

THE ADAMS CONJECTURE

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§0. INTRODUCTION

THIS paper contains a demonstration of the Adams conjecture [1] for real vector bundles. Unlike an old attempt of mine [12], which has recently been completed by Friedlander [8], and the proof of Sullivan [15], no use is made of the étale topology of algebraic varieties. The proof uses only standard techniques of algebraic topology together with some basic results on the representation rings of finite groups, notably the Brauer induction theorem and one of its well-known consequences: the fact that modular representations can be lifted to virtual complex representations.

The conjecture is demonstrated in the first section assuming some results which are treated in the later sections. Put briefly, one first shows that the conjecture is true for vector bundles with finite structural group and then using modular character theory one produces enough examples of virtual representations of finite groups to deduce the general case of the conjecture from this special case. The key step (Theorem 1.6) involves the partial computation of the mod l cohomology rings of the finite classical groups $GL_n(F_q)$ and $O_n(F_q)$ where l is a prime number not dividing q . I have included only what is needed here, but one could push further and obtain pretty complete information about these cohomology rings. However, more general and natural results can be obtained using étale cohomology as I plan to show in another paper.

There is an Appendix (Section 5) developing a modular character theory for orthogonal and symplectic representations of a finite group. This is needed to handle the Adams conjecture for *real* vector bundles.

This paper contains numerous suggestions of Michael Atiyah, especially 1.2 which he first proved for complex bundles. Section 3 owes much to conversations with R. J. Milgram about the cohomology of the symmetric group.

§1. PLAN OF THE DEMONSTRATION

Let X be a finite complex, let $KO(X)$ be the Grothendieck group of its virtual real bundles, and let $\text{Sph}(X)$ be the group of its stable spherical fibrations. The J -homomorphism $J: KO(X) \rightarrow \text{Sph}(X)$ is the map induced by associating to a vector bundle its underlying sphere bundle. The purpose of this paper is to prove the following conjecture of J. F. Adams [1].

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THEOREM 1.1. *Let k be an integer and let $x \in KO(X)$. Then $k^n J(\Psi^k x - x) = 0$ for some integer n .*

Because of the identity $\Psi^j \Psi^k = \Psi^{jk}$ satisfied by the Adams operations and the fact that the theorem is trivial for $k = 0, \pm 1$, it suffices to prove the theorem when k is a prime number which from now on will be denoted p . As $2x$ is in the image of the restriction of scalars map from complex K -theory $K(X)$ to $KO(X)$, one sees that 1.1 implies the analogous result for complex K -theory; moreover, if $p = 2$ then the real and complex cases are equivalent. Therefore, in the rest of the paper we shall concentrate on the situation with real K -theory and p odd; with trivial modifications the arguments will work for the complex case and all p , taking care of the real case with $p = 2$.

We say that a virtual bundle x over X admits a reduction of its structural group to a finite group G if there exists a principal G -bundle P over X such that x is in the image of the map from $RO(G)$, the real representation ring of G , to $KO(X)$ which is induced by sending a representation V into the bundle $P \times^G V$.

PROPOSITION 1.2. *The Adams conjecture is true for any virtual bundle whose structural group may be reduced to a finite group.*

This will be proved in the next section. It is the constructive part of the proof and uses Brauer's induction theorem to restrict to representations induced from one and two dimensional representations together with Adams' methods to handle this case. The rest of the proof consists in showing that there are enough virtual representations of finite groups so that this special case implies the general case.

Let k be an algebraic closure of the field with p elements and choose, once and for all, an embedding $\phi : k^* \rightarrow \mathbb{C}^*$. If G is a finite group and $\pi : G \rightarrow \text{Aut}(V)$ is a representation of G in a finite dimensional vector space over k , then the modular character of V (with respect to ϕ) is defined to be the complex-valued function on G given by the formula

$$(1.3) \quad \chi(g) = \sum_i \phi(\alpha_i)$$

where $\alpha_1, \dots, \alpha_n$ are the eigenvalues with multiplicity of $\pi(g)$. It is a basic consequence (Green [9, Theorem 1]) of the Brauer induction theorem that χ is the character of a unique element of the complex representation ring $R(G)$. We show in the appendix to this paper (5.5) that if p is odd and if G leaves invariant a non-degenerate symmetric bilinear form on V , then χ is the character of an element of the real representation ring $RO(G)$.

Let F_q be the subfield of k with q elements, let $GL_n(F_q)$ be the group of invertible n of n matrices with entries in F_q , and (p odd) let $O_n(F_q)$ be the subgroup of orthogonal matrices. Lifting the standard representations of these groups in k^n in the above way we obtain virtual representations in $R(GL_n(F_q))$ and $RO(O_n(F_q))$ respectively, which in turn give rise to maps

$$(1.4) \quad \begin{aligned} BGL_n(F_q) &\rightarrow BU \\ BO_n(F_q) &\rightarrow BO \end{aligned}$$

in the homotopy category. Now these virtual representations are evidently compatible as n tends to infinity and as the finite field tends toward k , hence by a standard limit argument

(compare [2, proof of 1.3]) the maps 1.4 give rise to unique maps in the homotopy category

$$(1.5) \quad \begin{aligned} BGL(k) &\rightarrow BU \\ BO(k) &\rightarrow BO \end{aligned}$$

where $GL(k)$ and $O(k)$ are the infinite general linear and orthogonal groups with entries in k .

THEOREM 1.6. *The maps 1.5 induce isomorphisms on cohomology with coefficients in \mathbb{Z}_d where d is any integer prime to p .*

This will be proved in Sections 3 and 4. We now prove the Adams conjecture.

