# Large Population Stochastic Control with a Common Noise

Model Approximation and Numerical Methods

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(Work in collaboration with R. Carmona)

# Motivation

- Model of inter-bank borrowing and lending
  - model introduced by Carmona, Fouque, Sun
  - N banks $i = 1, \ldots, N$ and a central bank
- Interaction between banks
  - bank $i$ lend to/borrow from bank $j$
- Control
  - banks control lending to/borrowing from central bank
- Cost for lending/borrowing fixed by the regulator
  - equilibriums?
  - $N$ large?

# Mean-field interaction

- Log-monetary reserve of bank $i \rightsquigarrow X_t^i$
  - borrow from/lend to from bank $j$ if $X_t^j > X_t^i$ / $X_t^j < X_t^i$
  - rate of borrowing/lending $\rightsquigarrow a(X_t^j - X_t^i) \rightsquigarrow (a/N)(X_t^j - X_t^i)$

$$dX_t^i = a\Bigg(\underbrace{\frac{1}{N}\sum_{j=1}^{N} X_t^j}_{\bar{X}_t^N} - X_t^i\Bigg)dt + \dots$$

  - mean-field interaction with attraction to the average
  - $\dfrac{dX_t^i}{dt} \rightsquigarrow$ instantaneous rate of lending/borrowing
  - $a \geq 0 \rightsquigarrow$ borrowing if $X_t^i < \bar{X}_t^N$ / lending if $X_t^i > \bar{X}_t^N$
- $\bar{X}_t^N$ used for systemic risk

# Dynamics

- Controlled rate of borrowing from/lending to central bank

$$dX_t^i = a(\bar{X}_t^N - X_t^i)\,dt + \alpha_t^i dt + \dots$$

  - $\alpha_t^i$ negative $\rightsquigarrow$ lending
    positive $\rightsquigarrow$ borrowing

- Noisy perturbations

$$dX_t^i = a(\bar{X}_t^N - X_t^i)\,dt + \alpha_t^i dt + \sigma d\tilde{W}_t^i$$

  - $\tilde{W}_t^i = \rho \underbrace{W_t^i}_{\text{independent}} + \sqrt{1-\rho^2}\,\underbrace{W_t^0}_{\text{common}}$
  - $((W_t^0)_t, (W_t^1)_t, \dots, (W_t^N)_t)$ : indep. Brownian motions
  - original paper $\rightsquigarrow$ role of $W^0$ in systemic risk

# Cost

- Cost functional $\Rightarrow$ small deviation from mean state

$$J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E}\left[g(X_T^i, \bar{X}_T^N) + \int_0^T f(X_t^i, \bar{X}_t^N, \alpha_t^i)dt\right]$$

  - depends on all the controls through $\bar{X}_t^N$
  - penalize high borrowing from/lending to central bank
  - incite borrowing/lending $\Rightarrow$ get closer to mean state
- Linear-quadratic functionals

$$f(x, m, \alpha) = \alpha^2 + \epsilon^2(m - x)^2 - 2q\epsilon\alpha(m - x)$$
$$g(x, m) = c^2(x - m)^2$$

  - $\bar{X}_t^N > X_t^i \Rightarrow$ lower cost if $\alpha_t > 0$
  - $q \in (0, 1) \rightsquigarrow$ fixed by the regulator

# Nash equilibrium

- Find strategies $(\alpha^{1,\star}, \ldots, \alpha^{N,\star})$ such that

$$J^i\big(\alpha^{1,\star}, \ldots, \alpha^{i-1,\star}, \alpha^i, \alpha^{i+1,\star}, \ldots, \alpha^{N,\star}\big) \geq J^i\big(\alpha^{1,\star}, \ldots, \alpha^{i-1,\star}, \alpha^{i,\star}, \alpha^{i+1,\star}, \ldots, \alpha^{N,\star}\big)$$

