De Rham Cohomology

I. Differential forms on $U \subset \mathbf{R}^n$

Definition 1. — Let $U \subset \mathbf{R}^n$ be an open subset and ω a 1-form on U. Let $\gamma : [0,1] \to U$ be a smooth path on U. We set

$$\int_\gamma \omega = \int_0^1 \omega_{\gamma(t)}(\gamma'(t))\, dt\,.$$

Exercise 1. — Let $U \subset \mathbf{R}^n$ be a connected open subset and ω a 1-form on U.

a) Show that if $\omega = df$ for some smooth function f on U, then $\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0))$.

b) Show that $\omega = df$ for some smooth function f if and only if $\int_\gamma \omega = 0$ for all loops in U. Hint: Fix a point $x_0 \in U$ and for $x \in U$ chose a path γ_x in U from x_0 to x. Consider the function $f(x) = \int_{\gamma_x} \omega$. Show that $f(x)$ does not depend on the choice of the path γ_x, then calculate ∇f.

Exercise 2. — In this exercise we will describe $H^1_{DR}(\mathbf{R}^2 \setminus (0,0))$. Consider the differential 1-form

$$\omega_0 = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

a) Show that $d\omega_0 = 0$.

b) Let $\gamma : [0,1] \to \mathbb{S}^1 \subset \mathbf{C} \simeq \mathbf{R}^2$ be the loop defined by $\gamma(t) = e^{2\pi i t}$. Calculate $\int_\gamma \omega_0$. Deduce that ω_0 cannot be equal to df for some smooth function f on U.

c) Let ω be a 1-form ω satisfying $d\omega = 0$ and $\int_\gamma \omega = 0$. Show that $\omega = df$ for some smooth function on $\mathbf{R}^2 \setminus (0,0)$. Hint: Consider the open subsets $U_1 = \{x > 0\}$, $U_2 = \{x < 0\}$, $U_3 = \{y > 0\}$, $U_4 = \{y < 0\}$ and a 1-form ω satisfying $d\omega = 0$. Show that we can find smooth functions f_i on U_i such that we have $\omega = df_i$ and $f_1 = f_3$ on $U_1 \cap U_3$, $f_2 = f_3$ on $U_2 \cap U_3$, $f_2 = f_4$ on $U_2 \cap U_4$ and $f_1 - f_4 = c$ is constant on $U_1 \cap U_4$. Calculate $\int_\gamma \omega$ using this description of ω to deduce that $c = 0$.

d) Deduce that integration along γ defines an isomorphism

$$\int_\gamma : \quad H^1_{DR}(\mathbf{R}^2 \setminus (0,0)) \longrightarrow \mathbf{R}.$$

Hence $H^1_{DR}(\mathbf{R}^2 \setminus (0,0))$ is a 1-dimensional vector space with basis $\{[\omega_0]\}$.

e) Describe $H^1_{DR}(\mathbf{R}^2 \setminus (0,0), (1,0))$.

Exercise 3. — In this exercise we will show that $H^2_{DR}(\mathbf{R}^2 \setminus (0,0)) \simeq 0$. Consider the smooth maps

$$\Phi : \mathbf{R}_{>0} \times \mathbf{R} \longrightarrow \mathbf{R}^2 \setminus \{(0,0)\} \qquad (r,\theta) \mapsto \varphi(r,\theta) = (r\cos(\theta), r\sin(\theta))$$

a) Show that $\Phi^* : \Omega^k(\mathbf{R}^2 \setminus (0,0)) \to \Omega^k(\mathbf{R}_{>0} \times \mathbf{R})$ is injective and identifies a k-form on $\mathbf{R}^2 \setminus (0,0)$ with a k-form on $\mathbf{R}_{>0} \times \mathbf{R}$ whose coefficients 2π-periodic with respect to θ.

b) Deduce that $H^2_{DR}(\mathbf{R}^2 \setminus (0,0)) \simeq 0$. Hint: To show that a closed form is exact consider integration with respect to r.

