

Additive and unstable algebra structures for the MU-cohomology of profinite spaces, resolutions and application to the cohomology of mapping spaces from $\mathbb{C}P^\infty$

A survey (March 2003)

1. Introduction

We consider division functors for MU-cohomology related to the MU-cohomology of mapping spaces, in the spirit of Lannes' T-functor for mod p -cohomology.

Let $(MU_n)_n$ denote the Ω -spectra representing the cobordism cohomology theory. The coefficient ring MU^* is the polynomial algebra over \mathbb{Z} generated by elements x_k of degree $2k$, $k \geq 1$. As we will consider spaces like $\mathbf{map}(\mathbb{C}P^\infty, MU_n)$ which already has infinite mod p cohomology group in each degree, we will use profinite completion and continuous cohomology. So we fix a prime number p and let $\hat{\mathcal{S}}$, respectively $h\hat{\mathcal{S}}$, denote the Quillen model category of profinite spaces, respectively the associated homotopy category, where the weak equivalences are the maps inducing an isomorphism in continuous mod p cohomology ([MO]).

For W a finite simplicial set, the mapping space $\mathbf{map}(W, Y) \in \hat{\mathcal{S}}$ is defined by the adjunction

$$\mathrm{Hom}_{\hat{\mathcal{S}}}(X, \mathbf{map}(W, Y)) \simeq \mathrm{Hom}_{\hat{\mathcal{S}}}(W \times X, Y) .$$

This extends to general simplicial set W by defining the external cartesian product $W \hat{\times} X$ as the colimit in $\hat{\mathcal{S}}$ of the $W_\alpha \times X$ and $\mathbf{map}(W, Y)$ as the limit in $\hat{\mathcal{S}}$ of the $\mathbf{map}(W_\alpha, Y)$, W_α spanning the simplicial finite subsets of W . We thus get a counit $W \hat{\times} \mathbf{map}(W, Y) \rightarrow Y$ in $\hat{\mathcal{S}}$. We still have a bijection when changing $\hat{\mathcal{S}}$ to the homotopy category $h\hat{\mathcal{S}}$ if Y is fibrant in $\hat{\mathcal{S}}$ and in particular a bijection $\pi_0 \mathbf{map}(W, Y) \simeq [W, Y]$, where $[W, Y]$ denotes the (profinite) set of homotopy classes of maps from W to the underlying space of Y .

We define the continuous MU-cohomology of a profinite space X in degree $n \in \mathbb{Z}$ as the set $\mathrm{Hom}_{h\hat{\mathcal{S}}}(X, \hat{M}U_n)$, where $\hat{M}U_n$ denotes the profinite completion of MU_n if $n \geq 1$ and the profinite space $\Omega^{1-n} \hat{M}U_1$ if $n \leq 0$; we denote it by $\hat{M}U^n X$. The ring structure on MU induces an MU^* -algebra structure on $\hat{M}U^n X$ for every profinite space X . Note that if \hat{X} is the profinite completion of some space X then the continuous mod p -cohomology of \hat{X} identifies with the ordinary mod p cohomology of X and the continuous MU-cohomology of \hat{X} identifies with the p -completed MU-cohomology of X .

We start by pointed out some facts concerning the continuous mod p cohomology of profinite spaces, which we denote by $H^*(-)$. We let \mathcal{E} denote the category of \mathbb{Z} -graded \mathbb{F}_p -vector spaces.

1) Let E be a non negatively graded \mathbb{F}_p -vector space, then there exists a profinite space $K(E)$ and a map $E \rightarrow H^*K(E)$ inducing a bijection $\mathrm{Hom}_{h\hat{\mathcal{S}}}(X, K(E)) \rightarrow \mathrm{Hom}_{\mathcal{E}}(E, H^*X)$ for any profinite space X .

The category \mathcal{K}_H of unstable algebras over the mod p Steenrod algebra is exactly such that the natural map $\mathrm{Hom}_{h\hat{\mathcal{S}}}(X, K(E)) \rightarrow \mathrm{Hom}_{\mathcal{K}_H}(H^*K(E), H^*X)$ is a bijection for all X , so $H^*K(E)$ is the "free unstable algebra on $E \in \mathcal{E}$ ".

2) We have a Künneth formula $H^*X \otimes H^*Y \simeq H^*(X \times Y)$ for all X and Y in $\hat{\mathcal{S}}$. Let W be a simplicial set whose mod p cohomology is degree-wise finite and let X be in $\hat{\mathcal{S}}$, then the map $H^*W \otimes H^*X \rightarrow H^*(W \hat{\times} X)$ is an isomorphism. In other words, the map $W \hat{\times} X \rightarrow \hat{W} \times X$ is a weak equivalence in $\hat{\mathcal{S}}$. So the set $\mathrm{Hom}_{h\hat{\mathcal{S}}}(W \hat{\times} X, K(E))$ is functorial in H^*X ; more over it is a representable functor in $H^*X \in \mathcal{K}_H$. Let then $(- : H^*W)$ denote the left adjoint of the functor $H^*W \otimes -$ in \mathcal{K}_H , then the canonical map $(H^*K(E) : H^*W) \rightarrow H^*\mathbf{map}(W, K(E))$ is an isomorphism.

