BP-cohomology of mapping spaces from the classifying space of a torus to some p-torsion free space

1. Introduction

Let p be a fixed prime number, V a group isomorphic to $(\mathbb{Z}/p)^d$ for some integer d and $\mathrm{B}V$ its classifying space. The exceptional properties of the mod p cohomology of $\mathrm{B}V$, as an unstable module and algebra over the Steenrod algebra, have led to the calculation of the mod p cohomology of the mapping spaces with source $\mathrm{B}V$ as the image by a functor T_V of the mod p cohomology of the target ([La2]). This determination is linked to the solution of the Sullivan conjecture concerning the space of homotopy fixed points for some action of a finite p-group ([Mi]). It is also an essential component of the homotopy theory of Lie groups initiated by Dwyer et Wilkerson ([DW]). We will see that we can deduce from the theory of the T functor (and from its equivariant version) a similar theory relative to the BP-cohomology of the mapping spaces with source the classifying space of a torus when the target is a space whose cohomology with p-adic coefficients is torsion free (p-torsion free space).

We start with recalling the theory of the T_V functor.

Let $\mathcal{K}$ be the category of unstable algebras over the Steenrod algebra (the mod p cohomology of a space X, which we denote by H^*X, is a typical object of $\mathcal{K}$). It is a subcategory of the abelian category $\mathcal{E}$ of graded $\mathbb{F}_p$-vector spaces. J. Lannes defines the functor T_V as the left adjoint of the functor $\mathcal{K} \to \mathcal{K}$, $N \mapsto \mathrm{H}^*\mathrm{B}V \otimes N$. The exceptional properties of the unstable algebra $\mathrm{H}^*\mathrm{B}V$ come down to the following statement:

Proposition 1.1. *Let $M' \to M$ and $M \to M''$ be formal linear combinations of morphisms of $\mathcal{K}$ such that the induced sequence $M' \to M \to M''$ of $\mathcal{E}$ is exact; then so is the sequence $\mathrm{T}_V M' \to \mathrm{T}_V M \to \mathrm{T}_V M''$.*

We will say that the functor T_V on $\mathcal{K}$ is exact.

Let X be a space (an object of the category $\mathcal{S}$ of simplicial sets, fibrant in what follows). We let $\mathbf{hom}(\mathrm{B}V, X)$ denote the space of maps from $\mathrm{B}V$ to X. The counit of the adjunction in $\mathcal{S}$, $\mathrm{B}V \times \mathbf{hom}(\mathrm{B}V, X) \to X$ induces a morphism $\mathrm{H}^*X \to \mathrm{H}^*\mathrm{B}V \otimes \mathrm{H}^*\mathbf{hom}(\mathrm{B}V, X)$ thus a morphism $\mathrm{T}_V\mathrm{H}^*X \to \mathrm{H}^*\mathbf{hom}(\mathrm{B}V, X)$. The properties of T_V are so that this last morphism is very often an isomorphism ([La2]); we have for example:

Theorem 1.2 ([La2], [DS]). *Let X be a fibrant degree wise finite simplicial set, having for every choice of the base point a finite number of non trivial homotopy groups, each of them being a finite p-group (we say that X is a finite p-space); then the natural morphism*

$$\mathrm{T}_V\mathrm{H}^*X \to \mathrm{H}^*\mathbf{hom}(\mathrm{B}V, X)$$

is an isomorphism.

We note that if H^*X is degree wise finite dimensional but not $\mathrm{T}_V\mathrm{H}^*X$ then $\mathrm{T}_V\mathrm{H}^*X$ and $\mathrm{H}^*\mathbf{hom}(\mathrm{B}V, X)$ do not have the same cardinal. Nevertheless, theorem 1.2 can be generalized to filtered limits of finite p-spaces (pro-p-spaces) and their limit cohomology. Thus replacing the space X by its pro-p-completion $\widehat{X}(-)$, one interprets $\mathrm{T}_V\mathrm{H}^*X$ as the limit mod p cohomology of the pro-p-space $\mathbf{hom}(\mathrm{B}V, \widehat{X}(-))$ ([Mo]). We will do it implicitly in what follows.