Exercise 4. — Consider the smooth map

$$S : \mathbf{R}^3 \setminus \{(0,0,0)\} \longrightarrow \mathbf{R}^3 \qquad S(x,y,z) = \frac{1}{\|(x,y,z)\|}(x,y,z)$$

and the differential 2-form $\omega_0 = z dx \wedge dy - y dx \wedge dz + x dy \wedge dz$. define

$$\Psi : \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}^3 \setminus \{0,0\} \quad ; \quad \Psi(\varphi,\theta) = (\cos(\theta)\sin(\varphi), \cos(\theta)\cos(\varphi), \sin(\theta))$$

a) Calculate the explicit formula of $S^*\omega_0$ and show that $S^*\omega_0$ is closed (note that ω_0 is not closed).

b) Calculate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}(S^*\omega_0)_{\Psi(\varphi,\theta)}(J_\Psi(\varphi,\theta))d\varphi d\theta$$

c) Let ω be an exact 2-form on $\mathbf{R}^3\setminus\{(0,0,0)\}$. Use Green's theorem to prove that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\omega_{\Psi(\varphi,\theta)}(J_\Psi(\varphi,\theta))d\varphi d\theta = 0.$$

d) Show that integration with respect to Ψ defines an isomorphism

$$\int_\Psi : H^2_{DR}(\mathbf{R}^3\setminus(0,0,0)) \longrightarrow \mathbf{R}.$$

e) Let ω be a closed 1-form, using Greens theorem and a slight adaptation of exercise 1 prove that ω is exact, hence $H^1_{DR}(\mathbf{R}^3\setminus(0,0,0))\simeq\{0\}$.

II. Homological algebra

***Exercise 5* (The 5 Lemma).** — Consider a commutative diagram of vector spaces

$$\begin{array}{ccccccccc} V_2 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & V_4 & \longrightarrow & V_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ W_1 & \longrightarrow & W_2 & \longrightarrow & W_3 & \longrightarrow & W_4 & \longrightarrow & W_5 \end{array}$$

Suppose that both lines are exact sequences, f_2, f_4 are isomorphisms, f_1 is surjective and f_5 is injective. Show that f_3 is an isomorphism.

Exercise 6. — Consider a bounded complex (V^*, d), that is $V^k\simeq 0$ for all but a finite number of k. Let $\chi(V)=\sum_{-\infty}^{\infty}(-1)^k\dim(V^k)$.

a) Suppose that V^* is a short exact sequence. Show that $\chi(V)=0$.
b) Use induction to show that if V^* is exact then $\chi(V)=0$.

Exercise 7. — Consider an exact sequence

$$V_1 \xrightarrow{f} V_2 \longrightarrow V_3 \longrightarrow V_4 \xrightarrow{g} V_5$$

a) Show that the sequence induces a short exact sequence

$$0 \longrightarrow V_2/\operatorname{im}(f) \longrightarrow V_3 \longrightarrow \ker(g) \longrightarrow 0$$

Deduce that $\dim(V_3)=\dim(V_2/\operatorname{im}(f))+\dim(\ker(g))$.

b) Fix $V_1=V_2=V_4=V_5=\mathbf{R}^2$. Suppose first that $f=g=\mathrm{id}$. Verify that $\dim(V_3)=0$. Explain why $\dim(V_3)\leqslant 4$, then give examples of f, g such that $\dim(V_3)=1,2,3,4$.

Exercise 8. — Consider a short exact sequence

$$0 \longrightarrow V \xrightarrow{f} W \xrightarrow{g} Z \longrightarrow 0$$

Let L be another vector space and consider the complexes

$$0 \longrightarrow V\otimes L \xrightarrow{f\otimes 1} W\otimes L \xrightarrow{g\otimes 1} Z\otimes L \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Hom}(Z,L) \xrightarrow{\circ g} \mathrm{Hom}(W,L) \xrightarrow{\circ f} \mathrm{Hom}(V,L) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Hom}(L,V) \xrightarrow{f\circ} \mathrm{Hom}(L,W) \xrightarrow{g\circ} \mathrm{Hom}(L,Z) \longrightarrow 0$$

Show that these complexes are short exact sequences.

Exercise 9. — Let $A_1^*, B_1^*, C_1^*, A_2^*, B_2^*$ and C_2^* be six cochain complexes. Assume that we have maps of cochains complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1^* & \longrightarrow & B_1^* & \longrightarrow & C_1^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_2^* & \longrightarrow & B_2^* & \longrightarrow & C_2^* & \longrightarrow & 0 \end{array}$$

such that the two rows are exact and the two squares commute.
Prove that there are induced maps between the long exact sequences :

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & H^p(A_1) & \longrightarrow & H^p(B_1) & \longrightarrow & H^p(C_1) & \longrightarrow & H^{p+1}(A_1) & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & H^p(A_2) & \longrightarrow & H^p(B_2) & \longrightarrow & H^p(C_2) & \longrightarrow & H^{p+1}(A_2) & \longrightarrow & \ldots \end{array}$$

such that all the squares commute.