3) If W is the classifying space of \mathbb{Z}/p then the isomorphisms in 1) and 2) generalize to isomorphisms

$$\mathrm{Hom}_{h\hat{\mathcal{S}}}(\mathbb{B}\mathbb{Z}/p, X) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{K}_H}(H^*X, H^*\mathbb{B}\mathbb{Z}/p)$$

and

$$(\mathbf{H}^* X : \mathbf{H}^* \mathbf{BZ}/p) \xrightarrow{\sim} \mathbf{H}^* \mathbf{map}(\mathbf{BZ}/p, X)$$

for all (fibrant) profinite space Y (cf [LA], [Mo]).

The purpose of this work is to provide analogues of the statements 1) and 2) for the continuous MU-cohomology. Motivation for this comes from some analogue of point 3) :

3') Let X be a 1-connected space whose ordinary homology is a free abelian group of finite type in each dimension then the map $[\mathbf{BS}^1, \hat{X}] \rightarrow \text{Hom}_{\mathcal{K}_K}(\mathbf{K}^* X, \mathbf{K}^* \mathbf{BS}^1)$ is a bijection, where \hat{X} denotes the p -completion of X , \mathcal{K}_K the category of p -adic lambda-rings and $\mathbf{K}^*(-)$ the p -completed \mathbf{K} -theory. The proof uses an MU-unstable resolution of the space X ([DL]).

Note that this map is also a bijection when X is the classifying space of a connected compact Lie group, even if its ordinary homology has torsion ([NS]).

We end this introduction by pointing out some difficulties that arise when using MU-cohomology instead of mod p cohomology.

Let \mathcal{M} denote the category of MU*-modules and let M be in \mathcal{M} . We should not expect to find a space $\mathbf{K}(M)$ and a bijection $\text{Hom}_{\mathbf{hS}}(X, \mathbf{K}(M)) \simeq \text{Hom}_{\mathcal{M}}(M, \text{MU}^* X)$ if the functor $X \mapsto \text{Hom}_{\mathcal{M}}(M, \text{MU}^* X)$ does not satisfy the exactness requirement for the Brown representability theorem. But if M is a free MU*-module on a graded set S , we can define $\mathbf{K}(M)$ as the product $\prod_{s \in S} \hat{\text{MU}}_{|s|}$ where $|s|$ refers to the degree of $s \in S$.

For the same reason that MU*-modules may have torsion, the continuous MU-cohomology of a product $X \times Y$ is not in general functorial in the continuous MU*-cohomology of Y . We will have instead a spectral sequence whose E^2 -term is functorial in $\text{MU}^* X$ and $\text{MU}^* Y$ and which is strongly convergent if X and Y are finite dimensional.

Finally if X and Y are ordinary spaces whose ordinary homology is a free abelian group of finite type in each dimension then the ordinary MU-cohomology of the product $X \times Y$ is the completion of the tensor product of the MU-cohomologies of X and Y with respect to the skeleton filtration, so some filtration on the MU-cohomology of a space is needed when the space is not finite dimensional.

2. Generalized Eilenberg - Mac Lane spaces and unstable algebra structure

We state the analogue of 1) for MU by defining the free unstable algebra on a graded set instead of an MU*-module. Let gr-Set denote the category of \mathbb{Z} -graded sets.

For S in gr-Set , we define $\mathbf{K}(S)$ as the product $\prod_{s \in S} \hat{\text{MU}}_{|s|}$ (where $|s|$ is the degree of s). The profinite space $\mathbf{K}(S)$ comes with a natural map $S \rightarrow \hat{\text{MU}}^* \mathbf{K}(S)$, which assigns to an element $s \in S$ the projection on the factor indexed by s : $\prod_{t \in S} \text{MU}_{|t|} \rightarrow \text{MU}_{|s|}$. This map induces for any profinite space X a bijection $\text{Hom}_{\mathbf{hS}}(X, \mathbf{K}(S)) \rightarrow \text{Hom}_{\text{gr-Set}}(S, \hat{\text{MU}}^* X)$. The functor $G : S \mapsto \hat{\text{MU}}^* \mathbf{K}(S)$ gets from this adjointness the structure of a monad on gr-Set . We denote by \mathcal{K}_{MU} the category of G -algebras of gr-Set and call its elements the MU-unstable algebras.

Examples.

