Mean-Field Games

Lectures at the Imperial College London

1st Lecture: Forward-backward SDEs

François Delarue (Nice – J.-A. Dieudonné)

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Joint works with R. Carmona; J.F. Chassagneux and D. Crisan; P. Cardaliaguet

Part I. Motivation

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a. General philosophy

Basic purpose

- Interacting particles/players
  - controlled particles / players / financial agents / neurons in mean-field interaction
  - particles have dynamical / non-static states ⇝ stochastic differential equation
  - mean-field ⇝ interaction of symmetric type / interaction with the whole population / no privileged interaction with some particles
- Associate cost functional with each player
  - find equilibria w.r.t. cost functionals
  - shape of the equilibria for a large population?

Different notions of equilibria

- Players may decide of the strategy on their own
  - no way that the particles minimize their own costs simultaneously
  - find a consensus inside the population?
  - no interest for a particle to leave the consensus
  - notion of Nash equilibrium in a game
- Center of decision may decide of the strategies for all the players
  - "Chief says what the companies will do"
  - minimize the global cost to the collectivity
  - different notion of equilibrium $\rightsquigarrow$ seek a minimizer
- Both cases $\rightsquigarrow$ asymptotic equilibria?

Asymptotic formulation

- Paradigm
  - mean-field/symmetry $\rightsquigarrow$ law of large numbers
propagation of chaos
  - reduce the asymptotic analysis to one typical player with interaction with a theoretical distribution of the population?
  - description of asymptotic equilibria in terms of

    player's private state
theoretical distribution of the population
  - decrease the complexity to solve asymptotic formulation first
- Program
  - Existence of asymptotic equilibria? Uniqueness? Shape?
  - Use asymptotic equilibria as quasi-equilibria in finite-player-systems
  - Prove convergence of equilibria in finite-player-systems

Different kinds of asymptotic formulation

- Asymptotic formulation of Nash equilibria
  - Mean-field games theory!

Lasry-Lions (2006)
Huang-Caines-Malhamé (2006)
Cardaliaguet, Achdou, Gomes, Porreta (PDE)
Bensoussan, Carmona, D., Kolokoltsov, Lacker, Yam (Probability)

  - PDE or probabilistic analysis $\rightsquigarrow$ both meet with the concept of master equation (last lecture)

- Central center of decision
  - optimal control of McKean-Vlasov stochastic differential equations
  - PDE point of view $\leftrightsquigarrow$ HJB equations in infinite dimension

Part I. Motivation

b. An introductory example

Purpose of the modeling

- Model for inter-bank borrowing and lending
  - model introduced by Fouque and Ichiba (2013), Carmona, Fouque and Sun (2014)
  - N banks $i = 1, \ldots, N$ and a central bank
- Interaction between banks
  - bank i may lend to / borrow from bank j
- Control
  - banks control lending to / borrowing from central bank
- Cost for lending / borrowing fixed by the regulator
  - Nash equilibria?
  - N large?

Mean-field interaction between the banks

- (Log)-monetary reserve of bank $i \rightsquigarrow X_t^i$
  - $\begin{array}{l}\text{borrow from}\\ \text{lend to}\end{array}$ bank j if $\begin{array}{l}X_t^j > X_t^i\\ X_t^j < X_t^i\end{array}$
  - rate a of $\begin{array}{l}\text{borrowing}\\ \text{lending}\end{array}$ $\rightsquigarrow a\,(X_t^j - X_t^i) \rightsquigarrow (a/N)\,(X_t^j - X_t^i)$

$$dX_t^i = a\Big(\underbrace{\frac{1}{N}\sum_{j=1}^{N} X_t^j}_{\bar{X}_t^N} - X_t^i\Big)\,dt + \ldots$$

  - $\dfrac{dX_t^i}{dt} \rightsquigarrow$ instantaneous rate of lending/borrowing
- $\bar{X}_N$ = empirical mean
  - mean-field interaction with reverting to the empirical mean
  - $\bar{X}_t^N$ mean state of the population (may be used for systemic risk)

Controlled stochastic dynamics

- Controlled rate of $\begin{array}{l}\text{borrowing from}\\ \text{lending to}\end{array}$ central bank

$$dX_t^i = a\,(\bar{X}_t^N - X_t^i)\,dt + \alpha_t^i\,dt + \dots$$

  - α_t^i $\begin{array}{l}\text{negative} \rightsquigarrow \text{lending}\\ \text{positive} \rightsquigarrow \text{borrowing}\end{array}$

- Noisy perturbations

$$dX_t^i = a\,(\bar{X}_t^N - X_t^i)\,dt + \alpha_t^i\,dt + \sigma\,d\tilde{W}_t^i$$

  - $\tilde{W}_t^i = \rho \underbrace{W_t^i}_{\text{independent}} + \sqrt{1-\rho^2}\underbrace{W_t^0}_{\text{common}}$

  - $((W_t^0)_t, (W_t^1)_t, \dots, (W_t^N)_t)$: indep. Brownian motions $\rightsquigarrow$ symmetric structure (original paper $\rightsquigarrow$ role of W^0 in systemic risk)

Cost functional

- Cost functional $\Rightarrow$ penalize high borrowing/lending activities

$$J^i(\alpha^1,\dots,\alpha^N) = \mathbb{E}\Big[g(X_T^i - \bar{X}_T^N) + \int_0^T f(\underbrace{X_t^i - \bar{X}_t^N}_{\text{global rate}}, \alpha_t^i)dt\Big]$$

  - depends on all the controls through $\bar{X}_t^N$
  - penalize high borrowing from/lending to central bank
  - incite borrowing/lending $\Rightarrow$ easier to borrow from central bank if low reserve

- Linear-quadratic functionals

$$f(x,m,\alpha) = \alpha^2 + \epsilon^2(m-x)^2 - 2q\epsilon\alpha(m-x)$$
$$g(x,m) = c^2(x-m)^2$$

  - $\bar{X}_t^N > X_t^i \Rightarrow$ lower cost if $\alpha_t > 0$
  - $q \in (0,1) \rightsquigarrow$ fixed by the regulator

Ansatz for the asymptotic Nash equilibrium

- Simplify $\rightsquigarrow$ no common noise W^0
  - law of large numbers $\Rightarrow \bar{X}_t^N$ stabilizes around some deterministic m_t
  - m_t should stand for the theoretical mean of any bank at the equilibrium
- Focus on one bank only with dynamics

$$dX_t = a\,(m_t - X_t)\,dt + \alpha_t\,dt + \sigma\,dW_t$$

  - the bank does not see the others anymore $\rightsquigarrow$ cost functional

$$J(\alpha) = \mathbb{E}\Big[g(X_T, m_T) + \int_0^T f(X_t, m_t, \alpha_t)dt\Big]$$