Recall [3, section 1] that for any finite complex X with basepoint we have $\text{Sph}(X) = \text{indlim} [X, BG_n]_o$, where G_n is the monoid of homotopy equivalences of S^{n-1} , and $[,]_o$ denotes the set of homotopy classes of basepoint-preserving maps. Let Z be the inductive limit in the sense of homotopy theory (infinite mapping cylinder) of the sequence

$$BG_p \rightarrow BG_{p^2} \rightarrow BG_{p^3} \rightarrow \dots$$

where the successive maps come from the p -fold Whitney sum of sphere fibrations. Then

$$\text{Sph}(X)[p^{-1}] = [X, Z]_o \tag{1.7}$$

and Z is connected with finite homotopy groups of order prime to p [3, 1.3]. Now quite generally $[X \cup pt, Z]_o$ is the quotient of $[X, Z]_o$ by the action of $\pi_1 Z$; as the left side of 1.7 is the same for $X \cup pt$ and X , this action is trivial, so $\pi_1 Z$ acts trivially on $\pi_n Z$ for all n . (This follows also from the fact that Z is an H -space, which is a consequence of the group structure on 1.7 and 1.8 below.) If X is an infinite complex which is the union of a sequence of finite complexes X_n , then the natural map

$$[X, Z] \simeq \text{invlim} [X_n, Z] \tag{1.8}$$

is an isomorphism, because in general it is surjective with kernel $R^1 \text{invlim} [SX_n, Z]$ and this is zero as the groups are finite. Applied to the skeleta of BO this implies that the map $x \mapsto J(\Psi^p x - x)$ from $\widetilde{KO}(?)$ to $\text{Sph}(?)[p^{-1}]$ induces a map μ in the diagram

$$\begin{array}{ccc} BO(k) & \xrightarrow{\alpha} & BO \rightarrow C \\ & \mu \downarrow & \nearrow \beta \\ & & Z \end{array} \tag{1.9}$$

where α is the map of 1.5 and C is its cone. The map α when restricted to any finite skeleton X of $BO(k)$ classifies a virtual bundle over X whose structural group is reduced to the finite group $O_n(F_q)$ for some n and q , hence $\mu\alpha$ restricted to X is null-homotopic by 1.2. By 1.8, $\mu\alpha$ is homotopic to zero and so there is dotted arrow β in 1.9 making the diagram homotopy commutative. The obstructions to deforming β to zero lie in $H^n(C, \pi_n Z)$, $n \geq 1$, where the local coefficient system is trivial as remarked above. As $\pi_n Z$ is a finite group of order prime to p and α induces an isomorphism on cohomology in such groups by 1.6, it follows that $H^n(C, \pi_n Z) = 0$. Therefore β , hence also μ , is null-homotopic, proving the Adams conjecture.

§2. ADAMS CONJECTURE FOR BUNDLES WITH FINITE STRUCTURAL GROUP

Let $f: Y \rightarrow X$ be a finite covering space of a finite complex X . If E is a vector bundle on Y let $f_* E$ be its trace with respect to f , i.e.

$$(f_* E)_x = \bigoplus_{y \in f^{-1}\{x\}} E_y \quad (2.1)$$

and denote also by $f_*: KO(X) \rightarrow KO(Y)$ the induced map on Grothendieck groups.

LEMMA 2.2. *If $y \in KO(Y)$ then $f_* \Psi^p y = \Psi^p f_* y$ in $KO(X)[p^{-1}]$.*

One knows that f_* coincides with the Gysin homomorphism for f in the generalized cohomology theory KO^* (see [10, p. 540]; the argument given there also works in the case of KO -theory). Since the normal bundle of f has a canonical trivialization, this Gysin homomorphism is essentially the composite of one suspension isomorphism and the inverse of another. As Ψ^p extends to a stable cohomology operation on the theory $KO^*(?)[p^{-1}]$, it commutes with suspension isomorphisms and hence with f_* , proving the lemma.

(Instead of using 2.2 some readers may prefer the following argument of Atiyah. It will be sufficient for the proof of 1.2 to show that if $i_*: RO(H) \rightarrow RO(G)$ is the induction homomorphism associated to a subgroup H of the finite group G , and if $y \in RO(H)$, then the difference

$$(*) \quad \Psi^p(i_* y) - i_*(\Psi^p y) \in RO(G)$$

goes to a p -torsion element under the homomorphism $RO(G) \rightarrow KO(X)$ associated to a principal G -bundle P over a finite complex X . If $j: G_l \rightarrow G$ is the inclusion of a Sylow l -subgroup of G and f denotes the covering map $P/G_l \rightarrow X$, then the kernel of $f_*: KO(X) \rightarrow KO(P/G_l)$ consists of elements of order prime to l , since the trace map f_* going the other way satisfies $f_*(f_* x) = (f_* 1)x$, where $f_* 1 = [G: G_l] +$ a nilpotent element. Consequently it is enough to show that the restriction j^* kills $(*)$ for all $l \neq p$. By the Mackey formula

$$j^* i_* = \sum_{G_l g H} (i_g)_* (j_g)^*$$

on is reduced to proving that $(*)$ is zero when G is of order prime to p . But this is clear either by direct calculation or by using the fact that Ψ^p coincides with the action of an element of the Galois group of a large cyclotomic extension of \mathbb{Q} (see [5, 3.2 and proof of 4.2].)

Denote by $[E]$ the element of the Grothendieck group associated to a vector bundle E .