- Linear quadratic model $\rightsquigarrow$ explicitly solvable
  - Optimal strategies in feedback form

$$\hat{\alpha}_t^i = \eta_t^N(\bar{X}_t^N - X_t^i)$$

  - $(\eta_t^N)_{t\geq 0} \rightsquigarrow$ deterministic coefficient (indep. of $\sigma$, $\rho$)
  - noticeable fact that

$$\hat{\alpha}_t^i = \hat{\alpha}\Big(t, \underbrace{X_t^i}_{\text{private state}}, \underbrace{\hat{X}_t^N}_{\text{average state}}\Big)$$

# Passage to the limit

- What do equilibriums become as $N \to \infty$?
- Implement optimal strategies

$$d\bar{X}_t^N = \underbrace{\frac{\sigma\rho}{N}\sum_{i=1}^{N} dW_t^i}_{\to 0} + \sigma\sqrt{1-\rho^2}dW_t^0$$

- Asymptotic mean: $d\bar{X}_t = \sigma\sqrt{1-\rho^2}dW_t^0$
- Asymptotic optimal path:

$$dX_t = (a + \eta_t^\infty)(\bar{X}_t - X_t)dt + \sigma\rho dW_t + \sigma\sqrt{1-\rho^2}dW_t^0$$

  - $(W_t)_t$ indep. of $(W_t^0)_t \Rightarrow \bar{X}_t = \mathbb{E}[X_t | W^0]$

# Conditional LLN

- What happened?
  - Optimal $(X^1, \ldots, X^N)$ are conditionally i.i.d. given $W^0$

$$\bar{X}_t^N \sim \mathbb{E}\big[X_t^1 | W^0\big]$$

  - conditional propagation of chaos towards conditional MKV

$$dX_t = (a + \eta_t^\infty)\big(\mathbb{E}[X_t | W^0] - X_t\big) + \sigma\rho dW_t + \sigma\sqrt{1-\rho^2} dW_t^0$$

- $(X_t)_t$ reads as optimal path in random environment
  - solve first the environment $\bar{X}_t = \mathbb{E}[X_t | W^0]$
  - minimize $J(\alpha) = \mathbb{E}\left[g(X_T^\alpha, \bar{X}_T) + \int_0^T f(X_t^\alpha, \bar{X}_t, \alpha_t) dt\right]$

$$dX_t^\alpha = a\big(\bar{X}_t - X_t^\alpha\big) dt + \alpha_t dt + \sigma\rho dW_t + \sigma\sqrt{1-\rho^2} dW_t^0$$

# General strategy

- empirical mean $\rightsquigarrow$ empirical measure $\bar{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$
  - drift $a(\bar{X}_t^N - X_t^i) + \alpha_t^i \rightsquigarrow b(X_t^i, \bar{\mu}_t^N, \alpha_t^i)$
  - cost $(f,g)(X_t^i, \bar{X}_t^N, \alpha_t^i) \rightsquigarrow (f,g)(X_t^i, \bar{\mu}_t^N, \alpha_t^i)$
- Asymptotic equilibriums $\rightsquigarrow$ mean-field games (Lasry-Lions)
  - fix the flow of random measures $(\mu_t)_{0\le t\le T}$
  - $\mu_t$ depends on $(W_s^0)_{0\le s\le t}$ only!
  - optimize

$$dX_t^\alpha = b(X_t^\alpha, \mu_t, \alpha_t)dt + \sigma\rho dW_t + \sigma\sqrt{1-\rho^2}dW_t^0$$

$$J(\alpha) = \mathbb{E}\left[g(X_T^\alpha, \mu_T) + \int_0^T f(X_t^\alpha, \mu_t, \alpha_t)dt\right]$$

  - solve the matching problem $\mu_t = \mathcal{L}(X_t^{\text{optimal}} | W^0)$

# PDE point of view: HJB

- Define the flow of measures $(\mu_t)_{0\le t\le T}$ at equilibrium
- No common noise $\Rightarrow$ Value function

$$U(t,x) = \inf_{\alpha} \mathbb{E}\Big[ g\big(X_T^{x,\alpha}, \mu_T\big) + \int_t^T f(X_s^{x,\alpha}, \mu_s, \alpha_s) ds \Big]$$