- Let S be a graded set; the natural transformation $G \circ G \rightarrow G$ makes $G(S)$ a G -algebra. The map from S to $G(S)$ induces a bijection $\text{Hom}_{\mathcal{K}_{\text{MU}}}(G(S), N) \rightarrow \text{Hom}_{\text{gr-Set}}(S, N)$ for any G -algebra N so that $G(S)$ is the free unstable algebra on S .
- Let X be a profinite space, then the map $X \rightarrow \mathbf{K}(\hat{\text{MU}}^* X)$, adjoint to the identity of $\hat{\text{MU}}^* X$, induces a map $G(\hat{\text{MU}}^* X) \rightarrow \hat{\text{MU}}^* X$ which makes $\hat{\text{MU}}^* X$ an MU-unstable algebra.

Let M be a G -algebra. From the monad structure on G we get a simplicial G -algebra

$$\cdots G^2(M) \rightrightarrows G(M),$$

image in MU -cohomology of a cosimplicial diagram in $\text{h}\hat{\mathcal{S}}$, and a morphism from it to the constant simplicial G -algebra M which is canonically a homotopy equivalence between simplicial graded sets. In particular the diagram $G^2(M) \rightrightarrows G(M) \rightarrow M$ is canonically a split coequalizer in gr-Set . Any natural structure on free G -algebras coming from a monad then induces a similar structure on M .

Our aim is to describe such a structure which is abelian and which gives the link between the continuous MU -cohomology of a profinite space and its continuous mod p cohomology. This is made possible by the fact that any free MU -unstable algebra is the continuous MU -cohomology of some ‘‘torsion free’’ profinite space for which we have a simple universal coefficient formula.

3. Additive structure on the continuous MU -cohomology of a torsion free profinite space

We choose a decreasing filtration $(f^n \text{MU}^*)$ of the ring MU^* such that $f^1 \text{MU}^*$ is the kernel of the map $\text{MU}^* \rightarrow \mathbb{Z}/p$ and $f^n \text{MU}^*/f^{n+1} \text{MU}^*$ is a finite graded \mathbb{F}_p -vector space. It induces a decreasing filtration on any MU^* -module M which we call the coefficient filtration. We let \mathcal{M}_f denote the category of MU^* -modules M with a decreasing filtration $(f^n M)_{n \in \mathbb{N}}$ less thin than the coefficient filtration, and $\hat{\mathcal{L}}$ the full subcategory of \mathcal{M}_f whose objects are those isomorphic to the completion for the coefficient filtration of some free MU^* -module, given with the limit filtration.

We will say that a space is torsion free if its continuous p -adic cohomology is torsion free.

PROPOSITION 3.1. *Let X be a profinite space of finite dimension (i.e. equal to some of its skeleton), then X is torsion free if and only if the continuous MU -cohomology of X with the coefficient filtration is in $\hat{\mathcal{L}}$. If so, the map $\hat{\text{MU}}^* X/f^1 \rightarrow \text{H}^* X$, induced by the orientation $\text{MU} \rightarrow \text{HZ}/p$, is an isomorphism.*

The proof uses an *ad hoc* Atiyah Hirzebruch spectral sequence.

Let X be a profinite space and let $\text{Sk}_s X$ denote its s -skeleton for any $s \geq 0$. We define the skeleton filtration $F_X^s \hat{\text{MU}}^* X$ of $\hat{\text{MU}}^* X$ has the kernel of the map $\hat{\text{MU}}^* X \rightarrow \hat{\text{MU}}^* \text{Sk}_s X$ and the skeleton closure of the coefficient filtration of $\hat{\text{MU}}^* X$ as the filtration given by $f^n \hat{\text{MU}}^* X = \bigcap_s (f^n \hat{\text{MU}}^* X + F_X^s \hat{\text{MU}}^* X)$. We obtain the following proposition by a limit argument :

PROPOSITION 3.2. *Let X be a torsion free profinite space, then the continuous MU -cohomology of X with the skeleton closure of the coefficient filtration is in $\hat{\mathcal{L}}$ and the map $\hat{\text{MU}}^* X/f^1 \rightarrow \text{H}^* X$ is an isomorphism.*

For M in $\hat{\mathcal{L}}$ and n an integer $F^n M$ denotes the sub-object of M generated by the elements of degree greater or equal to n , then the decreasing sequence $(F^n M)$ is a complete filtration of M in $\hat{\mathcal{L}}$. Let M and N be in $\hat{\mathcal{L}}$; we define the tensor product $M \hat{\otimes} N$ of M and N as the complete filtered MU^* -module given by $(M \hat{\otimes} N)/f^n = \lim_s ((M/F^s)/f^n \otimes_{\text{MU}^*} (N/F^s)/f^n)$, then $M \hat{\otimes} N$ is in $\hat{\mathcal{L}}$. For torsion free profinite spaces X and Y , the map $\hat{\text{MU}}^* X \otimes_{\text{MU}^*} \hat{\text{MU}}^* Y \rightarrow \hat{\text{MU}}^* X \times Y$ induces a map $\hat{\text{MU}}^* X \hat{\otimes} \hat{\text{MU}}^* Y \rightarrow \hat{\text{MU}}^* X \times Y$ which is an isomorphism if X and Y are torsion free.