LEMMA 2.3. *If L is a one or two dimensional bundle over Y , then 1.1 is true for $[f_* L]$.*

From the proof of the special case of 1.1 proved by Adams [1, Theorem 1.3], there is a vector bundle $L^{(p)}$ over Y such that $\Psi^p[L] = [L^{(p)}]$ and such that there is a (non-linear) map $L \rightarrow L^{(p)}$ of degree $\pm p^a$ on each fibre. Using 2.1 one sees easily that there is a (non-linear) map $f_* L \rightarrow f_* L^{(p)}$ of degree $\pm p^b$ on each fibre, hence both of these bundles yield the same element of $\text{Sph}(X)[p^{-1}]$ by the "mod p Dold theorem" [1, Theorem 1.1]. Using 2.2 we have that $\Psi^p[f_* L] = [f_* L^{(p)}]$ in $KO(X)[p^{-1}]$, hence the lemma is proved.

Denote by $RO(G)$ the real representation ring of a finite group G .

LEMMA 2.4. *Every element of $RO(G)$ is an integral linear combination of representations induced from one and two dimensional representations of subgroups of G .*

For the complex representation ring this is a well-known corollary of the Brauer induction theorem [14, pp. 11–29]. Recall that there is an extension of the induction theorem to representations over a field taking into account the action of the Galois group on the roots of unity [14, pp. 11–44]. For the real numbers it says that any element of $RO(G)$ is an integral linear combination of representations induced from subgroups H which are \mathbf{Z}_2 -elementary, i.e. either H is the direct product of a cyclic group and a group of prime power order or it is a semidirect product $P \widetilde{\times} C$, where C is cyclic of odd order and where P is a 2-group and P acts on C through a homomorphism $P \rightarrow \mathbf{Z}_2$ and the generator of \mathbf{Z}_2 acts on C as $-id$. Such a group H is of type (MP) , i.e. it has a chain of normal subgroups whose quotients are cyclic of prime order. Let V be an irreducible real representation of H and endow it with an invariant inner product. Then by Borel–Serre [6] H normalizes a torus in the orthogonal group of V . The eigenspaces of this torus form a system of imprimitivity in V , so by irreducibility, V is either of dimension one or induced from a representation of dimension two. By transitivity of induction the lemma is proved.

The Proposition 1.2 follows immediately from 2.3 and 2.4. Indeed one only has to note that if P is a principal G -bundle over X and if W is a representation of a subgroup H of G , then the bundle over X associated by P to the induced representation of G is isomorphic to the bundle f_*L , where f is the covering $P/H \rightarrow X$ and where L is the bundle on P/H associated to W .

§3. DETECTING COHOMOLOGY IN WREATH PRODUCTS

Let l be a fixed prime number and denote by $H^*(X)$ the singular cohomology ring of the space X with coefficients in \mathbf{Z}_l . Let C be a cyclic group of order l , and if Y is a space on which C acts, let Y_C be the associated fibre space over the classifying space BC with fibre Y . Let C act on the l -fold product X^l by permuting the factors and let the maps

$$X^l \xrightarrow{i} X^l_C \xleftarrow{j} BC \times X$$

be the inclusion of the fibre over the basepoint of BC and the map induced by the diagonal $X \rightarrow X^l$.

PROPOSITION 3.1. *The induced map on cohomology*

$$H^*(X^l_C) \xrightarrow{(i^*, j^*)} H^*(X^l) \oplus H^*(BC \times X)$$

is injective.

We may suppose that X is a CW -complex and in fact a finite complex because cohomology with coefficients in \mathbf{Z}_l transforms direct limits into inverse limits. Let $w \in H^2(BC)$ be a generator. Since the diagonal map $X \rightarrow X^l$ is the inclusion of the fixpoint set for the action of C , the localization theorem ([12], see also [4] for the argument in K -theory) implies that on inverting w the map j^* becomes an isomorphism:

$$H^*(X^l_C)[w^{-1}] \xrightarrow{\sim} H^*(BC \times X)[w^{-1}].$$

Consequently any element in the kernel of j^* is killed by a power of w .

On the other hand the spectral sequence of the fibration $X^l_C \rightarrow BC$

$$E^{pq} = H^p(BC, H^q(X^l)) \Rightarrow H^{p+q}(X^l_C)$$

degenerates on account of the isomorphism [11, Theorem 3.3]

$$H^*(BC, H^*(X^l)) \cong H^*(X^l_C).$$

A non-zero element y of $H^*(X^l_C)$ which is in the kernel of i^* has a non-trivial component z in E_2^{pq} for some $p > 0$. As the cohomology of the cyclic group C is periodic with periodicity map given by multiplying by w , it follows that z , and hence y , is not killed by any power of w . Thus the intersection of the kernels of i^* and j^* is zero, proving the proposition.

We shall say that a family $H_i \ i \in I$ of subgroups of a group G detects the cohomology of $G \pmod{l}$, or simply is a detecting family, if the map

$$H^*(BG) \rightarrow \prod_i H^*(BH_i)$$

given by the restriction homomorphisms is injective. It is clear from the Kunneth formula that if $H_j \ j \in J$ is also a detecting family for G' , then the family $H_i \times H_j \ (i, j) \in I \times J$ is a detecting family for $G \times G'$. Moreover, if H_i is a detecting family for G and if each H_i has a detecting family H_{ij} , then the subgroups H_{ij} for all i and j form a detecting family for G . By transfer theory a subgroup of finite index prime l in G detects the cohomology of G .