III. De Rham cohomology of manifolds

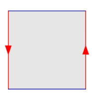
Exercise 10. — Let S^1 be the circle, and U, V be two "thick" open half circle that cover it.

a) Write down the Mayer-Vietoris sequence for de Rham cohomology associated to this cover. Deduce the de Rham cohomology of S^1.
Following the definition of the connecting morphism, give an explicit generator of $H^1(S^1)$.

b) Same question, but using de Rham cohomology with compact support instead.

c) Generalize to S^n, showing that a generator of $H^n(S^n)$ can be chosen with support arbitrary small.

Exercise 11. — The open Möbius strip M is the quotient of the semi-open square $[0,1]\times]0,1[$ via the following identifications: $\forall y \in]0,1[,\ (0,y) \simeq (1, 1-y)$.



a) Compute the de Rham cohomology of M.

b) Compute the de Rham cohomology with compact support of M.

Exercise 12. — **a)** Compute the de Rham cohomology of $\mathbf{R}^n - \{k \text{ points}\}$.

b) Same question for de Rham cohomology with compact support.

c) Same questions for $\mathbf{C} - \mathbf{Z}$.

Exercise 13. — Prove the Brouwer fix point theorem for smooth maps, *i.e.* prove that any smooth $f : D^n \to D^n$ has a fixed point.

Exercise 14. — Let $f : S^n \to S^n$ be a smooth map and $f^* : H^n(S^n) \to H^n(S^n)$ the induced map.

a) Prove that f^* is a homothety. We denote by $d(f)$ its scale factor.

b) For two maps f, g, prove that $d(f \circ g) = d(f)d(g)$.

c) Prove that if f is not surjective, then $d(f) = 0$.
[**Hint:** Use Exercise 10c).]

d) What is the degree of the map induced by an element of $O(n+1)$?
[**Hint:** Treat first the case of a reflexion: $(x_1, x_2, \ldots, x_{n+1}) \mapsto (-x_1, x_2, \ldots, x_{n+1})$ by induction on n.]

e) Same question for the map $S^1 \to S^1$, $z \mapsto z^n$?

Exercise 15. — Let n be an even integer. The goal is to prove that there is no non-vanishing smooth vector field on S^n. Suppose by contradiction that there exists such a vector field $X : S^n \to R^{n+1}$ such that $X(x) \cdot x = 0$.

a) Construct out of X a homotopy from id_{S^n} to $-\mathrm{id}_{S^n}$.
b) Conclude.
c) Does the extend to odd-dimensional spheres?

Exercise 16. — Let $p : Y \to X$ be a smooth Galois cover of manifolds, with Galois group G. Note that the group G acts on $\Omega^*(Y)$ and on $H^*(Y)$. We denote by $\Omega^*(Y)^G$ the subalgebra of differential forms that are fixed by all the elements in G.

a) Prove that p^* sends $\Omega^*(X)$ into $\Omega^*(Y)^G$.
b) Show that $p^* : \Omega^*(X) \to \Omega^*(Y)^G$ is an isomorphism of differential graded algebras.
[**Hint:** Construct explicitly its inverse.]
c) The inclusion $\Omega^*(Y)^G \subset \Omega^*(Y)$ induces a map $\iota : H^*(\Omega^*(Y)^G) \to H^*(Y)^G$. Prove that if G is finite, ι is an isomorphism.
[**Hint:** Whenever you have a finite group acting on some ω, it's a good idea to consider $\frac{1}{|G|} \sum_{g \in G} g.\omega$].
d) Apply this to compute the cohomology of the projective space $P^n(\mathbf{R})$.

Exercise 17. — Let $P^n(\mathbf{C})$ be the complex projective space.

a) Prove that $P^n(\mathbf{C})$ is compact. Prove that $P^1(\mathbf{C})$ is diffeomorphic to S^2.
b) Let $x := [0 : \cdots : 0 : 1]$ and $U := P^n(\mathbf{C}) - \{x\}$. Prove that U is homotopy equivalent to $P^{n-1}(\mathbf{C})$.
c) Use an inductive Mayer-Vietoris sequence to compute $H^*(P^n(\mathbf{C}))$.

