p]$

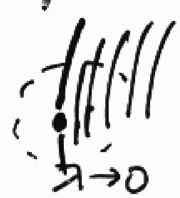
More generally we can consider parabolic bundles or bundles on DM-curves. For generic weights, all semistable Higgs bundles are stable.
($\Rightarrow$ unicity of $F^{\cdot}$ up to shift.)

10

Also Ramanan would want us to consider principal bundles for arbitrary (semisimple?) structure group.

<u>Q</u>. how to do the construction in this case?

For any situation like $(r,d)=1$ where the moduli space is smooth, M_{Hod} is also smooth $/\mathbb{A}^1$, so the $\mathbb{G}_m$-fixed point sets P_α are smooth and $G_\alpha \to P_\alpha$ are fibrations, topologically the same as $\widetilde{G}_\alpha \to P_\alpha$



<u>Conjecture</u> The stratifications G_α and $\widetilde{G}_\alpha$ are nested, i.e. $\overline{G_\alpha} - G_\alpha = \coprod G_\beta$

and furthermore the topological arrangement of the strata are the same for G_α, $\widetilde{G}_\alpha$.

<u>Thm</u> OK for $r=2$.

(11)

Deformation theory:

$\mathrm{Def.}(V,\nabla) = \mathbb{H}^1(\mathrm{End}\,V \to \mathrm{End}\,V\otimes\Omega^1_X)$

(if (V,∇) irred. $\mathbb{H}^0 = \mathbb{H}^2 = \mathbb{C}$)

The filtration $F^\cdot$ induces a "Hodge" filtration of the complex $\mathrm{End}\,V \to \mathrm{End}\,V\otimes\Omega^1_X$ and we can take the spectral sequence of this filtered complex.

If $(V,\nabla,F^\cdot)$ is gr-stable then

$\mathbb{H}^0(Gr_F(-)) = \mathbb{C}$

$\mathbb{H}^2(Gr_F(-)) = \mathbb{C}$ (and obs. $=0$)

These are the same as at the abutment

$\Rightarrow$ the spectral sequence degenerates.

Thus, $T(M_{DR})_{(V,\nabla)} = \mathbb{H}^1(-)$

has a filtration whose associated-graded is $\mathbb{H}^1(Gr^p_F(\mathrm{End}\,V) \xrightarrow{\Theta} Gr^{p-1}_F(\mathrm{End}\,V)\otimes\Omega^1_X)$

$F^0 = T(G_\lambda)_{(V,\nabla)}$ $\quad F^1 = T(\text{fiber of } G_\lambda \to P_\lambda)$

$\mathbb{H}^2 = F^1\mathbb{H}^2 \Rightarrow$ this F^1 has zero cup product with itself.

(12)

The assoc. graded of $\mathbb{H}^1(\cdots)$ depends only on $(Gr_F(V), \theta) \in P_\alpha \overset{VHS}{\simeq} G_\alpha$

So it can be viewed as a Hodge structure, $Gr_F^p \, \mathbb{H}^1 = \mathbb{H}^{p,1-p}$.

in particular $\dim Gr^p \mathbb{H}^1 = \dim Gr^{1-p} \mathbb{H}^1$.

$\Rightarrow$ The fibers of $G_\alpha \to P_\alpha$ are lagrangian.

If P_0 = the lowest stratum, the tangent space has only

$$\underbrace{\mathbb{H}^{1,0} = Gr_F^1}_{T^*P_0} \ , \quad \underbrace{\mathbb{H}^{0,1} = Gr_F^0}_{T P_0}$$

$$\Rightarrow \left.\begin{matrix} \widetilde{G}_0 \\ \downarrow \\ P_0 \end{matrix}\right\} \simeq \begin{matrix} T^*P_0 \\ \downarrow \\ P_0 \end{matrix} \qquad \begin{matrix} G_0 \\ \downarrow \\ P_0 \end{matrix} \ \text{twisted form}$$

This works even in the parabolic case where there can be no stable bundles $\mathcal{U}(r, X^{par}) = \emptyset$

P_0 = some kind of VHS's.

(depending on the weights)

(13)

Questions

- Princ. bundle version?
- what happens if $\dim X \geq 2$?
  (The destab. subsheaf can be torsion free, so this seems unclear.)
- Conj. that $\{G_\alpha\}$ nested
- variation of the strata G_α and fibers of $G_\alpha \to P_\alpha$, when X is deformed?
  for ex. when do the isomonodromy equations preserve points of G_α?
- in certain parabolic cases where $\mathcal{U}(r, X) = \emptyset$, should the lowest stratum P_0 replace "Bun" in geometric Langlands?
- understand the geometry of the stratification in low-dimensional cases: certain parabolic moduli spaces have $\dim = 2$ for example.
- does the collection of Lagrangian subspace fibers of $G_\alpha \to P_\alpha$, form a smooth foliation?

14+

Example of non-unicity

X $g \geq 3$ A, B line bundles of degree 0 distinct

$Ext^1(B, A) = H^1(X, B^{-1} \otimes A)$ dim $g-1$.

$\xi \neq 0$ V corresponding rank 2 bundle : semistable but not stable

$0 \to A \to V \to B \to 0$

$\varphi \in End(V)$ $\varphi(A) \subset A$ $A \in \ker(\varphi - \lambda)$

$\varphi - \lambda : B \to V$ doesn't exist $\Rightarrow \varphi = \lambda$.

So $End(V) = \mathbb{C} \Rightarrow V$ has a connection ∇_0.

$\{\nabla_0 + \phi\}$ dim $4g-3$ $H^0(End V \otimes \Omega^1_X)$

$\{\nabla$ preserving $A\}$ dim $\leq 3g-1$

$\Rightarrow$ there's ∇ not preserving A

$\Rightarrow \nabla$ irreducible.