We have an equivariant version of the T_V-functor theory ([La2]): Suppose that X is given with some action of V. We denote by $X^{\mathrm{h}V}$ the space of V-equivariant maps from the universal covering $\mathrm{E}V$ of $\mathrm{B}V$ to X, which we call the homotopy fixed points space of X under the action of V. Let

A part of the results stated below corresponds to joint work with Jean Lannes and has led to a common preprint ([DL]).

$X_{\mathrm{h}V}$ denote the Borel construction $\mathrm{E}V \times_V X$; it is the total space of a bundle over $\mathrm{B}V$. The space $X^{\mathrm{h}V}$ is then the fiber at the identity of $\mathrm{B}V$ of the fibration $\mathbf{hom}(\mathrm{B}V, X_{\mathrm{h}V}) \to \mathbf{hom}(\mathrm{B}V, \mathrm{B}V)$; this can be rephrased in cohomology:

Let $\mathrm{H}V$ denote the mod p cohomology of $\mathrm{B}V$ and $\mathrm{H}V - \mathcal{K}$ the category of unstable algebras under $\mathrm{H}V$ (the mod p cohomology of the space $X_{\mathrm{h}V}$ is a typical object of $\mathrm{H}V - \mathcal{K}$). The functor $\mathcal{K} \to \mathrm{H}V - \mathcal{K}$, $M \mapsto \mathrm{H}V \otimes M$ has a left adjoint, denoted by Fix_V, which is exact (we give sense to the exactness of Fix_V by considering $\mathrm{H}V - \mathcal{K}$ as a sub category of $\mathcal{E}$). Again we have a natural morphism

$$\mathrm{Fix}_V \mathrm{H}^* X_{\mathrm{h}V} \to \mathrm{H}^* X^{\mathrm{h}V}$$

which is an isomorphism under the same hypotheses as for a trivial action.

We denote by T, Fix and H the functors T_V, Fix_V and the unstable algebra $\mathrm{H}V$ for $V = \mathbb{Z}/p$.

2. Properties of the mod p cohomology of the mapping spaces with source the classifying space of a finite abelian p-group

Let π be a finite abelian p-group and κ a subgroup of index p. The classifying space $\mathrm{B}\pi$ appears as the homotopy quotient of the classifying space $\mathrm{B}\kappa$ for the action of $\pi/\kappa = \mathbb{Z}/p$ and the mapping space $\mathbf{hom}(\mathrm{B}\pi, X)$ as the homotopy fixed points space of $\mathbf{hom}(\mathrm{B}\kappa, X)$ for the action of $\mathbb{Z}/p$ at the source. The mod p cohomology of the space $\mathbf{hom}(\mathrm{B}\pi, X)$ is thus the image by the functor Fix of the mod p cohomology of the Borel construction $(\mathbf{hom}(\mathrm{B}\kappa, X))_{\mathrm{h}\mathbb{Z}/p}$, which is the target of the Serre spectral sequence with E_2-term $\mathrm{E}_2^{s,t} = \mathrm{H}^s(\mathrm{B}\mathbb{Z}/p, \mathrm{H}^t\mathbf{hom}(\mathrm{B}\kappa, X))$ (with untwisted coefficients). We give two examples:

- Let $p = 2$ and $X = \mathrm{B}\mathbb{Z}/2$. The space $\mathbf{hom}(\mathrm{B}\mathbb{Z}/2, \mathrm{B}\mathbb{Z}/2)$ is the disjoint union of two copies of $\mathrm{B}\mathbb{Z}/2$. The first comes with a trivial action of $\mathbb{Z}/2$ thus has a homotopy quotient equal to the product $\mathrm{B}\mathbb{Z}/2 \times \mathrm{B}\mathbb{Z}/2$ with projection on the first factor; thus the fiber at the identity of the fibration $\mathbf{hom}(\mathrm{B}\mathbb{Z}/2, (\mathrm{B}\mathbb{Z}/2)_{\mathrm{h}\mathbb{Z}/2}) \to \mathbf{hom}(\mathrm{B}\mathbb{Z}/2, \mathrm{B}\mathbb{Z}/2)$ identifies with the space $\mathbf{hom}(\mathrm{B}\mathbb{Z}/2, \mathrm{B}\mathbb{Z}/2)$.

  The second copy comes with $\mathbb{Z}/2$ acting non trivially. Its homotopy quotient is the space $\mathrm{B}\mathbb{Z}/4$ over $\mathrm{B}\mathbb{Z}/2$. The fiber at the identity of the fibration $\mathbf{hom}(\mathrm{B}\mathbb{Z}/2, \mathrm{B}\mathbb{Z}/4) \to \mathbf{hom}(\mathrm{B}\mathbb{Z}/2, \mathrm{B}\mathbb{Z}/2)$ is empty.

- Let $p = 2$ and $X = \mathrm{BSU}(2)$. The space $\mathbf{hom}(\mathrm{B}\mathbb{Z}/2, \mathrm{BSU}(2))$ is the disjoint union of two copies of $\mathrm{BSU}(2)$. The first comes with the trivial action of $\mathbb{Z}/2$ thus the corresponding fiber at the identity identifies with the space $\mathbf{hom}(\mathrm{B}\mathbb{Z}/2, \mathrm{BSU}(2))$. The second comes with $\mathbb{Z}/2$ acting non trivially and its homotopy quotient is the classifying space of the subgroup of $\mathrm{U}(2)$ consisting of the matrices of determinant ± 1. The corresponding fiber at the identity identifies with the classifying space of S^1. Note that the Serre spectral sequences converging to the cohomology of both Borel constructions $\mathrm{BSU}(2)_{\mathrm{h}\mathbb{Z}/2}$ are in even bidegrees so collapse so are identical. We deduce that the cohomologies of the Borel constructions are isomorphic as graded $\mathbb{F}_2$-algebras under H, but not as unstable algebras over the Steenrod algebra: The first is the polynomial algebra on one generator u of degree 1 and one generator v of degree 4 with the action of the Steenrod algebra given by the relation $\mathrm{Sq}^2 v = 0$. The second is the same polynomial algebra with the action of the Steenrod algebra given by the relation $\mathrm{Sq}^2 v = u^2 v$. They are distinguished by the functor Fix.

When the mod p cohomology of the space X is in even degrees, the mod p cohomology of the space $\mathbf{hom}(\mathrm{B}\pi, X)$ has exactness properties which are consequences of the proposition bellow:

Proposition 2.1 ([DL] section 2 and 3). *Let π be a finite abelian p-group, κ a subgroup of index p of π and X a space. We suppose that the mod p cohomology of the pro - mapping space $\mathbf{hom}(\mathrm{B}\kappa, \widehat{X}(-))$ is concentrated in even degrees; then:*

(a) *The Serre spectral sequence converging to the mod p cohomology of the Borel construction* $(\mathbf{hom}(\mathrm{B}\kappa, \widehat{X}(-)))_{\mathrm{h}\mathbb{Z}/p}$ *collapses.*

(b) *The mod p cohomology of the pro-space* $\mathbf{hom}(\mathrm{B}\pi, \widehat{X}(-))$ *is concentrated in even degrees.*

The statement (b) comes from the fact that the functor T preserves the property of being concentrated in even degrees or its equivariant analogue.

