Covering spaces

I. Covering spaces

Exercise 1. — Prove that a 1-fold covering space is a homeomorphism.

Exercise 2. — Let $p : E \to B$ be a map which is a local homeomorphism.

a) Give an example where p is not a covering.
b) Prove that if E is compact(1) and B is Hausdorff then p is a finite covering.
c) Assume B Hausdorff. Prove that if there exists an integer $n \geqslant 1$ such that for all $b \in B$, $p^{-1}(b)$ has n elements, then p is an n-fold covering.
d) Prove that for all $n \geqslant 1$ the maps $\begin{array}{ccc} S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$ are coverings.
e) Let P be a polynomial in $\mathbf{C}[X]$ of degree $n \geqslant 1$. Let S be the set of roots of the derivative P' and $R := P(S)$. Prove that the map $\begin{array}{ccc} \mathbf{C} - P^{-1}(R) & \to & \mathbf{C} - R \\ z & \mapsto & P(z) \end{array}$ is an n-fold covering.

Exercise 3. — Let X be a topological space equipped with a continuous action(2) of a (discrete) group G satisfying the following hypothesis:

"For every $x \in X$, there is a open neighbourhood U of x in X such that
for all $g \neq h \in G$, gU and hU are disjoint".

a) Prove that the quotient map $X \to G\backslash X$ is a covering.
b) If $H \subset G$ is a subgroup, prove that $H\backslash X \to G\backslash X$ is a covering.
c) Prove that the hypothesis is always satisfied when Y is Hausdorff, G is finite and acts without fixed point(3).
d) Deduce a covering map from the sphere S^n to the projective space $P^n(\mathbf{R})$.
e) Prove that if G is a topological group and H is a discrete subgroup, then $G \to H\backslash G$ is a covering space.
[**Hint:** Reduce to show the property at $x = e$. Let $\varphi : G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2^{-1}$. For U an open neighborhood of e in G, prove there exists a open V such that $\varphi(V \times V) \subset U$.]

Exercise 4. — Let $p : E \to B$ be a covering space with B locally connected. Show that the restriction of p to a connected component of E is still a covering space.

Exercise 5. — Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two connected covering spaces (with B locally connected) and $f : E_1 \to E_2$ be covering map. Show that f is a covering.

II. Lifting and universal covers

Exercise 6. — Prove (using covering spaces) that there is no continuous logarithm function $\log : \mathbf{C}^* \to \mathbf{C}$.
Prove that one can define it continuously on subsets of the form: $\mathbf{C} - \{\text{one half-line at } 0\}$

Exercise 7. — Let $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ be two universal covers.

a) Prove that for every $f : X \to Y$ there is a map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ such that $f \circ p = q \circ \widetilde{f}$.
b) **Warning:** This does not mean that the universal cover construction is functorial!
Let $e : \mathbf{R} \to S^1$ be the map $t \mapsto \exp(2i\pi t)$ and $f : S^1 \to S^1$ be the map $z \mapsto -z$.
Prove that there is not lift $\widetilde{f} : \mathbf{R} \to \mathbf{R}$ such that $\widetilde{f} \circ \widetilde{f} = \mathrm{id}_{\mathbf{R}}$

(1) Thus Hausdorff and quasi-compact.
(2) That is an action of G on X such that the map $X \times G \to X$ is continuous.
(3) *i.e.* one has $g \cdot x = x \implies g = e$.

Exercise 8. — Let $p : E \to B$ be a covering. Suppose that B is a complex manifold.

a) Prove that there is a unique complex structure on E such that p is analytic.
b) Let $f : X \to B$ be an analytic map. Under which condition does f lift to an analytic map $\widetilde{f} : X \to E$?
c) Apply this to exercice I.5 of the Complex Geometry course.

Exercise 9. — Let G be a connected and locally path-connected topological group. Let $p : \Gamma \to G$ be a connected cover and $\widetilde{e} \in p^{-1}(e) \subset \Gamma$.

a) Let $m : \Gamma \times \Gamma \to G$ be the map $(x, y) \mapsto p(x) \cdot p(y)$. Prove that $m_*(\pi_1(\Gamma \times \Gamma, (\widetilde{e}, \widetilde{e})))$ is contained in $p_*(\pi_1(\Gamma, \widetilde{e}))$.
[**Hint:** You can use exercise I.11 from the previous sheet.]
b) Prove that there is a unique group structure on Γ such that $\widetilde{e}$ is the unit and p is a group morphism.

III. The monodromy action

Exercise 10. — Let $p : E \to B$ be a covering and $f : X \to B$ be a map. Express the monodromy of $f^*(p) : X \times_B E \to X$ in terms of the monodromy of p.