2 choices of $F^\cdot$: — trivial, $Gr_F = V$ semistable vector bundle

— $F^0 = V$, $F^1 = A$ $Gr_F = A \oplus B$

$\theta : A \to B \otimes \Omega^1_X$

semistable Higgs bundle.

V and $(A \oplus B, 0)$ are S-equivalent as Higgs bundles (without the Hodge grading).

In general, since M_H is separated and its points represent S-equivalence classes, the limit $\lim_{t \to 0} t \cdot y$ is unique in M_H.

so $(\mathrm{gr}_F, 0)$ unique / S-equiv.

(14th)

Proof of nestedness (rank 2) gr-stable

given $E^1 \subset V \to V/E^1 = E^0$

$\theta : E^1 \to E^0 \otimes \Omega^1_X$ $\quad$ $\deg E^1 > 0$

$\deg E^0 < 0$

we would like to deform it to the next lower stratum ($\deg E^1$ lower by 1).

choose $L = E^1(-p)$ for some p

$$L \to E^1 \hookrightarrow V \quad \underbrace{L \to V}_{\varphi}$$

try to deform (V, ∇, L, φ) governed by

$$\begin{array}{ccc} \operatorname{End} V & \longrightarrow & \operatorname{End} V \otimes \Omega^1_X \\ \oplus & & \oplus \\ \vartheta = \operatorname{End} L & & \operatorname{Hom}(L, V) \end{array}$$

gr-stable $\Rightarrow$ degen. of the H-dR spectral sequence $\Rightarrow$

$\mathbb{H}^1(\operatorname{End} V \to \operatorname{End} V \otimes \Omega^1_X) \twoheadrightarrow H^1(\operatorname{Hom} E^1, E^0$

$$
\begin{tikzcd}
H^1(\mathrm{Hom}\,L, E^1) \arrow[r] & H^1(\mathrm{Hom}(L,V)) \arrow[r] & H^1(\mathrm{Hom}(L,E^0)) \arrow[r] & 0 \\
H^1(\mathrm{Hom}\,L,L) \arrow[u] & & H^1(\mathrm{Hom}(E^1,E^0)) \arrow[u, "\neq"'] & \\
& & \mathbb{H}^1(\mathrm{End}\,V\otimes\Omega^\bullet) \arrow[u] &
\end{tikzcd}
$$

So $\mathbb{H}^1(\mathrm{End}\,V\otimes\Omega^\bullet)\oplus H^1(\mathcal{O}) \twoheadrightarrow H^1(\mathrm{Hom}\,L,V)$

$\Rightarrow$ no obstruction to deforming (V, D, L, φ). We can choose a deformation which is nonzero in $H^1(\mathrm{Hom}(E^1,E^0)) = Gr\,F^{-1}$ so it goes out of our stratum, as the new bundle L will be the destabilizing subsheaf.

(14 ***)

Parabolic case

For example on $\mathbb{P}^1$ with 4 pts.

$V = V_0$

$0 \subsetneq W_p \subsetneq V_p \qquad p = p_1, p_2, p_3, p_4$

$W_p = V_{p,\beta} \qquad -\alpha_p \leq \beta \leq 0$

$\ker\left(V \to V_p / W_p\right) = V_{p,\beta} \quad -(1-\alpha) \leq \beta \leq \alpha$

$V(-p) = V_{p,\beta} \quad \beta < -(1-\alpha_p).$

so the weights are $-(1-\alpha_p)$, $-\alpha_p$

par. deg. $= \deg(V) + \sum \alpha_p + \sum 1-\alpha_p$
$= \deg(V)+4$

So take $\deg(V) = -4$ eg

$V = \mathcal{O}(-2) \oplus \mathcal{O}(-2)$.

subbundle $\mathcal{O}(-2) \subset V$

has par deg $\sum_p (1-\alpha_p)$ or α_p

For certain weights

e.g. $\frac{1}{4}$ $\frac{1}{2}-\varepsilon$ $\frac{1}{2}-\varepsilon$ $\frac{1}{2}-\varepsilon$

$\frac{3}{4}$ $\frac{1}{2}+\varepsilon$ $\frac{1}{2}+\varepsilon$ $\frac{1}{2}+\varepsilon$

all bundles are unstable:
take the $\mathcal{O}(-2) \subset V$ corresp. to the par. subspace for $\frac{3}{4}$,

$$\text{par.deg.} = -2 + \tfrac{3}{4} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} - 3\varepsilon > 0$$

In this case the bottom stratum will consist of non-unitary rhs's.

Conjecture of Etingof et. al. & Boalch:

Compare $r=2$ on $\mathbb{P}^1 - 4$ pts $(d=1)$

with $r = 2d$, parabolic flags
$=$ 1 subspace of dim d

i.e. par. wts are

$$\begin{pmatrix} \alpha_p & & & & & \\ & \ddots & & & & \\ & & \alpha_p & & & \\ & & & 1-\alpha_p & & \\ & & & & \ddots & \\ & & & & & 1-\alpha_p \end{pmatrix} \cdots \begin{pmatrix} + \; k/d & & \\ & \ddots & \\ & & k/d \end{pmatrix}$$

somewhere

they conjecture a natural (but weird) isomorphism between the corresponding M_B moduli spaces for different d.

If this is natural, it should be "motivic" and should also give isos $M_{DR}(\ldots 2d) \simeq M_{DR}(\ldots 2)$

$M_H \quad \ldots$

and should be compatible with the stratifications G_d.

For higher d the bottom stratum will consist of VHS's ...

Happy Birthday

to

Ramanan

from

Nicole Mestrano

André Hirschowitz

and

myself

!!!