We deduce an immediate generalization of the propositions 4.5 or 4.6 of [DL]:

Proposition 2.2. *Let π be a finite abelian p-group or a torus and X', X et X'' be spaces whose mod p cohomology is concentrated in even degrees. Let $X' \to X$ and $X \to X''$ be formal combinations of maps between spaces such that the induced sequence $\mathrm{H}^*X'' \to \mathrm{H}^*X \to \mathrm{H}^*X'$ of $\mathcal{E}$ is exact. Then so is the induced sequence*

$$\mathrm{H}^*\mathbf{hom}(\mathrm{B}\pi, \widehat{X''}(-)) \to \mathrm{H}^*\mathbf{hom}(\mathrm{B}\pi, \widehat{X}(-)) \to \mathrm{H}^*\mathbf{hom}(\mathrm{B}\pi, \widehat{X'}(-)) \ .$$

We want to apply this proposition to an unstable MU-resolution of X. Suppose that X is a pointed connected space whose ordinary homology is null in odd degree and a free finite dimensional $\mathbb{Z}$-module in each even degree. The results of Wilson ([Wi]) imply that the space $\mathrm{R}(X) = \Omega^\infty(\mathrm{MU} \wedge X)$ has the same properties. The functor $X \mapsto \mathrm{R}(X)$ brings the structure of a triple on the homotopy category of pointed connected spaces and allows one to associate to X a cosimplicial MU-resolution. We denote by $\mathrm{H}\mathbb{Z}/p$ the spectrum which represents the mod p cohomology. The standard orientation $\mathrm{MU} \to \mathrm{H}\mathbb{Z}/p$ makes the augmented simplicial unstable algebra $\mathrm{H}^*\mathrm{R}^\bullet(X) \to \mathrm{H}^*X$ a resolution of H^*X in $\mathcal{E}$. The previous proposition implies immediately ([DL] section 6):

Proposition 2.3. *Let π be a finite abelian p-group or a torus and X be a pointed connected space whose ordinary homology is null in odd degree and free and finite dimensional in each even degree. Then the augmented simplicial unstable algebra*

$$\mathrm{H}^*\mathbf{hom}(\mathrm{B}\pi, (\mathrm{R}^\bullet(X))\widehat{\ }(-)) \to \mathrm{H}^*\mathbf{hom}(\mathrm{B}\pi, \widehat{X}(-))$$

is a resolution of $\mathrm{H}^*\mathbf{hom}(\mathrm{B}\pi, \widehat{X}(-))$ *in $\mathcal{E}$.*

We remark that the mod p cohomology of the mapping spaces $\mathbf{hom}(\mathrm{B}\pi, \mathrm{R}^n(X))$ is never degree wise finite dimensional when π is non trivial, X differs from the point and n is positive. We are thus led to replace the spaces $\mathrm{R}^n(X)$ by their pro-p-completion.

We refer to [DL, section 7] or to [De] for a generalization of the propositions 2.1, 2.2 and 2.3 to p-torsion free spaces.

Proposition 2.3 indicates that, under the hypotheses made on X, the mod p cohomology of the pro - mapping space $\mathbf{hom}(\mathrm{B}\pi, \widehat{X}(-))$ is a functor of the MU-cohomology of X. We limit ourselves to the case $\pi = \mathrm{S}^1$. We first have to describe the structure of the MU-cohomology of X. As we complete everything at p, we consider the BP-cohomology of X instead of its MU-cohomology.

3. BP-cohomology of p-torsion free (profinite) spaces

We recall that the ring spectrum BP is a direct factor of the spectrum MU localized at p. The coefficient ring BP^* is a $\mathbb{Z}_{(p)}$-polynomial algebra on generators v_n of degree $-2(p^n-1)$, $n \geq 1$.