Exercise 11. — Let (B, b_0) be a pointed path-connected semi-locally simply-connected space.

a) Prove that in the monodromy correspondence, *path-connected* covers of B correspond to $\pi_1(B, b_0)$-sets with transitive action.
b) To which coverings do these $\pi_1(B, b_0)$-sets correspond to?

i) $\pi_1(B, b_0)$ with translation action

ii) A set F with trivial action

iii) $F_1 \times F_2$ with diagonal action for F_1, F_2 two $\pi_1(B, b_0)$-sets.

iv) $F_1 \sqcup F_2$ for F_1, F_2 two $\pi_1(B, b_0)$-sets.

Exercise 12. — Let $p : E \to B$ be a covering space with E path-connected. Let x be a point in X and e be a point in the fiber $p^{-1}(x)$.

a) Show that $p_* : \pi_1(E, e) \to \pi_1(X, x)$ is injective.
b) Show that if p_* is surjective then p is a homeomorphism.

Exercise 13. — Let G be a group and X_1, X_2 be G-sets.

a) Prove that X_1 and X_2 are isomorphic iff there exists a bijection $\varphi : X_1 \to X_2$ such that

$$\forall g \in G,\ \forall x_1 \in X_1,\ g \cdot x_1 = \varphi^{-1}(g \cdot \varphi(x)).$$

b) What are (up to isomorphism) the 2-fold and 3-fold covering spaces of S^1 and $S^1 \times S^1$?

Exercise 14. — Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two covering spaces.

a) Show that the fiber product $p : E_1 \times_B E_2 \to B$ is a covering of B. Express the monodromy of p in terms of that of p_1 and p_2.
b) In the case where $E_1 = E_2 = B = S^1$, $p_1(z) = z^2$ and $p_2(z) = z^3$, prove that the fiber product is isomorphic to $z \mapsto z^6$.
c) What do you think about the case where $E_1 = E_2 = B = S^1$, $p_1(z) = z^2$ and $p_2(z) = z^4$?

Exercise 15. — For $z \in \mathbf{C}$, recall that $\cos z = \frac{e^{\imath z} + e^{-\imath z}}{2}$.

a) For $z, z' \in \mathbf{C}$, prove the equivalence of:
(i) $\cos z = \cos z'$;
(ii) there exists $k \in \mathbf{Z}$ such that $z' = \pm z + 2k\pi$.
b) Show that $\cos 3z = 4\cos^3 z - 3\cos z$.
c) Prove that the map $p : \mathbb{C} - \{-1, -\frac{1}{2}, \frac{1}{2}, 1\} \to \mathbb{C} - \{-1, 1\}$, $z \mapsto 4z^3 - 3z$ is a 3-fold covering.

d) Let α and β be the loops

$$\alpha : t \mapsto \cos(\frac{\pi}{2}e^{\imath\pi t}) \text{ and } \beta : t \mapsto -\cos(\frac{\pi}{2}e^{\imath\pi t})$$

of $\mathbb{C} - \{-1, 1\}$ based at 0. Write down explicitly their lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ in $\mathbb{C} - \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$.

e) Deduce that the group $\pi_1(\mathbb{C} - \{-1, 1\}; 0)$ is not abelian.

Exercise 16. — Let $p_1 : Y_1 \to X$, $p_2 : Y_2 \to X$ be two covering spaces. Define H to be the set of pairs (x, f_x), with $x \in X$ and $f_x : p_1^{-1}(x) -> p_2^{-1}(x)$.
For each covering map $f : Y_1 \to Y_2$ and each open $U \subset X$, let $O_{U,f} := \{(x, f_{|p_1^{-1}(x)})\}_{x \in U}$. We put on H the topology generated by the $O_{U,f}$'s.

a) Show that the $O_{U,f}$'s form a neighborhood basis of the topology.
b) Show that the map $p : H \to X$, $(x, f_x) \mapsto x$ is a covering space.
c) Show that there is a bijective correspondence between covering maps $Y_1 \to Y_2$ and sections of $p : H \to X$.
d) Show that the evaluation map $Y_1 \times_X H \to Y_2$ is continuous.
e) Deduce that the monodromy action of $H \to X$ is the following:

$$(x, f_x) \cdot \alpha = (x, y \mapsto (f_x(y \cdot \alpha^{-1}))) \cdot \alpha$$

for $\alpha \in \pi_1(X, x_0)$.