  - minimize!
  - consensus means that optimal path has m_t as mean at time t

Case of a common noise

- If common noise W^0
  - law of large numbers becomes conditional law of large numbers

$$U^0,\ U^1,\dots,\ U^N,\ \dots \text{ i.i.d. r.v.'s (with values in } \mathbb{R})$$
$$\phi : \mathbb{R} \to \mathbb{R} \text{ bounded Borel measurable function}$$

$$\Rightarrow \mathbb{P}(\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N \phi(U^i, U^0) = \mathbb{E}[\phi(U^1, U^0)|U^0]) = 1$$

  - m_t should stand for the conditional mean of any bank at the equilibrium given the realization of W^0

- Focus on one bank only with dynamics

$$dX_t = a\,(m_t - X_t)\,dt + \alpha_t\,dt + \sigma\sqrt{1-\rho^2}dW_t + \sigma\rho dW_t^0$$

  - consensus means that optimal path has m_t as conditional mean at time t

Part I. Motivation

c. Toolbox for the solution

Program

- Solve standard optimization problem (1st Lecture)

  ◦ parameterized by some input (state of the population at equilibrium)

  ◦ consider the case when the input may be random (think of the case when $\rho \neq 0$ in the previous example)

  ◦ need for a nice characterization of the optimal state in terms of the input

  may use PDE arguments (HJB equation)
  may use probabilistic arguments (FBSDEs)

  ◦ finite horizon only!

- Solve a fixed point condition (2nd and 3rd Lectures)

  ◦ in order to characterize the asymptotic equilibrium

  ◦ fixed point condition of the McKean-Vlasov type $\rightsquigarrow$ need to revisit the theory of McKean-Vlasov SDEs (2nd Lecture)

Part II. Stochastic optimal control & FBSDEs

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a. Stochastic optimal control problem

Basic controlled dynamics

- Controlled stochastic dynamics

$$dX_t = b(X_t, \mu_t, \alpha_t)dt + \sigma(X_t, \mu_t, \alpha_t)dW_t, \quad t \in [0, T]$$

$(W_t)_{0 \le t \le T}$ B.M. with values in $\mathbb{R}^d$ on $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$

- ◦ may consider time dependent coefficients
- ◦ $X_t \rightsquigarrow$ state in $\mathbb{R}^d$ of the particle / agent at time t
- ◦ $(\mu_t)_{0 \le t \le T}$ denotes some environment (think of it as a the mean of a probability distribution or as the probability distribution itself)

may take value in a general Polish space $\mathcal{X}$
example: $\mathcal{X}$ = space of probability measures on $\mathbb{R}^d$ (see Lecture 2)

- ◦ $(\alpha_t)_{0 \le t \le T}$ denotes control process

  with values in $A \subset \mathbb{R}^k$, closed and convex

  and $\mathbb{F}$-progressively measurable

Basic controlled dynamics

- Controlled stochastic dynamics

$$dX_t = b(X_t, t, \alpha_t)dt + \sigma(X_t, t, \alpha_t)dW_t, \quad t \in [0, T]$$

$(W_t)_{0 \le t \le T}$ B.M. with values in $\mathbb{R}^d$ on $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$

  - may consider time dependent coefficients
  - $X_t \rightsquigarrow$ state in $\mathbb{R}^d$ of the particle agent at time t

may take value in a general Polish space $\mathcal{X}$
example: $\mathcal{X}$ = space of probability measures on $\mathbb{R}^d$ (see Lecture 2)

  - $(\alpha_t)_{0 \le t \le T}$ denotes control process
    with values in $A \subset \mathbb{R}^k$, closed and convex
    and $\mathbb{F}$-progressively measurable

Controlled dynamics in a random environment

- Allow $(\mu_t)_{0\le t\le T}$ to be random
  - Think of the case $\rho \neq 0$ in the introductory example
- Controlled stochastic dynamics

$$dX_t = b(X_t, \mu_t, \alpha_t)dt + \sigma(X_t, \mu_t, \alpha_t)dW_t + \sigma^0(X_t, \mu_t, \alpha_t)dW_t^0$$

$(W_t)_{0\le t\le T}$ B.M. with values in $\mathbb{R}^d$ on $(\Omega^1, \mathbb{F}^1 = (\mathcal{F}_t^1)_{0\le t\le T}, \mathbb{P}^1)$

$(W_t^0)_{0\le t\le T}$ B.M. with values in $\mathbb{R}^d$ on $(\Omega^0, \mathbb{F}^0 = (\mathcal{F}_t^0)_{0\le t\le T}, \mathbb{P}^0)$

- Equation set on $(\Omega, \mathbb{F}, \mathbb{P}) = (\Omega^0 \times \Omega^1, \mathbb{F}^0 \otimes \mathbb{F}^1, \mathbb{P}^0 \otimes \mathbb{P}^1)$
  - $(X_t)_{0\le t\le T}$ defined on Ω
  - $(\alpha_t)_{0\le t\le T}$ defined on Ω
  - $(\mu_t)_{0\le t\le T}$ defined on Ω^0

continuous and adapted to $\mathbb{F}^0$

Typical set of assumptions

- Coefficients
  - $(\sigma, \sigma^0)(x, \mu, \alpha) = (\sigma, \sigma^0)(x, \mu) \rightsquigarrow$ uncontrolled volatility
  - growth

$$
\begin{aligned}
&|b(x, \mu, \alpha)| + |\sigma(x, \mu, \alpha)| + |\sigma^0(x, \mu, \alpha)| \\
&\leq C(1 + |x| + d_{\mathcal{X}}(0_{\mathcal{X}}, \mu) + |\alpha|)
\end{aligned}
$$

where $d_{\mathcal{X}}$ distance on $\mathcal{X}$ and $0_{\mathcal{X}}$ some element in $\mathcal{X}$

  - b, σ and σ^0 Lipschitz in all the variables (too strong for the first lecture but useful for the sequel)
- Assumptions on the processes
  - control processes satisfy $\mathbb{E} \int_0^T |\alpha_t|^2 dt < \infty$
  - inputs (if random) satisfy $\mathbb{E}[\sup_{0 \leq t \leq T} (d_{\mathcal{X}}(0_{\mathcal{X}}, \mu_t))^2] < \infty$

Typical set of assumptions

- Coefficients
  - $(\sigma, \sigma^0)(x, \quad, \alpha) = (\sigma, \sigma^0)(x, \quad) \rightsquigarrow$ uncontrolled volatility
  - growth

$$|b(x,t,\alpha)| + |\sigma(x,t,\alpha)| + |\sigma^0(x,t,\alpha)| \\ \leq C(1 + |x| \qquad\qquad + |\alpha|)$$

