Large Population Stochastic Control with a Common Noise

Model Approximation and Numerical Methods

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(Work in collaboration with R. Carmona)
Motivation

• Model of inter-bank borrowing and lending
  ◦ model introduced by Carmona, Fouque, Sun
  ◦ N banks $i = 1, \ldots, N$ and a central bank

• Interaction between banks
  ◦ bank $i$ lend to/borrow from bank $j$

• Control
  ◦ banks control lending to/borrowing from central bank

• Cost for lending/borrowing fixed by the regulator
  ◦ equilibriums?
  ◦ $N$ large?
Mean-field interaction

- Log-monetary reserve of bank $i \sim X_t^i$
  - borrow from/lend to from bank $j$ if $X_t^j > X_t^i$ / $X_t^j < X_t^i$
  - rate of borrowing/lending $\sim a(X_t^j - X_t^i) \sim (a/N)(X_t^j - X_t^i)$

\[
dX_t^i = a\left(\frac{1}{N} \sum_{j=1}^{N} X_t^j - X_t^i\right) dt + \ldots
\]

- mean-field interaction with attraction to the average
  - $dX_t^i \sim$ instantaneous rate of lending/borrowing
    - $a \geq 0 \sim$ borrowing if $X_t^i < \bar{X}_t^N$ / lending if $X_t^i > \bar{X}_t^N$

- $\bar{X}_t^N$ used for systemic risk
Dynamics

- **Controlled** rate of borrowing from/lending to central bank

\[ dX_t^i = a(\bar{X}_t^N - X_t^i) dt + \alpha_t^i dt + \ldots \]

  - negative \( \sim \) lending
  - positive \( \sim \) borrowing

- **Noisy perturbations**

\[ dX_t^i = a(\bar{X}_t^N - X_t^i) dt + \alpha_t^i dt + \sigma d\tilde{W}_t^i \]

  - \( \tilde{W}_t^i = \rho \underbrace{W_t^i}_{\text{independent}} + \sqrt{1 - \rho^2} \underbrace{W_0^0}_{\text{common}} \)

  - \((W_0^0)_t, (W_1^1)_t, \ldots, (W_N^N)_t) : \text{ indep. Brownian motions} \)

  - original paper \( \sim \) role of \( W_0^0 \) in systemic risk
Cost

- Cost functional ⇒ small deviation from mean state

\[ J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E}\left[ g(X_T^i, \bar{X}_T^N) + \int_0^T f(X_t^i, \bar{X}_t^N, \alpha_t^i) dt \right] \]

  - depends on all the controls through \( \bar{X}_t^N \)
  - penalize high borrowing from/lending to central bank
  - incite borrowing/lending ⇒ get closer to mean state

- Linear-quadratic functionals

  \[ f(x, m, \alpha) = \alpha^2 + \epsilon^2 (m - x)^2 - 2q\epsilon\alpha (m - x) \]
  \[ g(x, m) = c^2 (x - m)^2 \]

  - \( \bar{X}_t^N > X_t^i \) ⇒ lower cost if \( \alpha_t > 0 \)
  - \( q \in (0, 1) \), fixed by the regulator
Nash equilibrium

- Find strategies $(\alpha^1, \star, \ldots, \alpha^N, \star)$ such that

$$J^i(\alpha^1, \star, \ldots, \alpha^{i-1}, \star, \alpha^i, \alpha^{i+1}, \star, \ldots, \alpha^N, \star) \geq J^i(\alpha^1, \star, \ldots, \alpha^{i-1}, \star, \alpha^i, \star, \alpha^{i+1}, \star, \ldots, \alpha^N, \star)$$

- Linear quadratic model $\Rightarrow$ explicitly solvable
  - Optimal strategies in feedback form
    $$\hat{\alpha}^i_t = \eta^N_t (\bar{X}^N_t - X^i_t)$$
    $$\eta^N_t \geq 0 \Rightarrow \text{deterministic coefficient (indep. of } \sigma, \rho)$$
    $$\Rightarrow \text{noticeable fact that}$$
    $$\hat{\alpha}^i_t = \hat{\alpha} \left(t, \hat{X}^N_t, \hat{X}^i_t\right)$$
    private state average state
Passage to the limit