Let H be a permutation group of degree n , i.e. a group endowed with an action on the set $\{1, \dots, n\}$, and let G be an arbitrary group. Then the wreath product $H \wr G$ is defined to be the semidirect product $H \tilde{\times} G^n$, where H permutes the factors according to the given permutation representation. If $EH \rightarrow BH$ and $EG \rightarrow BG$ are universal bundles for G and H respectively, then $EH \times (EG)^n$ is a contractible space on which $H \wr G$ has a natural free action, and hence the quotient space $EH \times {}^H(BG)^n$ is a classifying space for the wreath product. Taking H to be C with its standard permutation representation and identifying the maps i and j of 3.1 with the appropriate restriction homomorphisms we obtain

COROLLARY 3.2. *The two subgroups G^l and $C \times G$ detect the mod l cohomology of the wreath product $C \wr G$.*

COROLLARY 3.3. *If $A_i \ i \in I$ is a detecting family for G , then the family of subgroups of $C \wr G$*

$$\begin{aligned} A_{i_1} \times \cdots \times A_{i_n} &\subset G^l \subset C \wr G & i_1, \dots, i_n \in I \\ C \times A_i &\subset C \times G \subset C \wr G & i \in I \end{aligned}$$

is a detecting family.

Denote by \sum_n the symmetric group of degree n .

PROPOSITION 3.4. *Let G be a group whose mod l cohomology is detected by a family of abelian subgroups of exponent dividing l^a with $a \geq 1$. Then $\sum_n \wr G$ has the same property.*

The proof is by induction on n , the case $n = 1$ being clear. If n is not divisible by l , then the subgroup $(\sum_{n-1} \wr G) \times G$ detects cohomology because it is of index n which is prime to l . On the other hand if $n = ml$, then the evident embedding $\sum_m \wr (C \wr G) \subset \sum_n \wr G$ gives a

subgroup of index $n!/m!l^m$, which is prime to l by elementary number theory. Using the induction hypothesis and in the second case 3.3, one obtains detecting families for these subgroups by taking various products of abelian groups of exponent dividing l^a , hence $\sum_n \int G$ also has a detecting family of groups of this type.

COROLLARY 3.5. *The mod l cohomology of \sum_n is detected by its family of elementary abelian l -subgroups.*

§4. PROOF OF THEOREM 1.6

Continuing with our preceding notations, let l be a prime number different from the characteristic p of F_q . Write $n = 2m + e$ with $e = 0$ or 1 and let $T_m(F_q)$ be the subgroup of $O_n(F_q)$ which is the "Whitney sum" of m two-by-two blocks of the form

$$(4.1) \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix} a^2 + b^2 = 1$$

together with a trivial block of rank e . We suppose that 4 divides $q - 1$; choosing a square root i of -1 the block 4.1 becomes isomorphic to F_q^* by the map sending this matrix to $a + ib$. If N denotes the normalizer of $T_m(F_q)$ in $O_n(F_q)$, then, N is isomorphic to $(\sum_m \int O_2(F_q)) \times Z_2^e$.

LEMMA 4.2. *If $4, l$ divide $q - 1$, then the index of N in $O_n(F_q)$ is prime to l .*

One knows that the orders are

$$|O_{2m}(F_q)| = 2 \cdot q^{m(m-1)}(q^m - 1) \prod_{j=1}^{m-1} (q^{2j} - 1) \quad |N| = 2^m m! (q - 1)^m$$

$$|O_{2m+1}(F_q)| = 2 \cdot q^{m^2} \prod_{j=1}^m (q^{2j} - 1) \quad |N| = 2^{m+1} m! (q - 1)^m$$

and by number theory that $(q^l - 1)/r(q - 1)$ is a l -adic unit if l is odd and l divides $q - 1$ or if $l = 2$ and 4 divides $q - 1$ [7, pp. 45-46].

THEOREM 4.3. *The mod l cohomology of $GL_n(F_q)$ is detected by the subgroup of diagonal matrices if l divides $q - 1$ or if 4 divides $q - 1$ and $l = 2$. (Here p need not be odd.)*

(4.4) *If l is odd and $4l$ divides $q - 1$, then $T_m(F_q)$ detects the mod l cohomology of $O_n(F_q)$.*

(4.5) *If 4 divides $q - 1$, then the mod 2 cohomology of $O_n(F_q)$ is detected by its family of elementary abelian 2 -subgroups.*

We first prove 4.4. Using 4.2 and the fact that l is odd, we know that $O_n(F_q)$ has its (mod l) cohomology detected by the subgroup $\sum_m \int O_2(F_q)$. The Sylow l -subgroup of $O_2(F_q)$ is cyclic of order a power of l dividing $q - 1$, hence by 3.4 this wreath product and hence $O_n(F_q)$ has a detecting family consisting of abelian groups A of exponent dividing $q - 1$. But any such subgroup of $O_n(F_q)$ is conjugate to a subgroup of $T_m(F_q)$. Indeed let $V = F_q^n$ be regarded as an orthogonal representation of A and let L be an irreducible subspace on which A acts non-trivially. As the exponent of A divides $q - 1$, L is of dimension one, and as l is odd L is not isomorphic to its dual, so L is an isotropic subspace for the bilinear form on V . Choosing an invariant complementary subspace to the orthogonal space of L , which is

possible since A is of order prime to p , we can write V as an orthogonal direct sum of the hyperbolic orthogonal representation associated to L and another orthogonal representation. Continuing in this way V can be decomposed into a direct sum of A -invariant hyperbolic planes plus perhaps a trivial one-dimensional representation; these may then be transformed into the eigenspaces of $T_M(F_q)$ by an element of $O_n(F_q)$, thus giving the desired conjugacy. This concludes the proof of 4.4 and that of 4.3 is similar.