Suppose that X is a space whose ordinary homology is a free finite dimensional $\mathbb{Z}$-module in each degree, null in large degree. The BP-cohomology of X is then a free finite dimensional BP^*-module. We choose a decreasing filtration of BP^* by some BP^*-modules $\mathrm{f}^n\mathrm{BP}^*$ whose intersection is null, such that $\mathrm{f}^1\mathrm{BP}^*$ is the kernel of the canonical morphism $\mathrm{BP}^* \to (\mathrm{H}\mathbb{Z}/p)^*$

and such that $\mathrm{f}^n\mathrm{BP}^*/\mathrm{f}^{n+1}\mathrm{BP}^*$ is a finite dimensional (graded) $\mathrm{BP}^*/\mathrm{f}^1\mathrm{BP}^*$-vector space. The BP-cohomology of X inherits a filtration defined by $\mathrm{BP}^*X/\mathrm{f}^n\mathrm{BP}^*X = \mathrm{BP}^*/\mathrm{f}^n\mathrm{BP}^* \otimes_{\mathrm{BP}^*} \mathrm{BP}^*X$. The quotient $\mathrm{BP}^*X/\ \mathrm{f}^1\mathrm{BP}^*X$ identifies with the graded $\mathbb{F}_p$-vector space H^*X and the completion of BP^*X for its filtration identifies with the p-completed BP-cohomology of X.

Suppose now that X is a space whose ordinary homology is a free finite dimensional $\mathbb{Z}$-module in each degree. The BP-cohomology of X is then the limit of the BP-cohomologies of the skeletons X_s of X. We provide it with the limit filtration of the filtrations on the BP^*X_s, *i.e.* $\mathrm{BP}^*X/\mathrm{f}^n\mathrm{BP}^*X$ is the limit of the $\mathrm{BP}^*X_s/\mathrm{f}^n\mathrm{BP}^*X_s$. Again the quotient $\mathrm{BP}^*X/\mathrm{f}^1\mathrm{BP}^*X$ identifies with the mod p cohomology of X and the limit of the tower $(\mathrm{BP}^*X/\mathrm{f}^n\mathrm{BP}^*X)$ identifies with the p-completed BP-cohomology of X, $\widehat{\mathrm{BP}}^*X$. On the other hand there exists a graded set S such that the tower given in each degree n by the $\mathrm{BP}^*/\mathrm{f}^n\mathrm{BP}^*$-module $\mathrm{BP}^*X/\mathrm{f}^n\mathrm{BP}^*X$ is isomorphic to the tower given in each degree n by the free $\mathrm{BP}^*/\mathrm{f}^n\mathrm{BP}^*$-module with basis S. We denote by $\mathcal{M}\mathrm{a}$ the abelian category of towers indexed by n of $\mathrm{BP}^*/\mathrm{f}^n\mathrm{BP}^*$-modules and by $\mathcal{M}$ the full sub-category of $\mathcal{M}\mathrm{a}$ formed by the free towers on a graded set.

More generally, let $X(-)$ be a profinite space ([Mo]). For all integer v, the limit cohomology of $X(-)$ with coefficients in the group $\mathbb{Z}/p^v$ is the group $\mathrm{colim}_i\mathrm{H}^*(X(i),\mathbb{Z}/p^v)$. We say that $X(-)$ is p-torsion free if the mod p reduction $\mathbb{Z}/p^v \to \mathbb{Z}/p$ induces for all v a surjection $\mathrm{H}^*(X(-),\mathbb{Z}/p^v) \to \mathrm{H}^*(X(-),\mathbb{Z}/p)$. We denote by $\widehat{\mathrm{BP}}_n(-)$ the pro-p-completion of the n-th term BP_n of the Ω-spectrum BP for all $n > 0$. We define the limit BP-cohomology of $X(-)$ in each degree $n > 0$ as the group $\mathrm{Hom}_{\mathrm{h}\widehat{\mathcal{S}}}(X(-), \widehat{\mathrm{BP}}_n(-))$, where $\mathrm{h}\widehat{\mathcal{S}}$ stands for the homotopy category of profinite spaces. We extend this definition to integers $n \leq 0$ with the help of the suspension of profinite spaces. Then, if $X(-)$ is p-torsion free, the limit BP-cohomology of $X(-)$ is naturally an object of $\mathcal{M}$ and the quotient $\mathrm{BP}^*X(-)/\mathrm{f}^1\mathrm{BP}^*X(-)$ identifies with the limit mod p cohomology $\mathrm{H}^*X(-)$ ([De]).