- What do equilibriums become as $N \to \infty$?
- Implement optimal strategies

$$d\tilde{X}_t^N = \frac{\sigma \rho}{N} \sum_{i=1}^{N} dW_t^i + \sigma \sqrt{1 - \rho^2} dW_t^0 \to 0$$

- Asymptotic mean: $d\bar{X}_t = \sigma \sqrt{1 - \rho^2} dW_t^0$
- Asymptotic optimal path:

$$dX_t = (a + \eta_t^\infty)(\bar{X}_t - X_t) dt + \sigma \rho dW_t + \sigma \sqrt{1 - \rho^2} dW_t^0$$

- $(W_t)_t$ indep. of $(W_t^0)_t \Rightarrow \bar{X}_t = \mathbb{E}[X_t | W^0]$
Conditional LLN

• What happened?
  ◦ Optimal \((X^1, \ldots, X^N)\) are conditionally i.i.d. given \(W^0\)

\[
\bar{X}_t^N \sim \mathbb{E}[X_1^t | W^0]
\]

◦ conditional propagation of chaos towards conditional MKV

\[
dX_t = (a + \eta_t^\infty)(\mathbb{E}[X_t | W^0] - X_t) + \sigma \rho dW_t + \sigma \sqrt{1 - \rho^2} dW_0^t
\]

• \((X_t)_t\) reads as optimal path in random environment
  ◦ solve first the environment \(\bar{X}_t = \mathbb{E}[X_t | W^0]\)
  ◦ minimize \(J(\alpha) = \mathbb{E} \left[ g(X_T^\alpha, \bar{X}_T) + \int_0^T f(X_t^\alpha, \bar{X}_t, \alpha_t) dt \right]\)

\[
dX_t^\alpha = a(\bar{X}_t - X_t^\alpha) dt + \alpha_t dt + \sigma \rho dW_t + \sigma \sqrt{1 - \rho^2} dW_0^t
\]
General strategy

- empirical mean $\rightarrow$ empirical measure $\bar{\mu}_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta X_i^t$
  - drift $a(\bar{X}_t^N - X_t^i) + \alpha_t^i \sim b(X_t^i, \bar{\mu}_t^N, \alpha_t^i)$
  - cost $(f, g)(X_t^i, \bar{X}_t^N, \alpha_t^i) \sim (f, g)(X_t^i, \bar{\mu}_t^N, \alpha_t^i)$

- Asymptotic equilibriums $\rightarrow$ mean-field games (Lasry-Lions)
  - fix the flow of random measures $(\mu_t)_{0 \leq t \leq T}$
  - $\mu_t$ depends on $(W_s^0)_{0 \leq s \leq t}$ only!
  - optimize

$$dX_t^\alpha = b(X_t^\alpha, \mu_t, \alpha_t) dt + \sigma \rho dW_t + \sigma \sqrt{1 - \rho^2} dW_t^0$$

$$J(\alpha) = \mathbb{E} \left[ g(X_T^\alpha, \mu_T) + \int_0^T f(X_t^\alpha, \mu_t, \alpha_t) dt \right]$$

- solve the matching problem $\mu_t = \mathcal{L}(X_t^{\text{optimal}} | W^0)$
PDE point of view: HJB

- Define the flow of measures \((\mu_t)_{0 \leq t \leq T}\) at equilibrium
- No common noise \(\Rightarrow\) Value function

\[
U(t, x) = \inf_{\alpha} \mathbb{E} \left[ g(X_T^{x, \alpha}, \mu_T) + \int_t^T f(X_s^{x, \alpha}, \mu_s, \alpha_s) ds \right]
\]

- \(U\) \(\Rightarrow\) Backward HJB

\[
d_t U(t, x) + \left( \mathcal{L} U(t, x) + \inf_{\alpha} \left[ b(x, \mu_t, \alpha) \frac{\partial}{\partial x} U(t, x) + f(x, \mu_t, \alpha) \right] \right) dt = 0
\]

- Terminal boundary condition: \(U(T, \cdot) = g(\cdot, \mu_T)\)
PDE point of view: stochastic HJB