  - b, σ and σ^0 Lipschitz in all the variables

- Assumptions on the processes
  - control processes satisfy $\mathbb{E} \int_0^T |\alpha_t|^2 dt < \infty$

Cost functional

- Environment $(\mu_t)_{0\le t\le T}$ is fixed throughout the analysis
  - fix as well initial condition $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$
- Given an admissible control $\boldsymbol{\alpha} = (\alpha_t)_{0\le t\le T}$
  - Unique solution $(X_t^{\alpha})_{0\le t\le T}$ with $X_0^{\alpha} = \xi$
- Cost functional of the type

$$J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t)dt\Big]$$

  - $f \rightsquigarrow$ running cost, $g \rightsquigarrow$ terminal cost
  - assume f and g continuous and at most of quadratic growth

$$|f(x,\mu,\alpha)| + |g(x,\mu)| \le C\big(1 + |x| + d_{\mathcal{X}}(0_{\mathcal{X}}, \mu) + |\alpha|\big)^2$$

- Goal is to minimize $J(\alpha)$!

Cost functional

- fix as well initial condition $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$

- Given an admissible control $\boldsymbol{\alpha} = (\alpha_t)_{0 \le t \le T}$
  - Unique solution $(X_t^\alpha)_{0 \le t \le T}$ with $X_0^\alpha = \xi$
- Cost functional of the type

$$J(\alpha) = \mathbb{E}\Big[g(X_T, T) + \int_0^T f(X_t, t, \alpha_t) dt\Big]$$

  - $f \rightsquigarrow$ running cost, $g \rightsquigarrow$ terminal cost
  - assume f and g continuous and at most of quadratic growth

$$|f(x, t, \alpha)| + |g(x, T)| \le C(1 + |x| \qquad + |\alpha|)^2$$

- Goal is to minimize $J(\alpha)$!

Part II. Stochastic optimal control & FBSDEs

b. Interpretation of the value function

Backward representation of the cost functional

- Simplified assumption
  - Assume coefficients and σ^{-1} bounded in (x,μ) and A bounded
  - Case when $\mathbb{F}$ is generated by $(W_t, W_t^0)_{0\le t\le T}$ and ξ, μ_0 deterministic

- Dynamical version of the cost $\leadsto$ backward representation of the remaining cost functional
  - for a given $\boldsymbol{\alpha} = (\alpha_t)_{0\le t\le T}$

$$Y_t^{\boldsymbol{\alpha}} = \mathbb{E}\Big[g(X_T^{\boldsymbol{\alpha}}, \mu_T) + \int_t^T f(X_s^{\boldsymbol{\alpha}}, \mu_s, \alpha_s)ds \,\Big|\, \mathcal{F}_t\Big]$$

  - martingale representation of $g(X_T^{\boldsymbol{\alpha}}, \mu_T) + \int_0^T f(X_s^{\boldsymbol{\alpha}}, \alpha_s, \mu_s)ds$

$$Y_t^{\boldsymbol{\alpha}} = g(X_T^{\boldsymbol{\alpha}}, \mu_T) + \int_t^T f(X_s^{\boldsymbol{\alpha}}, \mu_s, \alpha_s)ds - \int_t^T Z_s^{\boldsymbol{\alpha}} dW_s - \int_t^T Z_s^{0,\boldsymbol{\alpha}} dW_s^0$$

  - where $\mathbb{E}\Big[\int_0^T \big(|Z_s^{\boldsymbol{\alpha}}|^2 + |Z_s^{0,\boldsymbol{\alpha}}|^2\big)ds\Big] < \infty$ (Z as a row vector)

A first backward SDE

- Consider another $\alpha^\star$ (candidate for optimality)
  - mimic equation of Y^α but turn it into a backward SDE

$$
\begin{aligned}
Y_t^{\alpha^\star} &= g(X_T^\alpha, \mu_T) + \int_t^T f(X_s^\alpha, \mu_s, \alpha_s^\star)\, ds \\
&\quad + \int_t^T Z_s^{\alpha^\star} \sigma^{-1}(X_s^\alpha, \mu_s)\Big(\underbrace{b(X_s^\alpha, \mu_s, \alpha_s^\star) - b(X_s^\alpha, \mu_s, \alpha_s)}_{\text{kind of default}}\Big)\, ds \\
&\quad - \int_t^T Z_s^{\alpha^\star}\, dW_s - \int_t^T Z_s^{0,\alpha^\star}\, dW_s^0
\end{aligned}
$$

  - coefficient is Lipschitz continuous in $Z^{\alpha^\star}$ $\rightsquigarrow$ extension of the martingale representation theorem $\rightsquigarrow$ existence and uniqueness of a solution (Pardoux and Peng, 1990)
  - $(Z_t^{\alpha^\star})_{0\le t\le T}$ and $(Z_t^{0,\alpha^\star})_{0\le t\le T}$ $\mathbb{F}$-progressively measurable with

$$
\mathbb{E}\Big[\int_0^T \Big(|Z_s^{\alpha^\star}|^2 + |Z_s^{0,\alpha^\star}|^2\Big) ds\Big] < \infty
$$

Change of probability measure

- Make use of Girsanov theorem

$$\frac{d\mathbb{P}^\star}{d\mathbb{P}} = \exp\Big(\int_0^T \big(b(X_s^\alpha,\mu_s,\alpha_s^\star) - b(X_s^\alpha,\mu_s,\alpha_s)\big)\,dW_s - \frac{1}{2}\int_0^T \big|b(X_s^\alpha,\mu_s,\alpha_s^\star) - b(X_s^\alpha,\mu_s,\alpha_s)\big|^2 ds\Big)$$

- Let $W_t^\star = W_t - \int_0^t \big(b(X_s^\alpha,\mu_s,\alpha_s^\star) - b(X_s^\alpha,\mu_s,\alpha_s)\big)ds$
  - Under $\mathbb{P}^\star$, $(W_t^\star, W_t^0)_{0\le t\le T}$ $2d$-dimensional B.M. w.r.t $\mathbb{F}$
- Connection with $(X_t^{\alpha^\star})_{0\le t\le T}$

$$dX_t^\alpha = b(X_t^\alpha,\mu_t,\alpha_t^\star)dt + \sigma(X_t^\alpha,\mu_t)dW_t^\star + \sigma^0(X_t^\alpha,\mu_t)dW_t^0$$

and $Y_0^{\alpha^\star} = \mathbb{E}^\star\Big[g(X_T^\alpha,\mu_T) + \int_0^T f(X_t^\alpha,\mu_t,\alpha_t^\star)dt\Big]$

- reminiscent of $(X_t^{\alpha^\star})_{0\le t\le T}$ under $\mathbb{P}$ and $J(\alpha^\star)$ (good point as aim at comparing with $J(\alpha)$)