As for the mod p cohomology of a (profinite) space, the BP-cohomology of a p-torsion free (profinite) space has an unstable algebra structure which translates the fact that BP-cohomology is represented in the category $\mathrm{h}\widehat{\mathcal{S}}$:

Proposition - Definition 3.1 ([De]).

(a) *Let $X(-)$ be a p-torsion free profinite space and M an object of $\mathcal{M}$; then there exists a p-torsion free profinite space* $\mathrm{K}(M)$ *functorial in M and a bijection*

$$\mathrm{Hom}_{\mathrm{h}\widehat{\mathcal{S}}}(X(-), \mathrm{K}(M)) \simeq \mathrm{Hom}_{\mathcal{M}}(M, \mathrm{BP}^*X(-))$$

natural in $X(-)$ and M.

(b) *The functor $M \mapsto \mathrm{BP}^*\mathrm{K}(M)$ defines a triple* G *on $\mathcal{M}$. We denote $\mathcal{M}(\mathrm{G})$ the category of* G*-algebras of $\mathcal{M}$ which we also call unstable* BP*-algebras.*

(c) *Likewise the functor $X(-) \mapsto \mathrm{K}(\mathrm{BP}^*X(-))$ defines a triple* R *on $\mathrm{h}\widehat{\mathcal{S}}$.*

The triple structure of G on $\mathcal{M}$ consists of natural transformations $\mathrm{G} \circ \mathrm{G} \to \mathrm{G}$ and $\mathrm{Id} \to \mathrm{G}$ satisfying the axioms of an associative monoid. A G-algebra consists of an object M of $\mathcal{M}$ and of a morphism $\mathrm{G}(M) \to M$ compatible with the triple structure of G ([Ma]).

The limit BP-cohomology of the pro-p-completion $\widehat{X}(-)$ of X identifies canonically with the p-completed BP-cohomology of X. When the profinite space $\widehat{X}(-)$ is p-torsion free, which is equivalent to the cohomology of X with p-adic coefficients being torsion free, this identification makes the p-completed BP-cohomology of X an object of $\mathcal{M}(\mathrm{G})$.

Comparison with the mod p cohomology

Let $X(-)$ be a profinite space and E an object of $\mathcal{E}$; then there exists a profinite space $\mathrm{K}(E)$ functorial in E and a bijection

$$\mathrm{Hom}_{\mathrm{h}\widehat{\mathcal{S}}}(X(-), \mathrm{K}(E)) \simeq \mathrm{Hom}_{\mathcal{E}}(E, \mathrm{H}^*X(-))$$

natural in $X(-)$ and E. The triple G resulting from this adjunction leads to the structure of G-algebra of $\mathcal{E}$ which coincides with the structure of unstable algebra over the Steenrod algebra; that is to say the category $\mathcal{K}$ and $\mathcal{E}(G)$ are the same.

The main difference for our purpose between the categories $\mathcal{M}(\mathrm{G})$ and $\mathcal{E}(\mathrm{G})$ comes from the fact that coequalizers in $\mathcal{M}\mathrm{a}$ of objects of $\mathcal{M}$ are not necessarily coequalizers in $\mathcal{M}$. The following proposition summarizes the link between unstable BP-algebras and algebras over the Steenrod algebra:

Proposition 3.2.