- Define the flow of random measures \((\mu_t)_{0 \leq t \leq T}\) at equilibrium
- common noise \(\Rightarrow\) Value function = random field

\[
U(t, x) = \inf_{\alpha} \mathbb{E} \left[ g(X_T^x, \alpha, \mu_T) + \int_t^T f(X_s^x, \alpha_s, \mu_s, \alpha_s) ds \bigg| \mathcal{F}_t^W \right]
\]

- \(U\) adapted \(\Rightarrow\) Backward Stochastic HJB

\[
d_t U(t, x) + \left( \mathcal{L} U(t, x) + \inf_{\alpha} \left[ b(x, \mu_t, \alpha) \partial_x U(t, x) + f(x, \mu_t, \alpha) \right] \right) + \sigma \sqrt{1 - \rho^2} \partial_x V(t, x) dt - V(t, x) dW_t^0 = 0
\]

- Terminal boundary condition: \(U(T, \cdot) = g(\cdot, \mu_T)\)

- No common noise \(\Rightarrow\) Value function = random field
Fokker-Planck

- Dynamics of $X$ at equilibrium
  
  $$dX_t = b(X_t, \mu_t, \hat{\alpha}(X_t, \partial_x U(t, X_t), \mu_t)) dt + \sigma dW_t$$

- **MKV constraint** $\mu_t = \mathcal{L}(X_t)$

- **extended Hamiltonian**
  
  $\circ H(x, y, \mu, \alpha) = b(x, \mu, \alpha)y + f(x, \mu, \alpha)$
  
  $\circ \hat{\alpha}(x, y, \mu) = \text{argmin}_\alpha H(x, y, \mu, \alpha)$

- **Law** $(X_t)_{0 \leq t \leq T}$ satisfies Fokker-Planck equation
  
  $$d_t \mu_t = -\text{div} \left( b(x, \mu_t, \hat{\alpha}(x, \mu_t, \partial_x U(t, x))) \right) dt + \frac{\sigma^2}{2} \partial_{xx} \mu_t dt$$
  
  $\hat{b}(t, x)$
Stochastic Fokker-Planck

- Dynamics of $X$ at equilibrium

\[
dX_t = b(X_t, \mu_t, \hat{\alpha}(X_t, \partial_x U(t, X_t), \mu_t)) dt + \sigma dW_t
\]
\[
+ \sigma \sqrt{1 - \rho^2} dW_0^t
\]

- MKV constraint $\mu_t = \mathcal{L}(X_t | (W_s^0)_{0 \leq s \leq t})$

- extended Hamiltonian

- $H(x, y, \mu, \alpha) = b(x, \mu, \alpha)y + f(x, \mu, \alpha)$

- $\hat{\alpha}(x, y, \mu) = \arg\min_{\alpha} H(x, y, \mu, \alpha)$

- Law $(X_t)_{0 \leq t \leq T}$ satisfies Fokker-Planck equation

\[
d_t \mu_t = -\text{div} \left(b(x, \mu_t, \hat{\alpha}(x, \mu_t, \partial_x U(t, x)))\right) dt + \frac{\sigma^2}{2} \partial_{xx} \mu_t dt
\]
\[
\hat{b}(t, x)
\]
\[
- \text{div} \left(\mu_t \sigma \sqrt{1 - \rho^2} \right) \circ dW_0^t
\]
Infinite dimensional FBSDE

- **Forward-backward coupling:**
  - *forward* equation: stochastic Fokker-Planck
  - *backward* equation: stochastic HJB

- **Well-posedness** $\Rightarrow$ decoupling field

- **Representation of value function**
  \[
  U(t, \cdot) = U(t, \mu_t, \cdot),
  \]
  \[
  \]
Infinite dimensional FBSDE

- **Forward-backward coupling:**
  - *forward* equation: stochastic Fokker-Planck
  - *backward* equation: stochastic HJB
- **Well-posedness** $\Rightarrow$ decoupling field
- Representation of value function
  $$U(t, x, \omega) = U(t, \mu_t, x),$$
  - $U: [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}$
  - $\mathcal{P}(\mathbb{R})$: space of probability measures
Infinite dimensional FBSDE

- **Forward-backward coupling:**
  - **forward** equation: stochastic Fokker-Planck
  - **backward** equation: stochastic HJB