Hamiltonian

- Compute

$$Y_0^{\alpha^\star} - Y_0^{\alpha} = \mathbb{E}\Big[\int_0^T \Big(H(X_t^\alpha, \mu_t, \alpha_t^\star, Z_t^\alpha \sigma^{-1}(X_t^\alpha, \mu_t)) - H(X_t^\alpha, \mu_t, \alpha_t, Z_t^\alpha \sigma^{-1}(X_t^\alpha, \mu_t))\Big)\, dt\Big]$$

  - $H(x, \mu, \alpha, z) = f(x, \mu, \alpha) + z \cdot b(x, \mu, \alpha)$ called Hamiltonian

- If

$$H(X_t^\alpha, \mu_t, \alpha_t^\star, Z_t^\alpha \sigma^{-1}(X_t^\alpha, \mu_t)) \leq H(X_t^\alpha, \mu_t, \alpha_t, Z_t^\alpha \sigma^{-1}(X_t^\alpha, \mu_t))$$

  - then $Y_0^{\alpha^\star} - Y_0^{\alpha} \leq 0$

- Recall $Y_0^{\alpha^\star} \leftrightsquigarrow J(\alpha^\star)$ (to be specified next) then optimality condition should read as

$$\begin{aligned} \alpha_t^\star &= \text{argmin}_{\alpha \in A} H(X_t^\alpha, \mu_t, \alpha, Z_t^\alpha \sigma^{-1}(X_t^\alpha, \mu_t)) \\ &= \alpha^\star(X_t^\alpha, \mu_t, Z_t^\alpha \sigma^{-1}(X_t^\alpha, \mu_t)) \end{aligned}$$

  - if $\alpha^\star(x, \mu, z) = \text{argmin}_{\alpha \in A} H(x, \mu, \alpha, z)$ uniquely defined

FBSDE for the optimal state

- Dynamics of $(X_t^\alpha)_{0\le t\le T}$ under $\mathbb{P}^\star$

$$dX_t^\alpha = b\Big(X_t^\alpha, \mu_t, \alpha^\star(X_t^\alpha, \mu_t, Z_t^{\alpha^\star}\sigma^{-1}(X_t^\alpha, \mu_t))\Big)\,dt + \sigma(X_t^\alpha, \mu_t)dW_t^\star + \sigma^0(X_t^\alpha, \mu_t)dW_t^0$$

  - coupled with the backward equation for $(Y_t^{\alpha^\star})_{0\le t\le T}$

$$Y_t^{\alpha^\star} = g(X_T^\alpha, \mu_T) + \int_t^T f\left(X_s^\alpha, \mu_s, \alpha^\star(X_s^\alpha, \mu_s, Z_s^{\alpha^\star}\sigma^{-1}(X_s^\alpha, \mu_s))\right) ds - \int_t^T Z_s^{\alpha^\star} dW_s^\star - \int_t^T Z_s^{0,\alpha^\star} dW_s^0$$

- Reformulate the equation under $(\mathbb{P}, (W_t, W_t^0)_{0\le t\le T})$ instead of $(\mathbb{P}^\star, (W_t^\star, W_t^0)_{0\le t\le T})$
  - Claim: the forward process should be the optimal state

Statement

- Assume that, on $(\Omega, \mathbb{F}, \mathbb{P})$, the FBSDE

$$X_t^\star = \xi + \int_0^t \phi(Z_s^\star) b\left(X_s^\star, \mu_s, \alpha^\star(X_s^\star, \mu_s, Z_s^\star \sigma^{-1}(X_t^\alpha, \mu_t))\right) ds + \int_0^t \sigma(X_s^\star, \mu_s) dW_s + \sigma^0(X_s^\star, \mu_s) dW_s^0$$

$$Y_t^\star = g(X_T^\star, \mu_T) + \int_t^T f\left(X_s^\star, \mu_s, \alpha^\star(X_s^\star, \mu_s, Z_s^\star \sigma^{-1}(X_s^\star, \mu_s))\right) ds - \int_t^T Z_s^\star dW_s - \int_t^T Z_s^{0,\star} dW_s^0$$

has a unique solution for any cut-off function ϕ with

- $Z^\star$ bounded by some C (indep. of ϕ)
- $\alpha^\star(x, \mu, z)$ is the unique minimizer of $\alpha \mapsto H(x, \mu, \alpha, z)$

- Then $(X_t^\star)_{0 \le t \le T}$ is the unique optimal path when $\phi(z) = z$ for $|z| \le C$

Sketch of proof

- Given an admissible $\alpha = (\alpha_t)_{0 \le t \le T}$, solve

$$
\begin{aligned}
Y_t^{\alpha^\star} &= g(X_T^\alpha, \mu_T) + \int_t^T f(X_s^\alpha, \mu_s, \alpha_s)\, ds \\
&+ \int_t^T \phi(Z_s^{\alpha^\star}) Z_s^{\alpha^\star} \sigma^{-1}(X_s^\alpha, \mu_s)\Big(b(X_s^\alpha, \mu_s, \alpha_s^\star) - b(X_s^\alpha, \mu_s, \alpha_s)\Big)\, ds \\
&- \int_t^T Z_s^{\alpha^\star}\, dW_s - \int_t^T Z_s^{0,\alpha^\star}\, dW_s^0
\end{aligned}
$$

  - with $\alpha_s^\star = \alpha^\star(X_s^\alpha, \mu_s, Z_s^{\alpha^\star} \sigma^{-1}(X_s^\alpha, \mu_s))$
- Under $(\mathbb{P}^\star, (W_t^\star, W_t^0)_{0 \le t \le T})$, get a solution to the FBSDE
  - Generalization of Yamada-Watanabe $\rightsquigarrow$ weak uniqueness

$$
\mathbb{P}^\star \circ (X_t^\alpha, Y_t^{\alpha^\star}, Z_t^{\alpha^\star})_{0 \le t \le T}^{-1} = \mathbb{P} \circ (X_t^\star, Y_t^\star, Z_t^\star)_{0 \le t \le T}^{-1}
$$

  - $J((\alpha^\star(X_t^\star, \mu_t, Z_t^\star \sigma^{-1}(X_t^\star, \mu_t))_{0 \le t \le T}) = Y_0^{\alpha^\star} \le Y_0^\alpha = J(\alpha)$ (strict)

Extension and complements

- Extension on the same model to the case when A is not bounded
  - Need to localize over the control or use quadratic BSDE
- Extension to the case when $\mathbb{F}$ is larger than the filtration generated by $(\xi, \mu_0, (W_t, W_t^0)_{0 \leq t \leq T})$
  - loose martingale representation theorem

$$\int_t^T Z_s dW_s + \int_t^T Z_s^0 dW_s^0 \rightsquigarrow \int_t^T Z_s dW_s + M_T - M_t$$