(a) *Let M be in $\mathcal{M}(\mathrm{G})$; then M/f^1M is naturally a* G*-algebra of $\mathcal{E}$ and the natural map*

$$\mathrm{Hom}_{\mathcal{M}(\mathrm{G})}(M, \widehat{\mathrm{BP}}^*) \to \mathrm{Hom}_{\mathcal{E}(\mathrm{G})}(M/\mathrm{f}^1M, (\mathrm{H}\mathbb{Z}/p)^*)$$

is a bijection.

(b) *Let $M' \to M \to M''$ be a complex of objects of $\mathcal{M}$; then $M' \to M \to M''$ is an exact sequence of $\mathcal{M}\mathrm{a}$ if and only if the induced sequence $M'/\mathrm{f}^1M' \to M/\mathrm{f}^1M \to M''/\mathrm{f}^1M''$ is an exact sequence of $\mathcal{E}$.*

4. BP-cohomology of mapping spaces from the classifying space of S^1 to some p-torsion free space

We are now in position to make a T functor theory for the BP-cohomology of the classifying space of S^1. For M and N in $\mathcal{M}$, we denote by $M\widehat{\otimes}N$ the term to term tensor product of the towers M and N, *i.e.* given by

$$(M\widehat{\otimes}N)/\mathrm{f}^n(M\widehat{\otimes}N) = M/\mathrm{f}^nM \otimes_{\mathrm{BP}^*} N/\mathrm{f}^nN \ .$$

If M and N are in $\mathcal{M}(\mathrm{G})$ then $M\widehat{\otimes}N$ comes naturally as the sum in $\mathcal{M}(\mathrm{G})$ of M and N. Let X be a space whose cohomology with p-adic coefficients is torsion free. The profinite space $\mathbf{hom}(\mathrm{BS}^1, \widehat{X}(-))$ is p-torsion free. The adjunction in $\widehat{\mathcal{S}}$ defining it can be rephrased in $\mathcal{M}(\mathrm{G})$:

Theorem 4.1 ([De]).

(a) *The functor $\mathcal{M}(\mathrm{G}) \to \mathcal{M}(\mathrm{G})$, $M \mapsto \widehat{\mathrm{BP}}^*\mathrm{BS}^1\widehat{\otimes}M$ has a left adjoint T_∞.*

(b) *The functor T_∞ is "exact" ($\mathcal{M}(\mathrm{G})$ is viewed as a subcategory of $\mathcal{M}\mathrm{a}$).*

(c) *Let X be a space whose cohomology with p-adic coefficients is torsion free; then the natural morphism*

$$\mathrm{T}_\infty\widehat{\mathrm{BP}}^*X \to \mathrm{BP}^*\mathbf{hom}(\mathrm{BS}^1, \widehat{X}(-))$$

is an isomorphism.

Sketch of proof

Let M be in $\mathcal{M}(\mathrm{G})$. When M is of the form $\mathrm{G}(N)$, $\mathrm{T}_\infty M$ is the image by G of the division of N by $\widehat{\mathrm{BP}}^*\mathrm{BS}^1$ in $\mathcal{M}$ (see [La1] for the case of the functor T) and identifies canonically with the BP-cohomology of the profinite space $\mathbf{hom}(\mathrm{BS}^1, \mathrm{K}(N))$. For general M, The triple G allows one to associate to M an augmented simplicial unstable BP-algebra $\mathrm{G}^\bullet(M) \to M$ which is a resolution of M in $\mathcal{M}\mathrm{a}$. The simplicial unstable algebra $\mathrm{G}^\bullet(M)$ is induced in BP-cohomology by a cosimplicial diagram in $\mathrm{h}\widehat{\mathcal{S}}$ of (p-torsion free) profinite spaces $\mathrm{R}^\bullet(M)$. The proposition 3.2 indicates that the simplicial unstable $\mathrm{H}\mathbb{Z}/p$-algebra $\mathrm{H}^*\mathrm{R}^\bullet(M)$ has, as a simplicial object of $\mathcal{E}$, a homology concentrated in degree 0 (which identifies with M/f^1M). Thus so is the simplicial unstable algebra $\mathrm{H}^*\mathbf{hom}(\mathrm{BS}^1, \mathrm{R}^\bullet(M))$ by proposition 2.2 and also the simplicial unstable BP-algebra