For the proof of 4.5 we can use the same method to reduce to showing that the mod 2 cohomology of $O_2(F_q)$ is detected by elementary abelian 2-subgroups, and this follows from

LEMMA 4.6. *The mod 2 cohomology of a dihedral group is detected by its family of elementary abelian 2-subgroups.*

Passing to the Sylow 2-subgroup, we can assume that the dihedral group D has generators x_1, x_2 subject to the relations $x_1^2 = x_2^{2^a} = 1, x_1 x_2 x_1^{-1} = x_2^{-1}$. The lemma is clear if $a = 1$, so suppose $a > 1$ and let $t_i \in H^1(BD) = \text{Hom}(D, \mathbb{Z}_2)$ be given by $t_i(x_j) = \sigma_{ij}$. Let $c \in H^2(BD)$ be the Euler class of the standard representation of D on the plane. Then

$$H^*(BD) = \mathbb{Z}_2[t_1, t_2, e]/(t_1^2 + t_1 t_2).$$

(This may be derived by considering the Hochschild–Serre spectral sequence of the extension obtained from the normal cyclic subgroup C generated by $x_2^{2^{a-1}}$. If u and v are the non-zero elements of the cohomology of BC in dimensions one and two respectively, then one can show that $d_2 u = t_1^2 + t_1 t_2$ and $d_3 v = 0$ so the spectral sequence collapses at E_3 .) This ring has no nilpotent elements. Since a cohomology class which restricts to zero on each elementary abelian 2-subgroup is necessarily nilpotent by the main theorem of [12], this proves the lemma. One can also compute directly that the elementary abelian 2-subgroups with generating sets $\{x_1^{2^{a-1}}, x_2\}$ and $\{x_1^{2^{a-1}}, x_1 x_2\}$ detect the cohomology of D . This concludes the proof of the lemma and hence that of the theorem.

Let k be an algebraic closure of F_p and let $\phi: k^* \rightarrow C^*$ be an embedding. If k_1 is a subfield of k then as in Section 1 we obtain elements

$$\sigma \in \text{invarlim}_{F_q \subset k_1} R(GL_n(F_q)), \quad \tau \in \text{invarlim}_{F_q \subset k_1} RO(O_n(F_q))$$

which in turn yield well-defined homotopy classes

$$\begin{aligned} BGL_n(k_1) &\rightarrow BU \\ BO_n(k_1) &\rightarrow BO. \end{aligned}$$

Denote by $c_i(\sigma)$, $p_i(\tau)$, and $w_i(\tau)$ the images of the universal Chern, Pontryagin, and Stiefel–Whitney classes respectively under these maps.

THEOREM 4.7. *Let d be a positive integer prime to p and suppose that k_1 contains the d^r -th roots of unity in k for all r . In the case of the orthogonal group we suppose also that p is odd and that k_1 contains a square root of -1 . Then*

$$\begin{aligned} H^*(BGL_n(k_1), \mathbb{Z}_d) &= \mathbb{Z}_d[c_1(\sigma), \dots, c_n(\sigma)] \\ H^*(BO_n(k_1), \mathbb{Z}_d) &= \mathbb{Z}_d[p_1(\tau), \dots, p_m(\tau)] \quad d \text{ odd}, m = [n/2] \\ H^*(BO_n(k_1), \mathbb{Z}_2) &= \mathbb{Z}_2[w_1(\tau), \dots, w_n(\tau)] \quad d = 2 \end{aligned}$$

where the notation means that the rings are polynomial rings with the indicated generators.

We can suppose that $d = l^n$ where l is a prime number different from p , and even that $d = l$ because the c_i, p_i are of even dimensions and hence the relevant Bockstein operations are zero. Consider the restriction homomorphism

$$(4.8) \quad H^*(BO_n(k_1)) \rightarrow H^*(BT_m(k_1))^n$$

where the cohomology has \mathbf{Z}_l coefficients and where $W = \sum_m \mathbf{Z}_2$ is the Weyl group. Now k_1^* is an increasing union of cyclic groups and is l -divisible by hypothesis; by passage to the limit in the known formula for the cohomology of a cyclic group we find $H^*(Bk_1^*) = \mathbf{Z}_l[x]$, where x is the first Chern class of the character of k_1^* obtained from ϕ . By Kunneth, $H^*(BT_m(k_1)) = \mathbf{Z}_l[x_1, \dots, x_n]$ where x_i is the image of x under the i -th projection $T_m(k_1) \rightarrow k_1^*$. The Weyl group acts by permuting and changing the signs of the x_i so if l is odd, the right side of 4.8 is the subring of symmetric functions of the x_i^2 . It is easy to compute that the restriction of $p_j(\tau)$ to $T_m(k_1)$ is the j -th elementary symmetric function of the x_i^2 . Indeed by means of characters one sees that the restriction of τ to $T_m(k_1)$ is the direct sum of the two-dimensional real representations with first Pontryagin classes x_i^2 , so one gets the elementary symmetric functions from the product formula for the Pontryagin classes. Therefore, the map 4.8 is surjective. By 4.4 it is injective, which proves the second formula of the theorem. The first formula is handled similarly.