Example. For any graded set S the profinite space $\text{K}(S)$ is torsion free by results of Wilson ([WI]) so the map $G(S)/f^1 \rightarrow \text{H}^* \text{K}(S)$ is an isomorphism.

4. Additive structure in the general case, free resolutions

For any graded set S , we define $\hat{L}(S)$ as the completion for the coefficient filtration of the free MU^* -module with basis S , equipped with the limit filtration. The filtered MU^* -module $\hat{L}(S)$ comes with a map $S \rightarrow \hat{L}(S)$ which induces a bijection $\text{Hom}_{\mathcal{M}_f}(\hat{L}(S), N) \rightarrow \text{Hom}_{\text{gr-Set}}(S, N)$ for any complete object N of \mathcal{M}_f ; so we again obtain a monad structure on $\hat{L} : \text{gr-Set} \rightarrow \text{gr-Set}$. We let $\hat{\mathcal{M}}$ denote the category of \hat{L} -algebras of gr-Set .

Example. Let M be a G -algebra, then the natural free \hat{L} -algebra structure on $G^2(M)$ and $G(M)$ induces a natural \hat{L} -algebra structure on M .

It turns out that the category $\hat{\mathcal{M}}$ is an abelian category. The objects of $\hat{\mathcal{L}}$ are exactly the projective objects of $\hat{\mathcal{M}}$. From the monad structure on \hat{L} , every \hat{L} -algebra M has a canonical free resolution $\hat{L}_*(M) \rightarrow M$.

For M and N in $\hat{\mathcal{M}}$, we define their tensor product as the coequalizer of the diagram $\hat{L}^2(M) \hat{\otimes} \hat{L}^2(N) \rightrightarrows \hat{L}(M) \hat{\otimes} \hat{L}(N)$, where $\hat{L}^2(M) \rightrightarrows \hat{L}(M) \rightarrow M$ and $\hat{L}^2(N) \rightrightarrows \hat{L}(N) \rightarrow N$ are the canonical free presentations of M and N . Note that if M and N are unstable algebras then $M \hat{\otimes} N$ is naturally an unstable algebra and as such the sum of M and N in \mathcal{K}_{MU} . We define the graded \hat{L} -algebra $\text{Tor}_*^{\hat{L}}(M, N)$ as the homology of the complex $\hat{L}_*(M) \hat{\otimes} N$, where $\hat{L}_*(M)$ is the canonical free resolution of M .

Example. For any \hat{L} -algebra M and any integer n , the quotient M/f^n inherits from M an \hat{L} -algebra structure. The natural map $M \hat{\otimes} (\text{MU}^*/f^n) \rightarrow M/f^n$ is an isomorphism in $\hat{\mathcal{M}}$.

PROPOSITION 4.1.

- (a) Let M be an \hat{L} -algebra, $M_* \rightarrow M$ a free resolution of M and n a non negative integer, then the cokernel of the morphism $M_{n+1} \rightarrow M_n$ is in $\hat{\mathcal{L}}$ if and only if the \hat{L} -algebra $\text{Tor}_{n+1}^{\hat{L}}(M, \text{MU}^*/f^1)$ is null.
- (b) Let M_* be a non negatively graded complex of $\hat{\mathcal{L}}$ and let M be the cokernel of the morphism $M_1 \rightarrow M_0$ in $\hat{\mathcal{M}}$. Then M is in $\hat{\mathcal{L}}$ and $M_* \rightarrow M$ is a free resolution of M if and only if the homology of the complex M_*/f^1 is concentrated in degree 0.

By definition the \hat{L} -algebra $\text{Tor}_1^{\hat{L}}(M, N)$ is null if M is in $\hat{\mathcal{L}}$. We need some finiteness assumptions to prove that it is also null if N is in $\hat{\mathcal{L}}$, that is, to prove that the tensor product by N is exact.

PROPOSITION 4.2. Let M and N be \hat{L} -algebras null in high degree and such that N is in $\hat{\mathcal{L}}$; then the \hat{L} -algebra $\text{Tor}_1^{\hat{L}}(M, N)$ is null.

The proof uses some change of ring formula related to the sub polynomial rings of $\hat{\text{MU}}^*$ generated by x_1, \dots, x_n , $n \in \mathbb{N}$. We also obtain from a syzygy theorem :

PROPOSITION 4.3. Let M be an \hat{L} -algebra and $M_1 \rightarrow M_0 \rightarrow M$ a free presentation of M . Suppose that M/f^1 and M/f^0 are concentrated in a finite range of degrees then M admits a free resolution of finite length.