$\mathrm{BP}^*\mathbf{hom}(\mathrm{BS}^1, \mathrm{R}^\bullet(M))$ viewed as a simplicial object of $\mathcal{M}$a. We deduce that the coequalizer in $\mathcal{M}$a of the diagram $\mathrm{T}_\infty\mathrm{G}^2(M) \rightrightarrows \mathrm{T}_\infty\mathrm{G}(M)$ is in $\mathcal{M}$ thus in $\mathcal{M}(\mathrm{G})$ and we define $\mathrm{T}_\infty M$ as this coequalizer.

We check by the same way that if M is the BP-cohomology of a p-torsion free (fibrant) profinite space $X(-)$, $\mathrm{T}_\infty M$ identifies with the BP-cohomology of the profinite space $\mathbf{hom}(\mathrm{BS}^1, X(-))$.

Let again M be in $\mathcal{M}(\mathrm{G})$ and let n be a positive integer. The simplicial unstable algebra $\mathrm{H}^*\mathbf{hom}(\mathrm{B}\mathbb{Z}/p^n, \mathrm{R}^\bullet(M))$ has, as a simplicial object of $\mathcal{E}$, a homology concentrated in degree 0. We denote by $\overline{\mathrm{T}}_n M$ this homology group; it has a natural structure of unstable algebra. We show by induction on n, using proposition 2.1, that the functor $\overline{\mathrm{T}}_n \colon \mathcal{M}(\mathrm{G}) \to \mathcal{K}$ is exact, which leads to the statement (b).

Statement (c) of the theorem and statement (a) of the proposition 3.2 leads to a "p-completed" version of the proposition 6.7 of [DL], where we denote by $[-,-]$ the set of homotopy classes of maps and $\widehat{X}$ the limit in $\mathcal{S}$ of the underlying diagram of the pro-p-completion of X (which is also its limit in h$\mathcal{S}$, see [Mo]):

Corollary 4.2. *Let X be a space whose cohomology with p-adic coefficients is torsion free; then the natural map*

$$[\mathrm{BS}^1, \widehat{X}] \to \mathrm{Hom}_{\mathcal{M}(\mathrm{G})}(\widehat{\mathrm{BP}}^*X, \widehat{\mathrm{BP}}^*\mathrm{BS}^1)$$

is a bijection.

On the other hand we deduce from statement (b) of the theorem the:

Corollary 4.3. *Let $M \to M'$ be a morphism of unstable* BP*-algebras which is a monomorphism in $\mathcal{M}$a; then every morphism $M \to \widehat{\mathrm{BP}}^*\mathrm{BS}^1$ in $\mathcal{M}(\mathrm{G})$ extends to a morphism $M' \to \widehat{\mathrm{BP}}^*\mathrm{BS}^1$.*

All the results stated above for the group S^1 extends immediately by induction to the case of a torus.

One example

Let G be a connected compact Lie group whose ordinary homology is torsion free. Its classifying space $\mathrm{B}G$ is then a p-torsion free space and the embedding of a maximal torus T in G induces an injection $\mathrm{H}^*\mathrm{B}G \to \mathrm{H}^*\mathrm{B}T$. Let T' be a torus. Corollary 4.3 indicates that every morphism $\widehat{\mathrm{BP}}^*\mathrm{B}G \to \widehat{\mathrm{BP}}^*\mathrm{B}T'$ extends to a morphism $\widehat{\mathrm{BP}}^*\mathrm{B}T \to \widehat{\mathrm{BP}}^*\mathrm{B}T'$. We can formulate this result in K-theory with the help of Hattori-Stong theorem ([DL] section 6) and recover the theorem 4.1 of [Wilk] in this special case.

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