If $l = 2$, one considers instead of 4.8 the restriction to the subgroup Q of diagonal matrices of $O_n(k_1)$. The appropriate Weyl group here is \sum_n acting by permuting the factors, and by the same argument one sees that the restriction map to Q analogous to 4.8 is surjective. Any maximal elementary abelian 2-subgroup A of $O_n(k_1)$ stabilizes an ordered orthogonal direct sum decomposition of k_1^n into one-dimensional subspace. Since k_1^* is 2-divisible by hypothesis, all one-dimensional quadratic spaces over k_1 are isomorphic, so it follows that A is conjugate to a subgroup of Q . Combining this with 4.5 one sees that the restriction map from $O_n(k_1)$ to Q is injective, proving the last formula of the theorem.

Theorem 1.6 follows from 4.7 by taking $k_1 = k$, letting n go to infinity, and using the known formulas for the cohomology of BU and BO .

§5. APPENDIX—LIFTING MODULAR ORTHOGONAL AND SYMPLECTIC REPRESENTATIONS

In this appendix we develop a modular character theory for orthogonal and symplectic representations of a finite group G in a field K of characteristic different from two. We suppose that K contains a primitive h -th root of unity, where h is the factor of the exponent of G which is prime to the characteristic of K ; this implies that all irreducible representations of G over K are absolutely irreducible.

(5.1). By an orthogonal (resp. symplectic) representation of G over K we mean a representation of G as linear transformations of a finite dimensional vector space over K which leave invariant a nondegenerate symmetric (resp. skew-symmetric) bilinear form. The Grothendieck group $RO_K(G)$ of orthogonal representations is defined (up to canonical isomorphism) as the target of a universal map $V \mapsto [V]$ from the set of isomorphism classes of orthogonal representations to an abelian group such that the following relations are satisfied.

(5.1.1) If V and V' are two orthogonal representations on the same vector space whose bilinear forms differ by a scalar factor, then $[V] = [V']$.

(5.1.2) If V is the orthogonal direct sum of two orthogonal representations V' and V'' , then $[V] = [V'] + [V'']$.

(5.1.3) If V is an orthogonal representation and if W is an invariant subspace which is isotropic, i.e. contained in its annihilator W° for the bilinear form, then $[V] = [W^\circ/W] + [W \oplus V/W^\circ]$, where W°/W and $W \oplus V/W^\circ$ are endowed with the natural bilinear forms induced from that of V .

One defines similarly the Grothendieck group $R\mathrm{Sp}_K(G)$ of symplectic representations.

Let $R_K(G)$ be the Grothendieck group of all representations of G over K , and to avoid confusion denote the element corresponding to a representation V by (V) . If $V_s, s \in S$ is a set of representatives for the isomorphism classes of irreducible representations, then $R_K(G)$ is a free abelian group with basis (V_s) . There are homomorphisms

$$(5.1.4) \quad \begin{array}{ccccc} RO_K(G) & \xrightarrow{f^+} & R_K(G) & \xrightarrow{h^+} & RO_K(G) \\ & & & & \\ R\mathrm{Sp}_K(G) & \xrightarrow{f^-} & R_K(G) & \xrightarrow{h^-} & R\mathrm{Sp}_K(G) \end{array}$$

where f^\pm come from forgetting the bilinear form and where h^\pm come from associating to a representation V the representation $V \oplus V^*$ with the "hyperbolic" bilinear form $B(x+y, x'+y') = (x, y') \pm (x', y)$, with (\cdot, \cdot) denoting the natural pairing of a vector space and its dual. One checks the formulas

$$(5.1.5) \quad \begin{aligned} h^+ f^+ &= h^- f^- = 2 \\ f^+ h^+ &= f^- h^- = * + id. \end{aligned}$$

If V is an irreducible representation isomorphic to its dual, then by Schur's lemma the subspace of invariants in $V^* \otimes V^* = \mathrm{Hom}(V, V^*)$ is one-dimensional. As the characteristic is different from 2, this subspace is contained either in $\Lambda^2 V^*$ or $S^2 V^*$, hence V has a non-zero invariant bilinear form which is either symmetric or skew-symmetric. As V is irreducible this form is nondegenerate and unique up to scalar multiples, so we obtain an element $[V]$ (well-defined by 5.1.1) of either the orthogonal or symplectic Grothendieck groups. Divide the set of irreducibles into a disjoint union

$$S = S_0 \cup S_0^* \cup S_+ \cup S_-$$

where S_+ and S_- are the subsets consisting of those s for which V_s is orthogonal or symplectic respectively, and where S_0 is a cross-section of the rest of S under the action of $*$.

PROPOSITION 5.1.6. $RO_K(G)$ is a free abelian group with basis $[V_s] s \in S_+$ and $h^+(V_s) s \in S_0 \cup S_-$. $R\mathrm{Sp}_K(G)$ is a free abelian group with basis $[V_s] s \in S_-$ and $h^-(V_s) s \in S_0 \cup S_+$. The maps f^\pm are injective and using them to identify $RO_K(G)$ and $R\mathrm{Sp}_K(G)$ with subgroups of $R_K(G)$ we have

$$(5.1.7) \quad \begin{aligned} RO_K(G) \cap R\mathrm{Sp}_K(G) &= \{x + x^* \mid x \in R_K(G)\} \\ RO_K(G) + R\mathrm{Sp}_K(G) &= \{x \in R_K(G) \mid x = x^*\}. \end{aligned}$$