Let X be a profinite space; a free resolution of X is a sequence of maps $C_n \rightarrow X_n \rightarrow C_{n+1}$, $n \geq 0$, between profinite spaces such that C_0 is X , the profinite space X_n is torsion free and the sequence $C_n \rightarrow X_n \rightarrow C_{n+1}$ is a cofiber sequence inducing an exact sequence $0 \rightarrow \hat{\text{MU}}^* C_{n+1} \rightarrow \hat{\text{MU}}^* X_n \rightarrow \hat{\text{MU}}^* C_n \rightarrow 0$ for all n . Such a resolution of X gives rise to a free resolution $\dots \rightarrow \hat{\text{MU}}^* X_1 \rightarrow \hat{\text{MU}}^* X_0 \rightarrow \hat{\text{MU}}^* X$ of $\hat{\text{MU}}^* X$. Every profinite space X admits a canonical free resolution defined inductively by $C_0 = X$, $X_n = \text{K}(\hat{\text{MU}}^* C_n)$ and $C_{n+1} = \text{Cofiber}(C_n \rightarrow \text{K}(\hat{\text{MU}}^* C_n))$. Moreover if X is finite dimensional then X admits a free resolution by finite dimensional profinite spaces : it suffices to replace inductively $\text{K}(\hat{\text{MU}}^* C_n)$ by its k -th skeleton in the construction above, where k is the dimension of C_n . We say that a free resolution (C_n, X_n) of X is of finite length if C_n is the point for some n . From the proposition above and the proposition 3.1 we deduce (cf [AD] or [CS]) :

PROPOSITION 4.4. *Let X be a profinite space of finite dimension, then X admits a free resolution of finite length.*

PROPOSITION 4.5. *Let X and Y be finite dimensional profinite spaces, then there exists a spectral sequence whose E^2 term is given by $E_{s,*}^2 = \text{Tor}_s^{\hat{\mathcal{L}}}(\hat{\text{M}}\text{U}^*X, \hat{\text{M}}\text{U}^*Y)$ and which strongly converges to $\hat{\text{M}}\text{U}^*X \times Y$.*

PROPOSITION 4.6. *Let X be a finite dimensional profinite space, then there exists a spectral sequence whose E^2 term is given by $E_{s,*}^2 = \text{Tor}_s^{\hat{\mathcal{L}}}(\hat{\text{M}}\text{U}^*X, \text{MU}^*/f^1)$ and which strongly converges to H^*X*

Both spectral sequence are formed by using a free resolution of X and by considering the resulting exact couple $\hat{\text{M}}\text{U}^*C_{n+1} \times Y \rightarrow \hat{\text{M}}\text{U}^*X_n \times Y \rightarrow \hat{\text{M}}\text{U}^*C_n \times Y$ for the first one, $\tilde{\text{H}}^*C_{n+1} \rightarrow \tilde{\text{H}}^*X_n \rightarrow \tilde{\text{H}}^*C_n$ for the second one (cf [AD] for the case of finite CW-complexes or spectra).

We use some limit argument to approach the $\hat{\mathcal{L}}$ -algebra $\text{Tor}^{\hat{\mathcal{L}}}(M, N)$ when M or N are $\hat{\mathcal{L}}$ -algebras not necessarily bounded above. We obtain :

PROPOSITION 4.7. *Let L be in $\hat{\mathcal{L}}$ such that L/f^1 is degree-wise finite, then :*

- (a) *The $\hat{\mathcal{L}}$ -algebra $\text{Tor}_1^{\hat{\mathcal{L}}}(M, L)$ is null for all $M \in \hat{\mathcal{M}}$.*
- (b) *Let M_s be a tower of $\hat{\mathcal{L}}$ -algebras then the map $M_\infty \hat{\otimes} L \rightarrow \lim_s (M_s \hat{\otimes} L)$ and $(\lim_s^1 M_s) \hat{\otimes} L \rightarrow \lim_s^1 (M_s \hat{\otimes} L)$ are isomorphisms.*

We can generalize the previous proposition, replacing L by an $\hat{\mathcal{L}}$ -algebra M such that M admits a free resolution $M_* \rightarrow M$ with M_k/f^1 degree-wise finite for all k .

Applications.