  - $(M_t)_{0 \leq t \leq T}$ is a square-integrable martingale orthogonal to $\sigma((W_t)_{0 \leq t \leq T})$
- Scope of application $\rightsquigarrow$ σ invertible and H strictly convex in α
  - f strictly convex in α and b linear in α
- Connection with HJB equation when no common noise $\rightsquigarrow$ next section

Part II. Stochastic optimal control & FBSDEs

c. Stochastic Pontryagin principle

Perturbation in deterministic control

- First order optimality condition when no noise ($\sigma \equiv \sigma^0 \equiv 0$)
  - find $(\alpha_t^\star)_{0 \le t \le T}$ such that

$$\frac{dJ^\varepsilon}{d\varepsilon} \ge 0 \quad \text{with } J^\varepsilon = J\Big((\alpha_t^\star + \varepsilon(\beta_t - \alpha_t^\star))\Big)$$

  - $(\beta_t)_{0 \le t \le T}$ is another A valued control

- Let (formally) $x_t^\star = X_t^{\alpha^\star}$, $\partial x_t^\star = \frac{d}{d\varepsilon} X_t^{\alpha^\star + \varepsilon(\beta - \alpha^\star)}$ and $d = k = 1$

$$\frac{dJ^\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} = \partial_x g(x_T^\star, \mu_T)\partial_x x_T^\star + \int_0^T \big(\partial_x f(x_t^\star, \mu_t, \alpha_t^\star)\partial_x x_t^\star + \partial_\alpha f(x_t^\star, \mu_t, \alpha_t^\star)(\beta_t - \alpha_t^\star)\big)dt$$

  - with $\partial_x x_t^\star = (\partial_x b(x_t^\star, \mu_t, \alpha_t^\star)\partial_x x_t^\star + \partial_\alpha b(x_t^\star, \mu_t, \alpha_t^\star)(\beta_t - \alpha_t^\star))dt$

Deterministic Hamiltonian system

- C^1 path $(y_t)_{0 \le t \le T}$ s.t. $y_T = \partial_x g(x_T^\star, \mu_T) \rightsquigarrow$ integration by parts

$$\frac{dJ^\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} = \int_0^T \left(\dot{y}_t + \partial_x f(x_t^\star, \mu_t, \alpha_t^\star) + y_t \partial_x b(x_t^\star, \mu_t, \alpha_t^\star)\right)\partial_x x_t^\star \, dt$$
$$+ \int_0^T \left(\partial_\alpha f(x_t^\star, \mu_t, \alpha_t^\star) + y_t \partial_\alpha b(x_t^\star, \mu_t, \alpha_t^\star)\right)(\beta_t - \alpha_t^\star) dt$$

  - recognize $\partial_x H(x_t, \mu_t, \alpha_t^\star, y_t)$ and $\partial_\alpha H(x_t, \mu_t, \alpha_t^\star, y_t)$

- Solve $y_t = \partial_x g(x_T^\star, \mu_T) + \int_t^T \partial_x H(x_s^\star, \mu_s, \alpha_s^\star, y_s) ds$

$$\frac{dJ^\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} = \int_0^T \partial_\alpha H(x_t^\star, \mu_t, \alpha_t^\star, y_t)(\beta_t - \alpha_t^\star) dt$$

- $\frac{dJ^\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} \ge 0$ for any $(\beta_t)_{0 \le t \le T}$ if and only if

$$\forall \beta \in A \quad \partial_\alpha H(x_t^\star, \mu_t, \alpha_t^\star, y_t)(\beta - \alpha_t^\star) \ge 0$$
$$\text{with } \dot{x}_t^\star = \partial_y H(x_t^\star, \mu_t, \alpha_t^\star, y_t)$$

Stochastic version

• With $(\alpha_t^\star)_{0\le t\le T}$ candidate for being the optimal control, associate $(Y_t^{\alpha^\star})_{0\le t\le T}$

$$Y_t^{\alpha^\star} = \partial_x g(X_T^{\alpha^\star}, \mu_T) + \int_t^T \partial_x H(X_t^{\alpha^\star}, \mu_t, \alpha_t^\star, Y_t^{\alpha^\star})dt - \int_t^T Z_t^{\alpha^\star} dW_t - \int_t^T Z_t^{0,\alpha^\star} dW_t^0$$

∘ martingale component $\rightsquigarrow$ the dual variable is adapted!

∘ backward equation in $(Y_t^{\alpha^\star})_{0\le t\le T}$ $\rightsquigarrow$ existence and uniqueness if Lipschitz (quite natural)

• Require now $\alpha_t^\star = \alpha^\star(X_t^{\alpha^\star}, \mu_t, Y_t^{\alpha^\star})$

∘ $\alpha^\star(x,\mu,y)$ is the unique minimizer of $\alpha \mapsto H(x,\mu,\alpha,y)$

∘ implicit condition $\rightsquigarrow$ new FBSDE

∘ is the forward component an optimal path? Turn the first-order necessary condition into a sufficient condition $\rightsquigarrow$ convexity

Statement

- Let σ and σ^0 indep. of x and $\mathbb{F}$ generated by $(\xi, \mu_0, (W_t, W_t^0)_{0 \le t \le T})$
- Assume that, on $(\Omega, \mathbb{F}, \mathbb{P})$, the FBSDE

$$
\begin{aligned}
X_t^\star &= \xi + \int_0^t b\left(X_s^\star, \mu_s, \alpha^\star(X_s^\star, \mu_s, Y_s^\star)\right) ds \\
&\quad + \int_0^t \sigma(\mu_s) dW_s + \sigma^0(\mu_s) dW_s^0 \\
Y_t^\star &= \partial_x g(X_T^\star, \mu_T) + \int_t^T \partial_x H\left(X_s^\star, \mu_s, \alpha^\star(X_s^\star, \mu_s, Y_s^\star)\right) ds \\
&\quad - \int_t^T Z_s dW_s - \int_t^T Z_s^0 dW_s^0
\end{aligned}
$$

has a solution with

- ∘ H convex in (x, α) and strictly in α and g convex in x
- ∘ $\alpha^\star(x, \mu, z)$ is the unique minimizer of $\alpha \mapsto H(x, \mu, \alpha, z)$

- Then $(X_t^\star)_{0 \le t \le T}$ unique optimal path with $\alpha_t^\star = \alpha^\star(X_t^\star, \mu_t, Y_t^\star)$