If V is an orthogonal representation having an irreducible invariant subspace W , then either the bilinear form restricted to W is nondegenerate, whence by 5.1.2 $[V] = [V_s] + [W^\circ]$ with $s \in S_+$, or W is isotropic and by 5.1.3 $[V] = h^+(V_s) + [W^2/W]$ with $s \in S$. By induction on the length of V one sees that $[V]$ is a positive linear combination of the elements $[V_s]$ with $s \in S_+$ and $h^+(V_s)$ with $s \in S_0 \cup S_-$, hence these elements generate $RO_K(G)$. But these elements are independent in $R_K(G)$, so they form a basis for $RO_K(G)$, proving the first assertion of the proposition as well as the injectivity of f^+ . The assertions about the symplectic case follow similarly, while the formulas 5.1.7 result from looking at the bases, so the proposition is proved. An immediate consequence of 5.1.7 is

COROLLARY 5.1.8. *There is an exact sequence*

$$R_K(G) \xrightarrow{\ast-id} R_K(G) \xrightarrow{(h^+, -h^-)} RO_K(G) \oplus RSp_K(G) \xrightarrow{f^+ + f^-} R_K(G) \xrightarrow{\Psi \ast-id} R_K(G).$$

When K is the field of complex numbers any representation V possesses an invariant hermitian metric $(,)$. If V is orthogonal (resp. symplectic) with bilinear form B , then this metric may be chosen so that the conjugate linear operator J defined by $B(v, v') = (v, Jv')$ satisfies $J^2 = id$ (resp. $J^2 = -id$). Hence V is the complexification of a representation over the real numbers (resp. the restriction of a quaternionic representation). Using this one sees that $RO_K(G)$ and $RSp_K(G)$ are respectively isomorphic to the Grothendieck groups of real and symplectic representations and that the maps 5.1.4 are given by the appropriate extension and restriction of scalars.

(5.2). Given an orthogonal representation V and a symplectic representation V' , consider the direct sum $V \oplus V'$ as a representation of $G \times \mathbf{Z}_2$ where the generator of \mathbf{Z}_2 acts trivially on the first factor and as $-id$ on the second. This construction gives rise to a homomorphism

$$(5.2.1) \quad RO_K(G) \oplus RSp_K(G) \rightarrow R_K(G \times \mathbf{Z}_2)$$

which is injective, since on composing this with the homomorphism $R_K(G \times \mathbf{Z}_2) \rightarrow R_K(G) \oplus R_K(G)$ giving the two eigenspaces for the \mathbf{Z}_2 -action one obtains the map $f^+ \oplus f^-$. Now if two representations V and V' are endowed with nondegenerate bilinear forms, then the representations $V \otimes V'$ and $\Lambda^j V$ inherit natural nondegenerate forms. Using this one sees that the left side of 5.2.1 is a sub λ -ring of the right; in fact, it is a \mathbf{Z}_2 -graded λ -ring in the evident sense (e.g. $RO_K(G)$ is stable under the λ -operations while λ^i and Ψ^i carry $RSp_K(G)$ into $RO_K(G)$ or $RSp_K(G)$ depending on whether i is even or odd).

Since the map $f^+ + f^-$ of 5.1.8 is the composition of 5.2.1 and the restriction homomorphism from $G \times \mathbf{Z}_2$ to G , it is a λ -homomorphism. Hence by additivity of Ψ^i and the identities 5.1.5, one sees that the exact sequence of 5.1.8 is compatible with the action of the Adams operations.

(5.3). Let A be a discrete valuation ring with quotient field K and residue field k both satisfying the assumptions made at the beginning of Section 5.

Let

$$(5.3.1) \quad d: R_K(G) \rightarrow R_K(G)$$

be the decomposition homomorphism [14, pp. 111-10]; it sends (V) into $(L \otimes_A k)$ where L is

an invariant lattice (i.e. free A -submodule of maximum rank) in V . It is clear that d is a homomorphism of λ -rings. We now wish to show that this homomorphism carries $RO_K(G)$ and $RSp_K(G)$, viewed as subgroups of $R_K(G)$, into the corresponding subgroups over k .

LEMMA 5.3.2. *Let π generate the maximal ideal of A , and let V be an orthogonal (resp. symplectic) representation of G over K with bilinear form B . Then there exists an invariant lattice L in V such that $\pi L^* \subset L \subset L^*$, where $L^* = \{v \in V \mid B(v, L) \subset A\}$.*

By choosing an invariant lattice and multiplying it by some high power of π we can find an invariant lattice L such that $\pi^n L^* \subset L \subset L^*$ for some positive integer n . Suppose that n is the least integer ≥ 1 for which such an L exists, and write $n = 2j - e$ with $e = 0$ or 1 . Then

$$\begin{aligned}(L + \pi^j L^*)^* &= L^* \cap \pi^{-j} L \supset L + \pi^j L^* \\ \pi^j (L + \pi^j L^*)^* &= \pi^j L^* \cap L \subset L + \pi^j L^*\end{aligned}$$

so that lattice $L_1 = L + \pi^j L^*$ has the same properties as L but with n replaced by j . By minimality of n , $n = j = 1$, proving the lemma.

Suppose now that V is an orthogonal representation of G over K and let L be a lattice in V as in the lemma, so that there is an exact sequence of representations over k

$$0 \rightarrow L^*/L \rightarrow L \otimes_A k \rightarrow L/\pi L^* \rightarrow 0$$

where the first map is induced by multiplying by π . Then the map $v, v' \mapsto B(v, v')$ (resp. $v, v' \mapsto \pi^{-1} B(v, v')$) followed by reduction modulo π gives rise to a nondegenerate invariant symmetric bilinear form on $L/\pi L^*$ (resp. L^*/L), hence $(L \otimes_A k)$ lies in the subgroup $RO_k(G)$ of $R_k(G)$. Thus we have shown that $d(RO_K(G)) \subset RO_k(G)$ and the proof for the symplectic case is the same.