- (a) Let X and Y be profinite spaces such that X is torsion free and H^*X is degree-wise finite, then the map $\hat{\text{M}}\text{U}^*X \hat{\otimes} \hat{\text{M}}\text{U}^*Y \rightarrow \hat{\text{M}}\text{U}^*X \times Y$ is an isomorphism.
- (b) Let n be an integer and X a profinite space. Then

$$\text{Tor}_k^{\hat{\mathcal{L}}}(\hat{\text{M}}\text{U}^*X, \text{MU}^*/f^n) \rightarrow \lim_s \text{Tor}_k^{\hat{\mathcal{L}}}(\hat{\text{M}}\text{U}^*X/F_X^s, \text{MU}^*/f^n)$$

is an isomorphism for all k if $\lim_s^1 \text{Tor}_k^{\hat{\mathcal{L}}}(\hat{\text{M}}\text{U}^*X/F_X^s, \text{MU}^*/f^n)$ is null for all k , in particular if X is the profinite completion of some space.

- (c) We take $M = \hat{\text{M}}\text{U}^*\text{B}\mathbb{Z}/p^n$ for some n . The $\hat{\mathcal{L}}$ -algebra $\hat{\text{M}}\text{U}^*\text{B}\mathbb{Z}/p^n$ admits a free resolution of length 1. We let N_s be a tower of $\hat{\mathcal{L}}$ -algebras such that each N_s has a free presentation $N_{s,1} \rightarrow N_{s,0} \rightarrow N_s$ with $N_{s,1}/f^1$ and $N_{s,0}/f^1$ concentrated in a finite range of degrees. We can prove that $\text{Tor}_1^{\hat{\mathcal{L}}}(\hat{\text{M}}\text{U}^*\text{B}\mathbb{Z}/p^n, N_s)$ is null so we obtain an exact sequence $0 \rightarrow \hat{\text{M}}\text{U}^*\text{B}\mathbb{Z}/p^n \hat{\otimes} N_\infty \rightarrow \lim_s (\hat{\text{M}}\text{U}^*\text{B}\mathbb{Z}/p^n \hat{\otimes} N_s) \rightarrow \text{Tor}_1^{\hat{\mathcal{L}}}(\hat{\text{M}}\text{U}^*\text{B}\mathbb{Z}/p^n, \lim_s^1 N_s) \rightarrow 0$. If X is the profinite completion of some space then taking $N_s = \hat{\text{M}}\text{U}^*X_s$ where X_s is the s -th skeleton of X , we obtain an isomorphism $\hat{\text{M}}\text{U}^*\text{B}\mathbb{Z}/p^n \hat{\otimes} \hat{\text{M}}\text{U}^*X \rightarrow \hat{\text{M}}\text{U}^*\text{B}\mathbb{Z}/p^n \times X$. In general we are unable to prove such a formula (see also [RWY]).
- (d) Let X be a space such that the $\hat{\mathcal{L}}$ -algebra $\lim_s^1 \text{Tor}_k^{\hat{\mathcal{L}}}(\hat{\text{M}}\text{U}^*X/F_X^s, \text{MU}^*/f^1)$ is null for all k then the filtration of $\hat{\text{M}}\text{U}^*X$ coming from a free resolution of X coincides with the skeleton closure of the coefficient filtration.

5. Back to MU-unstable algebra

From the isomorphism $G(S)/f^1 \simeq H^*K(S)$ we get an unstable algebra structure over the mod p Steenrod algebra on $G(S)/f^1$. As any MU-unstable algebra M is the split coequaliser in gr-Set of the diagram $G^2(M) \rightrightarrows G(M)$, we obtain an unstable algebra structure on M/f^1 for all M .

PROPOSITION 5.1.

- (a) For all $M \in \mathcal{K}_{\text{MU}}$, the morphism $\text{Hom}_{\mathcal{K}_{\text{MU}}}(M, \hat{\text{MU}}^*) \rightarrow \text{Hom}_{\mathcal{K}_{\text{H}}}(M/f^1, \mathbb{Z}/p)$ is a bijection.
- (b) For all profinite space X , the morphism $\text{Hom}_{\mathcal{K}_{\text{MU}}}(\hat{\text{MU}}^*X, \hat{\text{MU}}^*) \rightarrow \text{Hom}_{\mathcal{K}_{\text{H}}}(\text{H}^*X, \mathbb{Z}/p) \simeq \pi_0 X$ is a bijection.

Let P be a free $\hat{\mathcal{L}}$ -algebra such that P/f^1 is degree-wise finite, then the functor $\hat{\mathcal{M}} \rightarrow \hat{\mathcal{M}}, M \mapsto P \hat{\otimes} M$ has a left adjoint $(- : P)_{\hat{\mathcal{M}}}$. If S and S' are graded sets with S' concentrated in a finite number of degrees and finite in each degree, then $(\hat{\mathcal{L}}(S) : \hat{\mathcal{L}}(S'))_{\hat{\mathcal{M}}}$ is isomorphic to $\hat{\mathcal{L}}(S \times S'^-)$, where S'^- is the graded set obtained from S' by reversing the sign of the degrees.

By adjunction we obtain :

PROPOSITION 5.2.