- $U$ $\Rightarrow$ Backward HJB

$$d_t U(t,x) + \Big( \underbrace{\mathcal{L}U(t,x)}_{\text{Laplace generator}} + \underbrace{\inf_{\alpha}\big[ b(x,\mu_t,\alpha)\partial_x U(t,x) + f(x,\mu_t,\alpha)\big]}_{\text{standard Hamiltonian in HJB}} \Big) dt = 0$$

- Terminal boundary condition: $U(T,\cdot) = g(\cdot,\mu_T)$

# PDE point of view: stochastic HJB

- Define the flow of random measures $(\mu_t)_{0\le t\le T}$ at equilibrium
- common noise $\Rightarrow$ Value function = random field

$$U(t,x) = \inf_{\alpha} \mathbb{E}\Big[g(X_T^{x,\alpha}, \mu_T) + \int_t^T f(X_s^{x,\alpha}, \mu_s, \alpha_s)ds \Big| \mathcal{F}_t^{W^0}\Big]$$

- $U$ adapted $\Rightarrow$ Backward Stochastic HJB

$$\begin{aligned} & d_t U(t,x) \\ & \quad + \Big( \underbrace{\mathcal{L}U(t,x)}_{\text{Laplace generator}} + \underbrace{\inf_{\alpha}\big[b(x,\mu_t,\alpha)\partial_x U(t,x) + f(x,\mu_t,\alpha)\big]}_{\text{standard Hamiltonian in HJB}} \\ & \quad + \underbrace{\sigma\sqrt{1-\rho^2}\,\partial_x V(t,x)}_{\text{Ito Wentzell cross term}}\Big)dt - \underbrace{V(t,x)dW_t^0}_{\text{backward term}} = 0 \end{aligned}$$

- Terminal boundary condition: $U(T,\cdot) = g(\cdot,\mu_T)$

# Fokker-Planck

- Dynamics of $X$ at equilibrium

$$dX_t = b\big(X_t, \mu_t, \hat{\alpha}(X_t, \partial_x U(t, X_t), \mu_t)\big)dt + \sigma dW_t$$

  - MKV constraint $\mu_t = \mathcal{L}\big(X_t \qquad\qquad\qquad\qquad \big)$

- extended Hamiltonian
  - $H(x, y, \mu, \alpha) = b(x, \mu, \alpha)y + f(x, \mu, \alpha)$
  - $\hat{\alpha}(x, y, \mu) = \text{argmin}_\alpha H(x, y, \mu, \alpha)$

- Law $(X_t)_{0 \le t \le T}$ satisfies Fokker-Planck equation

$$d_t\mu_t = -\text{div}\big(\underbrace{b(x, \mu_t, \hat{\alpha}(x, \mu_t, \partial_x U(t, x)))}_{\hat{b}(t,x)}\big)dt + \frac{\sigma^2}{2}\partial^2_{xx}\mu_t dt$$

# Stochastic Fokker-Planck

- Dynamics of $X$ at equilibrium

$$dX_t = b\big(X_t, \mu_t, \hat{\alpha}(X_t, \partial_x U(t, X_t), \mu_t)\big)dt + \sigma dW_t + \sigma\sqrt{1-\rho^2}dW_t^0$$

  - MKV constraint $\mu_t = \mathcal{L}\big(X_t | (W_s^0)_{0\le s\le t}\big)$

- extended Hamiltonian
  - $H(x, y, \mu, \alpha) = b(x, \mu, \alpha)y + f(x, \mu, \alpha)$
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# Infinite dimensional FBSDE

- Forward-backward coupling:
  - forward equation: stochastic Fokker-Planck
  - backward equation: stochastic HJB
- Well-posedness $\Rightarrow$ decoupling field
- Representation of value function

$$U(t, \cdot) = \mathcal{U}(t, \mu_t, \cdot),$$

  - 
  - 

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$$U(t, x) = \mathcal{U}(t, \mu_t, x),$$