Sketch of proof

- Consider an arbitrary control $\boldsymbol{\alpha} = (\alpha_t)_{0 \le t \le T}$
  - write

$$J(\boldsymbol{\alpha}) - J(\boldsymbol{\alpha}^\star) = J(\boldsymbol{\alpha}) - J(\boldsymbol{\alpha}^\star) - \mathbb{E}[(X_T^\alpha - X_T^\star) \cdot \partial_x g(X_T^\star, \mu_T)] + \mathbb{E}[(X_T^\alpha - X_T^\star) \cdot Y_T^\star]$$

- Itô expansion of the last term

$$\begin{aligned} &J(\boldsymbol{\alpha}) - J(\boldsymbol{\alpha}^\star) \\ &= \mathbb{E}\Big[g(X_T^\alpha, \mu_T) - g(X_T^\star, \mu_T) - (X_T^\star - X_T^\alpha) \cdot \partial_x g(X_T^\star, \mu_T) \\ &\quad + \int_0^T \Big[H(X_t^\alpha, \mu_t, \alpha_t, Y_t^\star) - H(X_t^\star, \mu_t, \alpha_t^\star) \\ &\quad - (X_t^\alpha - X_t^\star) \cdot \partial_x H(X_t^\star, \mu_t, \alpha_t^\star) - \underbrace{0}_{\le (\alpha_t - \alpha_t^\star) \cdot \partial_\alpha H(X_t^\star, \mu_t, \alpha_t^\star)} \Big] dt \Big] \ge 0 \end{aligned}$$

Extension and complements

• Extension to the case when $\mathbb{F}$ is larger than the filtration generated by $(\xi, \mu_0, (W_t, W_t^0)_{0 \leq t \leq T})$

∘ loose martingale representation theorem

$$\int_t^T Z_s dW_s + \int_t^T Z_s^0 dW_s^0 \rightsquigarrow \int_t^T Z_s dW_s + M_T - M_t$$

∘ $(M_t)_{0 \leq t \leq T}$ is a square-integrable martingale orthogonal to $\sigma((W_t)_{0 \leq t \leq T})$

• Scope of application $\rightsquigarrow$ no need for σ invertible but indep. of x and H convex in (x, α)

∘ f convex in (x, α) and b linear in (x, α)

• Connection with HJB equation when no common noise $\rightsquigarrow$ next section

Part III. Analysis of FBSDEs

Part III. Analysis of FBSDEs

a. Small time analysis

General form of the FBSDE

- On $(\Omega, \mathbb{F}, \mathbb{P})$ with $\mathbb{F}$ generated by $(\xi, \mu_0, (W_t, W_t^0)_{0\leq t\leq T})$

$$X_t = \xi + \int_0^t b\left(X_s, \mu_s, Y_s, Z_s\right) ds + \int_0^t \sigma(X_s, \mu_s, Y_s) dW_s + \sigma^0(X_s, \mu_s, Y_s) dW_s^0$$

$$Y_t = g(X_T, \mu_T) + \int_t^T f\left(X_s, \mu_s, Y_s, Z_s\right) ds - \int_t^T Z_s dW_s - \int_t^T Z_s^0 dW_s^0$$

  - no Z in σ and σ^0!
  - $(X_t, Y_t, Z_t) : \Omega \to \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times m}$
- Call a $(X_t, Y_t, Z_t)_{0\leq t\leq T}$ a solution if progressively-measurable and

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} (|X_t|^2 + |Y_t|^2) + \int_0^T |Z_t|^2 dt\Big] < \infty$$

Cauchy-Lipschitz theory in small time

• Assume that the coefficients are at of most of linear growth

$$\left|(b,f,\sigma,\sigma^0,g)(x,\mu,y,z)\right| \le C(1+|x|+d_{\mathcal{X}}(0_{\mathcal{X}},\mu)+|y|+|z|)$$

• Assume that the coefficients are measurable in all the variables and L-Lipschitz continuous in (x,y,z) (uniformly in μ)

• There exists $c(L)$ such that unique solution for any initial condition provided that

$$T \le c(L)$$

• Two-point-boundary problem $\rightsquigarrow$ no way to expect better

$$\dot{x}_t = y_t, \quad \dot{y}_t = -x_t, \quad y_T = x_T, \quad T = \pi/4$$

∘ $\ddot{x}_t = -x_t, \quad \begin{array}{l} x_t = A\cos(t) + B\sin(t) \\ y_t = -A\sin(t) + B\cos(t) \end{array} \Rightarrow A = 0 \Rightarrow x_0 = 0$

∘ no solution if $x_0 \neq 0$ and ∞ many if $x_0 = 0$

Sketch of proof

- Construction a contraction mapping
  - With $(X_t)_{0\le t\le T}$ solve the backward equation
  - With $(Y_t, Z_t)_{0\le t\le T}$, solve the forward equation

$$
\begin{aligned}
X'_t = \xi + \int_0^t b\left(X'_s, \mu_s, Y_s, Z_s\right) ds \\
+ \int_0^t \sigma(X'_s, \mu_s, Y_s) dW_s + \sigma^0(X'_s, \mu_s, Y_s) dW^0_s
\end{aligned}
$$

  - Seek $(X_t)_{0\le t\le T}$ such that $X \equiv X'$
- Forward-backward constraints $\Rightarrow$ no way to use Gronwall!
  - Given $(X^1_t)_{0\le t\le T}$ and $(X^2_t)_{0\le t\le T}$, prove that for $T \le 1$

$$
\mathbb{E}[\sup_{0\le t\le T} |X^{1\prime}_t - X^{2\prime}_t|^2] \le c(L)\, T\, \mathbb{E}[\sup_{0\le t\le T} |X^1_t - X^2_t|^2]
$$

- Denote solution by $(X^\xi_t, Y^\xi_t, Z^\xi_t)_{0\le t\le T}$

Decoupling field

- Stability estimates for $T \leq c(L)$

$$\mathbb{E}[|Y_0^{\xi} - Y_0^{\xi'}|^2 | \mathcal{F}_0] \leq C\,|\xi - \xi'|^2$$

- Let $u(0,x) = Y_0^x$, $x \in \mathbb{R}^d$
  - $x \mapsto u(0,x)$ is a random field, $\mathcal{F}_0^0$-measurable (reduce the filtration $u(0,x,\mu_0)$), deterministic if no common noise
  - Lipschitz continuous
- Choose $\xi' = \sum_{i=1}^N 1_{A_i} x_i$, with $A_i \in \mathcal{F}_0$
  - $Y_0^{\xi'} = \sum_{i=1}^N 1_{A_i} Y_0^{x_i} = \sum_{i=1}^N 1_{A_i} u(0,x_i) = u(0,\xi')$
  - approximation argument $\rightsquigarrow$ $Y_0^{\xi} = u(0,\xi)$
- Extension to any time $t \in [0,T]$, $u(t,x) = Y_t^{t,x}$ is $\mathcal{F}_t^0$-measurable
  - $(X_s^{t,\xi}, Y_s^{t,\xi}, Z_s^{t,\xi})_{t \leq s \leq T}$ solution with $X_t^{t,\xi} = \xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$
  - $Y_t^{t,\xi} = u(t,\xi)$