- **Well-posedness** $\Rightarrow$ decoupling field

- **Representation of random value function**
  \[
  U(t, x, \omega) = U(t, \mu_t(\omega), x),
  \]

  - $U : [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}$
  - $\mathcal{P}(\mathbb{R})$: space of probability measures
Infinite dimensional FBSDE

- **Forward-backward coupling:**
  - *forward* equation: stochastic Fokker-Planck
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- **Well-posedness** $\Rightarrow$ decoupling field
- **Representation of random value function**
  \[
  U(t, x, \omega) = U(t, \mu_t(\omega), x),
  \]
  - $U : [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}$
  - $\mathcal{P}(\mathbb{R})$: space of probability measures
- **Write second-order (master?) PDE**
  - in general: PDE in infinite dimension (diff. calc. on $\mathcal{P}(\mathbb{R})$)
  - interbank lending/borrowing: finite dimension
  \[
  U(t, x, \omega) = U(t, m_t, x), \quad m_t = \int_{\mathbb{R}} x' d\mu_t(x')
  \]
Optimal feedback

- **Implement** the decoupling field
  - optimal feedback $\hat{\alpha}(x, \partial_x U(t, x, \mu), \mu)$
  - with $\hat{\alpha}(x, y, \mu) = \arg\min_\alpha [b(x, \mu, \alpha)y + f(x, \mu, \alpha)]$

- **MFG equilibrium path:**
  
  $$dX_t = b(X_t, \mu_t, \hat{\alpha}(X_t, \partial_x U(t, X_t, \mu_t), \mu_t)) \, dt + \sigma \, d\tilde{W}_t$$

  - **MKV constraint:** $\mu_t = \mathcal{L}(X_t | W^0)$

- **Dual process** $Y_t = \partial_x U(t, X_t, \mu_t) \rightsquigarrow** Pontryagin principle**:
  
  $$dY_t = -\partial_x H(X_t, Y_t, \mu_t, \hat{\alpha}(X_t, Y_t, \mu_t)) + Z_t dW_t + Z_t^0 dW_t^0$$
Optimal feedback

- **Implement** the decoupling field
  - optimal feedback $\hat{\alpha}(x, \partial_x U(t, x, \mu), \mu)$
  - with $\hat{\alpha}(x, y, \mu) = \arg\min_\alpha [b(x, \mu, \alpha)y + f(x, \mu, \alpha)]$

- **Pontryagin principle:**
  \[
  dX_t = b(X_t, \mu_t, \hat{\alpha}(X_t, Y_t, \mu_t)) dt + \sigma \rho dW_t + \sigma \sqrt{1 - \rho^2} dW_t^0
  
  dY_t = -\partial_x H(X_t, Y_t, \mu_t, \hat{\alpha}(X_t, Y_t, \mu_t)) + Z_t dW_t + Z_t^0 dW_t^0
  
  Y_T = \partial_x g(X_T, \mu_T)
  
  - MKV constraint: $\mu_t = \mathcal{L}(X_t|W^0)$
  - convexity $\Rightarrow$ characterization of the equilibriums
  - yields another strategy to tackle MFG (Carmona, D.)
Smoothing effect of common noise

- **Simple example**
  - $b(x, \mu, \alpha) = -x + b(m) + \alpha$, $m = \int x' d\mu(x')$
  - $f(x, \mu, \alpha) = \frac{1}{2} \left[ (x + f(m))^2 + \alpha^2 \right]$
  - $g(x, \mu) = \frac{1}{2} (x + g(m))^2$

- **Stochastic Pontryagin** $\sim Y_t = X_t + \chi_t$

  $dm_t = (b(m_t) - 2m_t - \chi_t) dt + \sigma \sqrt{1 - \rho^2} dW^0_t,$
  $d\chi_t = -(f + b)(m_t) dt + Z^0_t dW^0_t, \quad \chi_T = g(m_T)$

  - $m_t = \mathbb{E}[X_t|\mathcal{W}^0]$

  - $b, f, g$ smooth bounded $+$ noise $\Rightarrow \exists$ and !

  - without noise $\Rightarrow$ ! may fail

- **Prospects**: higher dimension, zero noise limits?