IV. Galois coverings

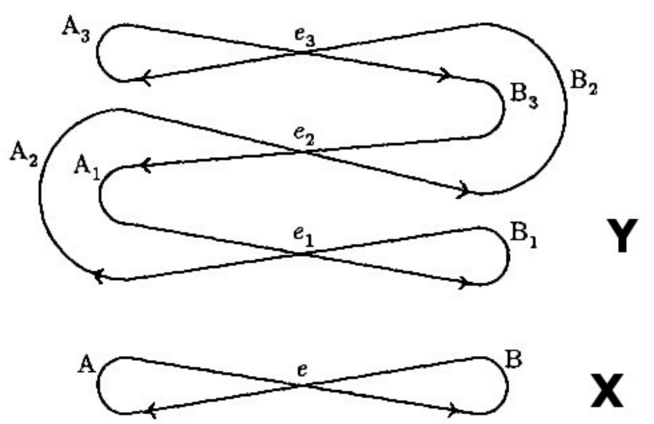
Exercise 17. — For an integer $n \geqslant 1$, let p_n be the covering $\begin{array}{ccc} S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$.

a) For which n, m are there covering morphisms $p_n \to p_m$?
b) What is the group of automorphisms of p_n?

Exercise 18. — **a)** Prove that a connected 2-fold cover is a Galois cover.
b) Prove that a connected 3-fold cover is Galois iff its automorphism group is isomorphic to $\mathbf{Z}/3\mathbf{Z}$.
c) Prove that the automorphism group of a connected 3-fold cover is either $\mathbf{Z}/3\mathbf{Z}$ or trivial.

Exercise 19. — Let $p : E \to B$ be a connected cover.
Prove that p is Galois if and only if the pull-back cover $p^*(p) : E \times_B E \to E$ is trivial.
[**Hint:** If p is Galois, let G be the (discrete) group $\mathrm{Aut}(p)$. Prove that $p^*(p)$ is isomorphic to the trivial cover $E \times G \to E$. Conversely, use $E \times F \cong E \times_B E \xrightarrow{pr_2} E$ to construct automorphisms.]

Exercise 20. — Let X and Y be the spaces represented in the picture below and let $p : Y \to X$ be the continuous map sending homeomorphically each arc A_i (resp. B_i) over A (resp. B) according to the orientation indicated.



a) Show that p is a covering space.

b) What is its automorphism group? Deduce that p is not Galois.

c) Construct a 2-fold cover $q : \widetilde{Y} \to Y$ such that $p \circ q : \widetilde{Y} \to X$ is Galois.

Exercise 21. — Let (B, b) be a pointed connected, semi-locally simply connected space. Prove that in the monodromy correspondence connected Galois covers correspond to $\pi_1(B, b)$-sets with the following property: an element of $\pi_1(B, b)$ acts either trivially (as the identity permutation) or without fixed point.

Exercise 22. — Let E be a connected space, G be a (discrete) group acting properly discontinuously on E and $H \subset G$ be a subgroup.

a) Prove that $p : H\backslash E \to G\backslash E$ is a connected covering space.

b) Prove that p is Galois iff H is normal.
[**Hint:** Prove that every automorphism of p is of the form $[e] \mapsto [g \cdot e]$.]

Exercise 23. — Prove that the covering in Exercise 15 is non Galois.

V. Van Kampen's theorem

Exercise 24. — Let X be a wedge of two circles (or a "figure-eight").

a) Determine all 2-folded covers of X (up to isomorphism).

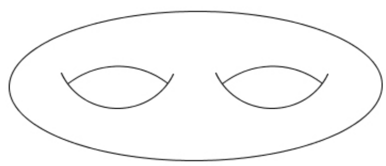
b) How many subgroups H of index 2 are there in the free group $F_2 =< a, b >$?

c) Give the exact list of these subgroups H and show that each one is free.

Exercise 25. — What is the fundamental group of $\mathbf{R}^3$ minus the three coordinate axes?

Exercise 26. — **a)** Show that the fundamental group of a torus minus a point is a free group with two generators.

b) What is the fundamental group of the surface pictured below?



Exercise 27. — Let $\pi : S^2 \to P^2(\mathbf{R})$ be the quotient map. Let $\widetilde{U}_1$ be the open upper hemisphere $z > 0$ and $\widetilde{U}_2$ be the "equatorial region" $-1/4 < z < 1/4$ in S^2.

a) Prove that $U_1 := \pi(\widetilde{U}_1)$ and $U_2 := \widetilde{U}_2$ form an open cover of $P^2(\mathbf{R})$. Prove that the intersection $U_1 \cap U_2$ is path connected.

b) Show that U_2 is homeomorphic to a Möbius strip. What is its fundamental group?

c) What is the fundamental group of $U_1 \cap U_2$? What is the map induced on fundamental groups by the inclusion $U_1 \cap U_2 \hookrightarrow U_2$?

d) Deduce from Van Kampen's theorem the fundamental group of $P^2(\mathbf{R})$.