Connection with PDE

- Assume that no common noise W^0 ($\sigma^0 = 0$)
- Write $Y_t^{0,\xi} = Y_t^{t,X_t^{0,\xi}} = u(t, X_t^{0,\xi})$
- If u smooth enough $\rightsquigarrow$ expand as a semi-martingale and compare with $dY_t^{0,\xi}$
  - compare dW_t terms $\rightsquigarrow Z_t^{0,\xi} = \partial_x u(t, X_t^{0,\xi})\sigma(x, \mu_t)$
  - compare dt terms $\rightsquigarrow$ nonlinear PDE

$$\begin{aligned}\partial_t u(t,x) + \frac{1}{2}\text{trace}(\sigma\sigma^\dagger(x,\mu_t,u(t,x))\partial_{xx}^2 u)\\ + \partial_x u(t,x) b(x,\mu_t,u(t,x),\partial_x u(t,x)\sigma(x,\mu_t))\\ + f(x,\mu_t,u(t,x),\partial_x u(t,x)\sigma(x,\mu_t)) = 0\end{aligned}$$

  - terminal boundary condition $u(T,x) = g(x,\mu_T)$
- If $(\mu_t)_{0\le t\le T}$ random $\rightsquigarrow$ backward SPDE!

Examples

- Revisit the FBSDEs of Section II when $\sigma^0 \equiv 0$
- Interpretation of the value function
  - PDE writes

$$\partial_t u(t,x) + \frac{1}{2}\mathrm{trace}\big(\sigma\sigma^\dagger(x,\mu_t)\partial^2_{xx}u\big) + \inf_{\alpha\in A}\Big[\partial_x u(t,x)\cdot b(x,\mu_t,\alpha) + f(x,\mu_t,\alpha)\Big] = 0$$

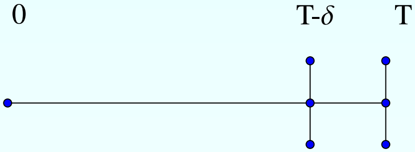
  - HJB equation describing minimal cost when $X_t = x$
  - optimal control $\alpha^\star_t = \alpha^\star(X^\star_t, \mu_t, \partial_x u(t, X^\star_t))$ has Markov feedback form!
- Use of the Stochastic Pontryagin principle
  - Same shape for the Markov feedback form $\rightsquigarrow$ decoupling field must be $\partial_x u(t,x)$!
  - PDE is the derivative of HJB

Part III. Analysis of FBSDEs

b. From small to long times

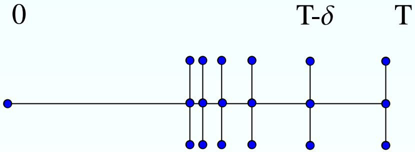
Principle for an iterative construction

- Let T arbitrary $\rightsquigarrow$ construct the decoupling field close to T



$\circ$ for $t \in [T-\delta, T] \rightsquigarrow$ unique solution with $X_t = \xi \rightsquigarrow$ define decoupling field on $[T-\delta, T]$

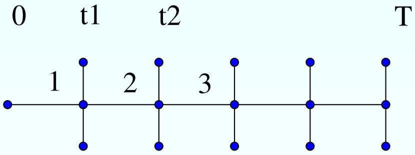
- Consider on $[0, T-\delta]$ new FBSDE with $u(T-\delta, \cdot)$ instead of $g(\cdot, \mu_T)$ as terminal condition (forget $\mu_{T-\delta}$)



$\circ$ need to control the Lipschitz constant of u along the induction

Construction of a solution from the decoupling field

- Construction of the decoupling field by backward induction
- Construction of a solution by forward induction



◦ solve first on 1 with $X_0 = \xi$ as initial condition and $u(t1, \cdot)$ as terminal condition

◦ restart at $t1$ with X_{t1} as new initial condition and $u(t2, \cdot)$ as terminal condition ...

- Uniqueness by backward induction

Part III. Analysis of FBSDEs

c. Convex framework

Revisiting the Pontryagin principle

- Assume
  - σ, σ^0 constant
  - $b(x,\mu,\alpha) = b_0(\mu) + b_1 x + b_2 \alpha$
  - $\partial_x f, \partial_\alpha f, \partial_x g$ L-Lipschitz in (x,α)
  - f convex in (x,α) with λ-convexity in α

$$f(x',\alpha') - f(x,\alpha) - (x'-x)\cdot\partial_x f(x,\alpha) - (\alpha'-\alpha)\cdot\partial_\alpha f(x,\alpha) \geq \lambda|\alpha'-\alpha|^2$$

- Unique minimizer $\alpha^\star(x,\mu,z) = \text{argmin}_{\alpha\in A} H(x,\mu,z,\alpha)$
  - implicit function theorem $\rightsquigarrow$ $\alpha^\star$ is Lipschitz
- Existence and uniqueness hold in small time
  - control of the decoupling field?

Using convexity

- Let $t \in [0,T]$ and $x, x' \in \mathbb{R}^d$

$$
\begin{aligned}
&\mathbb{E}\Big[g(X_T^{\star,t,x'},\mu_T) + \int_t^T f(X_s^{\star,t,x'},\mu_t,\alpha_t^{\star,t,x'})dt \,|\mathcal{F}_t\Big] \\
&\qquad - \mathbb{E}\Big[g(X_T^{\star,t,x},\mu_T) + \int_t^T f(X_s^{\star,t,x},\mu_t,\alpha_t^{\star,t,x})dt \,|\mathcal{F}_t\Big] \\
&\geq (x'-x)\cdot Y_t^{\star,t,x} \\
&\quad + \mathbb{E}\Big[g(X_T^{\star,t,x'},\mu_T) - g(X_T^{\star,t,x},\mu_T) - (X_T^{\star,t,x'} - X_T^{\star,t,x})\cdot \partial_x g(X_T^{\star,t,x},\mu_T) \\
&\quad + \int_0^T \Big[H(X_t^{\star,t,x'},\mu_t,\alpha_t^{\star,t,x'},Y_t^{\star,t,x}) - H(X_t^{\star,t,x},\mu_t,\alpha^{\star,t,x},Y_t^{\star,t,x}) \\
&\qquad\qquad - (X_t^{\star,t,x'} - X_t^{\star,t,x})\cdot \partial_x H(X_t^{\star,t,x},\mu_t,\alpha_t^{\star,t,x}) \\
&\qquad\qquad - (\alpha_t^{\star,t,x'} - \alpha_t^{\star,t,x})\cdot \partial_\alpha H(X_t^{\star,t,x},\mu_t,\alpha_t^{\star,t,x})\Big]\, dt \,\Big|\mathcal{F}_t\Big] \\
&\geq (x'-x)\cdot Y_t^{\star,t,x} + \lambda \mathbb{E}\Big[\int_t^T \big|\alpha_s^{\star,t,x'} - \alpha_s^{\star,t,x'}\big|^2 ds \,|\mathcal{F}_t\Big]
\end{aligned}
$$