- (a) Let P be in $\mathcal{K}_{\text{MU}} \cap \hat{\mathcal{L}}$ such that P/f^1 is degree-wise finite, then the functor $\mathcal{K}_{\text{MU}} \rightarrow \mathcal{K}_{\text{MU}}, M \mapsto P \hat{\otimes} M$ has a left adjoint $(- : P)_{\mathcal{K}_{\text{MU}}}$.
- (b) Let S be a graded set and W a simplicial space whose p -adic cohomology is torsion free and finite dimensional in each degree, then the counit $W \hat{\times} \mathbf{map}(W, K(S)) \rightarrow K(S)$ induces an isomorphism

$$(G(S) : \hat{\text{MU}}^*W)_{\mathcal{K}_{\text{MU}}} \rightarrow \hat{\text{MU}}^* \mathbf{map}(W, K(S)) .$$

This is the analogue of statement (2) for continuous MU-cohomology.

6. Application to the continuous MU-cohomology of mapping spaces from the classifying space of \mathbb{Z}/p^n or S^1 .

Recall that the map $\text{colim}_n B\mathbb{Z}/p^n \rightarrow BS^1$ is a mod p homology equivalence. For M in \mathcal{K}_{MU} and n in $\mathbb{N} \cup \{\infty\}$, we define the unstable algebra $T_n(M)$ as the coequalizer of the diagram

$$\hat{\text{MU}}^* \mathbf{map}(B\mathbb{Z}/p^n, K(G(M))) \rightrightarrows \hat{\text{MU}}^* \mathbf{map}(B\mathbb{Z}/p^n, K(M))$$

induced by the diagram $G^2(M) \rightrightarrows G(M)$ in \mathcal{K}_{MU} . The last proposition states that T_∞ is left adjoint to $\hat{\text{MU}}^* BS^1 \hat{\otimes} -$ in \mathcal{K}_{MU} . A weaker form of this adjunction can be shown for T_n for finite n . At least we still have a map $T_n \hat{\text{MU}}^* X \rightarrow \hat{\text{MU}}^* \mathbf{map}(B\mathbb{Z}/p^n, X)$ for any profinite space X and a bijection $\text{Hom}_{\mathcal{K}_{\text{MU}}}(T_n M, \hat{\text{MU}}^*) \simeq \text{Hom}_{\mathcal{K}_{\text{MU}}}(M, \hat{\text{MU}}^* B\mathbb{Z}/p^n)$.

Let X be a torsion free profinite space. The augmented simplicial G -algebra $G_\bullet(\hat{\text{MU}}^* X) \rightarrow \hat{\text{MU}}^* X$ is the image in continuous MU-cohomology of a coaugmented cosimplicial profinite space $X \rightarrow \mathbf{R}^\bullet X$. As $\hat{\text{MU}}^* X$ is in $\hat{\mathcal{L}}$ and $G_\bullet(\hat{\text{MU}}^* X) \rightarrow \hat{\text{MU}}^* X$ is acyclic in $\hat{\mathcal{M}}$, the augmented simplicial \mathbb{F}_p -vector space $H^* \mathbf{R}^\bullet X \rightarrow H^* X$ is acyclic. We use the following key-input from [DL] :

PROPOSITION 6.1. *The augmented simplicial \mathbb{F}_p -vector space*

$$H^* \mathbf{map}(B\mathbb{Z}/p^n, \mathbf{R}^\bullet X) \rightarrow H^* \mathbf{map}(B\mathbb{Z}/p^n, X)$$

is acyclic

As the morphism $T_n \hat{\text{MU}}^* \mathbf{R}^k X \rightarrow \hat{\text{MU}}^* \mathbf{map}(B\mathbb{Z}/p^n, \mathbf{R}^k X)$ is iso for all k , we obtain

THEOREM 6.2. *Let X be a torsion free profinite space, then the morphism*

$$T_n \hat{\text{MU}}^* X \rightarrow \hat{\text{MU}}^* \mathbf{map}(B\mathbb{Z}/p^n, X)$$

is an isomorphism.

For general X we have the following proposition :

PROPOSITION 6.3. *Let $X \rightarrow X_0 \rightarrow C$ be a cofiber sequence in $\hat{\mathcal{S}}$, then the induced sequence*

$$\mathbf{map}(\mathbb{B}\mathbb{Z}/p^n, X) \rightarrow \mathbf{map}(\mathbb{B}\mathbb{Z}/p^n, X_0) \rightarrow \mathbf{map}(\mathbb{B}\mathbb{Z}/p^n, C)$$

is a cofiber sequence.