COROLLARY 5.3.3. *The decomposition homomorphisms give rise to a map of exact sequences*

$$\begin{array}{ccccccccc} R_K(G) & \rightarrow & R_K(G) & \rightarrow & RO_K(G) \oplus RSp_K(G) & \rightarrow & R_K(G) & \rightarrow & R_K(G) \\ \downarrow d & & \downarrow d & & \downarrow d & & \downarrow d & & \downarrow d \\ R_k(G) & \rightarrow & R_k(G) & \rightarrow & RO_k(G) \oplus RSp_k(G) & \rightarrow & R_k(G) & \rightarrow & R_k(G) \end{array}$$

as in 5.1.8, where all maps commute with the Adams operations and where the vertical maps are λ -ring homomorphisms.

(5.4). Suppose now that K is of characteristic zero and that k is of characteristic p . Define the modular character of an element of $R_K(G)$ to be the K -valued central function on G given on elements of the form (V) by

$$(5.4.1) \quad (V)(g) = \sum \alpha_i^*$$

where $\alpha_1, \dots, \alpha_n$ are the eigenvalues counted with multiplicity of the linear transformation of V produced by g , and where α_i^* is the unique root of unity of order prime to p which reduces to α_i modulo the maximal ideal of A . The following result (valid even for $p = 2$) summarizes those facts about the Brauer theory of modular characters that we need.

PROPOSITION 5.4.2. *If $m = p^a h$, $(p, h) = 1$ is the exponent of G , let q be a power of p such that p^a divides q and h divides $q - 1$. Then Ψ^q is an idempotent operator on $R_K(G)$ whose image is mapped isomorphically onto $R_k(G)$ by the decomposition homomorphism. Moreover*

if $y \in R_k(G)$, then the character of the unique element $x \in \Psi^q(R_k(G))$ with $d(x) = y$ coincides with the modular character of y .

Recall that by associating to an element its character or modular character one obtains isomorphisms of $K \otimes_{\mathbb{Z}} R_k(G)$ and $K \otimes_{\mathbb{Z}} R_k(G)$ with the ring of K -valued central functions on G and G_{reg} respectively [14, pp. 111–35]. Serre supposes K complete, but the results apply here since $R_k(G) = R_{\hat{K}}(G)$ where \hat{K} is the completion of K . Moreover, the decomposition homomorphism is given in terms of characters by restriction of a function on G to G_{reg} . By computing on cyclic subgroups one obtains the formula

$$(\Psi^q x)(g) = x(g^q) = x(g_r)$$

valid for x in either $R_k(G)$ or $R_k(G)$, where g_r is the p -regular component of g . It follows that Ψ^q acts trivially on $R_k(G)$ and that it is idempotent on $R_k(G)$ with image the set of x such that $x(g) = x(g_r)$ for all g . Since d is surjective [14, pp. 111–113] and a λ -homomorphism, the map $\Psi^q(R_k(G)) \rightarrow R_k(G)$ induced by d is surjective. But on tensoring with K this map becomes an isomorphism, hence it is an isomorphism as claimed. Finally the last assertion of the proposition follows immediately from the character description of d .

COROLLARY 5.4.3. Ψ^q is an idempotent operator on $RO_k(G)$ and $RSp_k(G)$ and the decomposition homomorphism induces a λ -ring isomorphism.

$$\Psi^q(RO_k(G) \oplus RSp_k(G)) \simeq RO_k(G) \oplus RSp_k(G).$$

Moreover, if $y \in R_k(G)$ comes from $RO_k(G)$ (resp. $RSp_k(G)$), then the unique element x of $R_k(G)$ whose character coincides with the modular character of y comes from $RO_k(G)$ (resp. $RSp_k(G)$).

Since the orthogonal and symplectic Grothendieck groups over K both admit injections into $R_k(G)$ commuting with Ψ^q , it follows that Ψ^q is an idempotent operator on these former groups. Passing to the image of this idempotent operator acting on the top row of the diagram of 5.3.3 gives an exact sequence which is mapped by d to the bottom row. Since there are isomorphisms at the four outer places by the above proposition, the middle map is an isomorphism by the five-lemma. This proves the first part of the corollary, and the rest is clear.

(5.5). It remains to connect up 5.4.3 with what is used in Section 1. Let k_1 be a field of odd characteristic p containing a primitive h -th root of unity, where $m = p^a h$ is the exponent of G , and suppose given an embedding ϕ of μ_h , the group of h -th roots of unity in k_1 , into \mathbb{C}^* . Then given a representation V of G over k_1 we can define its modular character χ with respect to ϕ using formula 1.3, and we want to show that if V is an orthogonal (resp. symplectic) representation then χ is the character of a virtual real (resp. quaternionic) representation. By localizing the ring of cyclotomic integers $\mathbb{Z}[\exp 2\pi i/m]$ with respect to a maximal ideal containing p , we find a discrete valuation ring contained in \mathbb{C} with residue field k isomorphic to a subfield of k_1 containing μ_h . Consider the modular character χ' of V defined by using the embedding $\alpha \mapsto \alpha^*$ of 5.4.1. Since μ_h is cyclic there is an integer b prime to h , which one can suppose to be odd, such that $\phi(x) = (x^*)^b$ for all $x \in \mu_h$, consequently if x and x' are the elements of $R(G)$ with characters χ and χ' , then $x = \Psi^b x'$. Using 5.4.3 one finds

that x' lies in $RO(G)$ (resp. $RSp(G)$) if V is an orthogonal (resp. symplectic) representation, hence the same is true for x as these subgroups are stable under Ψ^b .

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