Lipschitz estimate in the convex setting

- Exchange the roles of x and x' and make the sum

$$0 \geq (x' - x) \cdot (Y_t^{\star,t,x} - Y_t^{\star,t,x'}) + \lambda \mathbb{E}\Big[\int_t^T \big|\alpha_s^{\star,t,x'} - \alpha_s^{\star,t,x'}\big|^2 ds \,|\mathcal{F}_t\Big]$$

- Stability of the forward equation

$$\mathbb{E}[\sup_{t \leq s \leq T} |X_s^{\star,t,x} - X_s^{\star,t,x'}|^2 | \mathcal{F}_t] \leq C\mathbb{E}\Big[\int_t^T \big|\alpha_s^{\star,t,x'} - \alpha_s^{\star,t,x'}\big|^2 ds \,|\mathcal{F}_t\Big]$$

- Stability of the backward equation

$$\mathbb{E}[\sup_{t \leq s \leq T} |Y_s^{\star,t,x} - Y_s^{\star,t,x'}|^2 | \mathcal{F}_t] \leq C\mathbb{E}\Big[\int_t^T \big|\alpha_s^{\star,t,x'} - \alpha_s^{\star,t,x'}\big|^2 ds \,|\mathcal{F}_t\Big]$$
$$\leq C(x' - x) \cdot (Y_t^{\star,t,x'} - Y_t^{\star,t,x})$$

- Deduce $|u(t,x') - u(t,x)|^2 \leq C(x' - x) \cdot (u(t,x') - u(t,x))$

Part III. Analysis of FBSDEs

d. Non-degenerate case

A simple case

- Assume (for simplicity)
  - $\sigma = \mathrm{Id}$, $\sigma^0 = 0$ (no common noise)
  - $b(x, \mu, \alpha) = \alpha$
  - $f(x, \mu, \alpha) = f_0(x, \mu) + \frac{1}{2}|\alpha|^2$
  - f_0 and g bounded and Lipschitz
- Compute $\alpha^\star(x, \mu, z) = -\pi_A(z)$ ($\perp$ projection onto A)
  - consider a cut-off function ϕ

$$dX_s^{\star,t,x} = -\phi(Z_s^{\star,t,x})\pi_A(Z_s^{\star,t,x})ds + dW_s + \sigma^0(X_s^{\star,t,x}, \mu_s)dW_s^0$$

$$dY_s^{\star,t,x} = -f_0(X_s^{\star,t,x}, \mu_s)ds - \frac{1}{2}|\pi_A(Z_s^{\star,t,x})|^2 ds + Z_s^{\star,t,x}dW_s$$

$$Y_T^{\star,t,x} = g(X_T^{\star,t,x}, \mu_T)$$

  - unique solution in small time

Change of probability

- Let

$$\frac{d\mathbb{P}^{\star,t,x}}{d\mathbb{P}} = \exp\Big(\int_t^T (\phi\pi_A)(Z_s^{\star,t,x})dW_s - \frac{1}{2}\int_t^T \big|(\phi\pi_A)(Z_s^{\star,t,x})\big|^2 ds\Big)$$

  ∘ $\Big(W_s^{\star,t,x} = W_s - \int_t^s (\phi\pi_A)(Z_r^{\star,t,x})dW_r\Big)_{t\le s\le T}$ B.M. under $\mathbb{P}^{\star,t,x}$

- Under $\mathbb{P}^{\star,t,x}$

$$dX_s^{\star,t,x} = dW_s^{\star,t,x}$$

$$dY_s^{\star,t,x} = -f_0(X_s^{\star,t,x},\mu_s)ds + \Big((\phi\pi_A)(Z_s^{\star,t,x})\cdot Z_s^{\star,t,x} - \frac{1}{2}|Z_s^{\star,t,x}|^2\Big)ds + Z_s^{\star,t,x}dW_s^{\star,t,x}$$

$$Y_T^{\star,t,x} = g(X_T^{\star,t,x},\mu_T)$$

  ∘ same system but under $\mathbb{P}$ $\leadsto (\tilde{X}_s^{\star,t,x},\tilde{Y}_s^{\star,t,x},\tilde{Z}_s^{\star,t,x})_{t\le s\le T}$

  ∘ same joint distribution $\leadsto \tilde{Y}_t^{\star,t,x} = u(t,x)$ (PDE is the same)

Quadratic BSDE

- Consider $x, x' \in \mathbb{R}^d$ and let

$$(\delta\tilde{X}^\star_s, \delta\tilde{Y}^\star_s, \delta\tilde{Z}^\star_s) = (\tilde{X}^{\star,t,x'}_s - \tilde{X}^{\star,t,x'}_s, \tilde{Y}^{\star,t,x'}_s - \tilde{Y}^{\star,t,x'}_s, \tilde{Z}^{\star,t,x'}_s - \tilde{Z}^{\star,t,x'}_s)$$

- Dynamics

$$d(\delta\tilde{Y}^\star_s) = -\delta_x f_s \delta\tilde{X}^\star_s ds - \delta_z f_s \delta\tilde{Z}^\star_s ds + \delta\tilde{Z}^\star_s dW_s$$

  - $|\delta\tilde{X}^\star_s|^2 \leq C|x - x'|^2$
  - $|\delta_x f_s| \leq C$, $|\delta_z f_s| \leq C(1 + |Z^{\star,t,x}_s| + |Z^{\star,t,x'}_s|)$
- New Girsanov argument to remove $\delta\tilde{Z}^\star$
  - get a bound on Lip. $x \mapsto u(t, x)$
  - recall $Z^{\star,t,x}_s = \partial_x u(s, X^{t,x}_s)$ to get a bound on $Z^{\star,t,x}$

Extension

- A may not be bounded
- presence of common noise
- b, f and g bounded in (x, μ), C Lipschitz in x
- Regularity in α
  - b linear in α and f $\begin{array}{l}\text{strictly convex}\\ \text{at most quadratic growth}\end{array}$ in α
  - f loc. Lip in α, with Lip(f) at most of linear growth in α
- then FBSDE characterizing optimizer is uniquely solvable (forget cut-off and focus on solutions with bounded $Z^\star$)
  - decoupling field is Lipschitz and $Z^\star$ is bounded
  - forward path is the unique optimal path