We can apply this to the beginning of a free resolution $X = C_0 \rightarrow X_0 \rightarrow C_1$ of X . As the associated long exact sequence in MU-cohomology is short, we obtain a coequalizer diagram $\hat{\mathbf{M}}\mathbf{U}^*C_1 \hat{\otimes} \hat{\mathbf{M}}\mathbf{U}^*X_0 \rightrightarrows \hat{\mathbf{M}}\mathbf{U}^*X_0 \rightarrow \hat{\mathbf{M}}\mathbf{U}^*X$. Using the fact that T_n commutes by construction to reflexive coequalizers, we obtain :

PROPOSITION 6.4. *Suppose that the morphism $(T_n\hat{\mathbf{M}}\mathbf{U}^*Y)/f^1 \rightarrow (\hat{\mathbf{M}}\mathbf{U}^*\mathbf{map}(\mathbb{B}\mathbb{Z}/p^n, Y))/f^1$ is epi for $Y = X$ and $Y = C_1$ then $T_n\hat{\mathbf{M}}\mathbf{U}^*X \rightarrow \hat{\mathbf{M}}\mathbf{U}^*\mathbf{map}(\mathbb{B}\mathbb{Z}/p^n, X)$ is an isomorphism*

Example. Let X be a profinite space whose continuous MU-cohomology is null in odd degree, then we can choose X_0 having the same property. If $M \in \mathcal{K}_{\text{MU}}$ is null in odd degree then the same is true for T_nM . Suppose that $\hat{\mathbf{M}}\mathbf{U}^*\mathbf{map}(\mathbb{B}\mathbb{Z}/p^n, Y)$ is null in odd degree for $Y = X$ and $Y = C_1$, then the proposition applies and $T_n\hat{\mathbf{M}}\mathbf{U}^*X \rightarrow \hat{\mathbf{M}}\mathbf{U}^*\mathbf{map}(\mathbb{B}\mathbb{Z}/p^n, X)$ is an isomorphism. This applies to the case of a product of Eilenberg - Mac Lane spaces for mod p -cohomology : their MU-cohomology are null in odd degrees by a result of Ravenel-Wilson-Yagita ([RWY]).

To conclude, recall that we have a bijection $\text{Hom}_{\mathcal{K}_{\text{MU}}}(T_nM, \hat{\mathbf{M}}\mathbf{U}^*) \simeq \text{Hom}_{\mathcal{K}_{\text{MU}}}(M, \hat{\mathbf{M}}\mathbf{U}^*\mathbb{B}\mathbb{Z}/p^n)$. So, letting $[\mathbb{B}\mathbb{Z}/p^n, X]$ denote the homotopy classes of maps from $\mathbb{B}\mathbb{Z}/p^n$ to the underlying space of a profinite space X , we have :

PROPOSITION 6.5. *Let X be a profinite space such that $T_n\hat{\mathbf{M}}\mathbf{U}^*X \rightarrow \hat{\mathbf{M}}\mathbf{U}^*\mathbf{map}(\mathbb{B}\mathbb{Z}/p^n, X)$ is an isomorphism, then the map*

$$[\mathbb{B}\mathbb{Z}/p^n, X] \rightarrow \text{Hom}_{\mathcal{K}_{\text{MU}}}(\hat{\mathbf{M}}\mathbf{U}^*X, \hat{\mathbf{M}}\mathbf{U}^*\mathbb{B}\mathbb{Z}/p^n)$$

is a one to one correspondance.

References

- [Ad] J. F. ADAMS, *Lectures on Generalized Cohomology Theories*, Springer L. N. M., **99**, 1969.
- [CS] P. E. CONNER and L. SMITH, On the complex bordism of finite complexes, *Publ. Math. I. H. E. S.*, **37** (1969), 117-221.
- [DE] F.-X. DEHON, Cobordisme complexe des espaces profinis et foncteur T de Lannes, preprint (2001-03).
- [DL] F.-X. DEHON and J. LANNES, Sur les espaces fonctionnels dont la source est le classifiant d'un groupe de Lie compact commutatif, *Publ. Math. I. H. E. S.*, **89** (1999).
- [LA] J. LANNES, Sur les espaces fonctionnels dont la source est le classifiant d'un p -groupe abélien élémentaire, *Publ. Math. I. H. E. S.*, **75** (1992), 135-244.
- [Mo] F. MOREL, Ensembles profinis simpliciaux et interprétation géométrique du foncteur T, *Bull. Soc. Math. France*, **124** (1996), 347-373.
- [NS] D. NOTBOHM and L. SMITH, Fake Lie groups and maximal tori I, *Math. Ann.*, **288** (1990), 637-661.
- [RWY] D. C. RAVENEL, W. S. WILSON and N. YAGITA, Brown-Peterson cohomology from Morava K -theory, *K-Theory*, **15** (1998), 147-199.
- [Wi] W. S. WILSON, The Ω -spectrum for Brown-Peterson cohomology, Part I, *Comm. Math. Helv.*, **48** (1973), 45-55.

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