Exercise 18. — Compute the de Rham cohomology of the following product manifolds:

a) $(S^1)^n$ **b)** $S^1 \times S^2 \times S^3$

Exercise 19. — Let $\mathbf{Z}$ be the discrete space of integers.

a) Compute the de Rham cohomology of $\mathbf{Z}$. In particular, prove that $H^0(\mathbf{Z}) \cong \mathbf{R}^{\mathbf{Z}}$ has uncountable dimension.
b) Prove that the natural Kunneth map $H^*(\mathbf{Z}) \otimes H^*(\mathbf{Z}) \to H^*(\mathbf{Z} \times \mathbf{Z})$ is not an isomorphism.
[**Hint:** Prove that in $\mathbf{R}^{\mathbf{Z} \times \mathbf{Z}}$, the sequence $a_{n,m} = \delta_{n,m}$ is not in the image.]

IV. Orientation and integration. Poincaré duality

Exercise 20. — Let $\mathbf{P}^2(\mathbf{R})$ be the real projective plane.

a) Prove that $\mathbf{P}^2(\mathbf{R})$ is a manifold that is not orientable.
b) Prove that the Möbius band is non orientable.
c) How does b) implies a)?

Exercise 21. — Prove that a complex analytic manifold is orientable.

Exercise 22. — **a)** Prove that the two manifolds $S^2 \times S^4$ and $P^3(\mathbf{C})$ have isomorphic de Rham cohomology groups in each degree.
b) Prove that these two manifolds are not homotopy equivalent.
[**Hint:** Compare the algebra structure.]

Exercise 23. — Let M be a connected, compact oriented n-dimensional manifold.
Prove that a form $\omega \in \Omega^n(M)$ is closed iff $\int_M \omega = 0$.

Exercise 24. — The goal of this exercise is to present the "correct" proof of Poincaré duality. Let M be an oriented manifold of dimension n. We fix $k \leqslant n$. For every open $O \subset M$, let $\iota_O : H^k(O) \to (H_c^{n-k}(O))^*$ be the map defined by integration. The goal is thus to prove that ι_M is an isomorphism.

Let $\mathcal{O}$ be a set of opens subsets of M which is closed under intersections (*i.e.* if O_1, O_2 are opens in $\mathcal{O}$ then $O_1 \cap O_2$ also). Let $\mathcal{O}_f$ be the set of opens in M which are finite unions of opens in $\mathcal{O}$ and let $\mathcal{O}_d$ be the set of opens in M which are arbitrary disjoint unions of opens in $\mathcal{O}$.

a) Prove that if for every $O \in \mathcal{O}$, ι_O is an isomorphism, then ι_U is also an isomorphism for every open $U \in \mathcal{O}_f$ (resp. $U \in \mathcal{O}_d$).
[**Hint:** Note that the dual of a direct sum $\bigoplus_{i \in I} E_i$ is isomorphic to the product $\prod_{i \in I} E_i^*$.]

The crux of the proof is the following fact on the topology of a manifold.
Suppose that $\mathcal{O}$ as above is a basis of the topology of M such that the closure of each $O \in \mathcal{O}$ is compact. Then there exist two opens $V_1, V_2 \in (\mathcal{O}_f)_d$ such that $M = V_1 \cup V_2$.
To prove this: start with a cover $U_1 \subset U_2 \subset \dots$ of M such that every U_i has compact closure $\overline{U_i} \subset U_{i+1}$.

b) Construct new opens W_i of M satisfying:
(i) $\forall i,\ \overline{U_i} \subset \bigcup_{j \leqslant i} W_j \subset U_{i+1}$.
(ii) $\forall i,\ W_i \in \mathcal{O}_f$.
(iii) $\forall i,\ W_i \cap W_{i+2} = \emptyset$.
[**Hint:** Construct the W_i by induction, covering $\overline{U_i} - \bigcup_{j<i} W_j$ by a finite number of opens in $U_{i+1} - \overline{U_{i-1}}$.]

c) Show that one can take $V_1 := \bigcup_{i \text{ odd}} W_i$ and $V_2 := \bigcup_{i \text{ even}} W_i$.

Conclude, proving that:

d) ι_O is an isomorphism for every open in $O \subset \mathbf{R}^n$.
[**Hint:** One can take as basis $\mathcal{O}$ the set of all open "boxes".]

e) Then that ι_M is an isomorphism.

Exercise 25. — **a)** Using the same lemma as in Exercise 24, give a "correct proof" of the Kunneth isomorphism for compactly supported de Rham cohomology.

b) Where does the proof goes wrong for de Rham cohomology?

c) How can we save it when one of the manifolds has finite dimensional cohomology?