  - $\mathcal{U} : [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}$
  - $\mathcal{P}(\mathbb{R})$: space of probability measures

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$$U(t, x, \omega) = \mathcal{U}\big(t, \mu_t(\omega), x\big),$$

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  - $\mathcal{U} : [0,T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}$
  - $\mathcal{P}(\mathbb{R})$: space of probability measures
- Write second-order (master?) PDE
  - in general: PDE in infinite dimension (diff. calc. on $\mathcal{P}(\mathbb{R})$)
  - interbank lending/borrowing: finite dimension

$$U(t,x,\omega) = \mathcal{U}\big(t, m_t, x\big), \quad m_t = \int_{\mathbb{R}} x' d\mu_t(x')$$

# Optimal feedback

- Implement the decoupling field
  - optimal feedback $\hat{\alpha}\big(x, \partial_x \mathcal{U}(t, x, \mu), \mu\big)$
  - with $\hat{\alpha}(x, y, \mu) = \text{argmin}_\alpha \big[b(x, \mu, \alpha) y + f(x, \mu, \alpha)\big]$
- MFG equilibrium path:

$$dX_t = b\big(X_t, \mu_t, \hat{\alpha}\big(X_t, \partial_x \mathcal{U}(t, X_t, \mu_t), \mu_t\big)\big) dt + \sigma d\tilde{W}_t$$

  - MKV constraint: $\mu_t = \mathcal{L}(X_t | W^0)$
- Dual process $Y_t = \partial_x \mathcal{U}(t, X_t, \mu_t) \rightsquigarrow$ Pontryagin principle:

$$dY_t = -\partial_x H\big(X_t, Y_t, \mu_t, \hat{\alpha}(X_t, Y_t, \mu_t)\big) + Z_t dW_t + Z_t^0 dW_t^0$$

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  - with $\hat{\alpha}(x, y, \mu) = \text{argmin}_\alpha \big[b(x, \mu, \alpha)y + f(x, \mu, \alpha)\big]$
- Pontryagin principle:

$$
\begin{aligned}
dX_t &= b\big(X_t, \mu_t, \hat{\alpha}\big(X_t, Y_t, \mu_t\big)\big)dt + \sigma\rho dW_t + \sigma\sqrt{1-\rho^2}dW_t^0 \\
dY_t &= -\partial_x H\big(X_t, Y_t, \mu_t, \hat{\alpha}(X_t, Y_t, \mu_t)\big) + Z_t dW_t + Z_t^0 dW_t^0 \\
Y_T &= \partial_x g(X_T, \mu_T)
\end{aligned}
$$

  - MKV constraint: $\mu_t = \mathcal{L}(X_t | W^0)$
  - convexity $\Rightarrow$ characterization of the equilibriums
  - yields another strategy to tackle MFG (Carmona, D.)

# Smoothing effect of common noise

- Simple example
  - $b(x,\mu,\alpha) = -x + b(m) + \alpha$, $m = \int x' d\mu(x')$
  - $f(x,\mu,\alpha) = \frac{1}{2}\left[\left(x + f(m)\right)^2 + \alpha^2\right]$
  - $g(x,\mu) = \frac{1}{2}\left(x + g(m)\right)^2$
- Stochastic Pontryagin $\rightsquigarrow$ $Y_t = X_t + \chi_t$

$$
\begin{aligned}
dm_t &= \left(b(m_t) - 2m_t - \chi_t\right)dt + \sigma\sqrt{1-\rho^2}\,dW_t^0, \\
d\chi_t &= -\left(f + b\right)(m_t)dt + Z_t^0 dW_t^0, \quad \chi_T = g(m_T)
\end{aligned}
$$

  - $m_t = \mathbb{E}[X_t | W^0]$
  - $b$, $f$, $g$ smooth bounded + noise $\Rightarrow$ $\exists$ and !
  - without noise $\Rightarrow$ ! may fail
- Prospects: higher dimension, zero noise limits?