Dynamical systems 2 : Nodes, saddle, spirals, and centers

Type of equilibria : the Poincaré classification

Henri Poincaré has introduced a classification of linear vector fields of the plan which collects all these vector fields in a finite number of classes, according to their qualitative behaviour. This classification is important because it can be used in the study of the non-linear systems near their equilibria, as we will see below.

\[
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix} = \begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} \begin{pmatrix}
  x \\
  y
\end{pmatrix}
\]

with \( A \) a real \( 2 \times 2 \) matrix. Poincaré assumes that \( A \) is non-degenerate which means that 0 is not an eigenvalue. Let us denote by \( \lambda \) and \( \mu \) the two eigenvalues of \( A \); when they are complex we will also write \( \alpha \pm i\omega \). One knows that it exists a basis of \( \mathbb{R}^2 \) in which the associate linear map associate with \( A \) becomes one of the following:

\[
\begin{pmatrix}
  \lambda & 0 \\
  0 & \mu
\end{pmatrix}
\quad
\begin{pmatrix}
  \lambda & \alpha \\
  \mu & \omega
\end{pmatrix}
\quad
\begin{pmatrix}
  \alpha & \omega \\
  \omega & -\alpha
\end{pmatrix}
\]

If we denote by \( U \) and \( V \) the coordinates in this basis, it is easy to solve the differential system satisfied by \( U \) and \( V \) : in the first case one has \( (U, V) = (e^{\lambda t}U_0, e^{\mu t}V_0) \), in the second \( (U, V) = e^{\lambda t}(U_0 + tV_0, V_0) \) and finally in the last case

\[
\begin{pmatrix}
  U \\
  V
\end{pmatrix} = e^{\lambda t} \begin{pmatrix}
  \cos \omega t & \sin \omega t \\
  -\sin \omega t & \cos \omega t
\end{pmatrix} \begin{pmatrix}
  U_0 \\
  V_0
\end{pmatrix}.
\]

It is then easy to find the qualitative behaviour of solutions. Figure 1 gives this behaviour, according to the values of \( \lambda, \mu \) and \( \alpha \), so has the usual translation of the names given by Poincaré for these various cases.

Linearization of a non-linear differential system near an equilibrium

Let \((x^*, y^*)\) be an equilibrium of the differential system

\[
\begin{align*}
x' &= f(x, y) \\
y' &= g(x, y)
\end{align*}
\]

, so a common zero of \( f \) and \( g \). Let \( \varepsilon > 0 \) be a small parameter. Performing the changes of unknown \( X := \frac{x-x^*}{\varepsilon}, \ Y := \frac{y-y^*}{\varepsilon} \) corresponds to look at the equilibrium \((x^*, y^*)\) through a magnifying glass, as limited \((\bar{X}, \bar{Y})\) corresponds to small \( x - x^* \) et \( y - y^* \). Elementary computations show that the corresponding system in \( X \) and \( Y \) is of kind

\[
\begin{align*}
X' &= aX + bY + o_1(\varepsilon) \\
Y' &= cX + dY + o_2(\varepsilon)
\end{align*}
\]

where \( o_1(\varepsilon) \) and \( o_2(\varepsilon) \) are of order \( \varepsilon \). So, up to \( \varepsilon \), limited solutions \((X, Y)\) are close to solutions of

\[
\begin{pmatrix}
  X' \\
  Y'
\end{pmatrix} = \begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} \begin{pmatrix}
  X \\
  Y
\end{pmatrix}
\]

where matrix \( A \) is called the jacobian matrix at the equilibrium \((x^*, y^*)\) and can be easily computed through elementary calculus, as its coefficients are just partial derivatives of \( f \) and \( g \) computed at \((x^*, y^*)\). Indeed, we have

\[
A = A(x^*, y^*) = \begin{pmatrix}
  \frac{\partial f}{\partial x}(x^*, y^*) & \frac{\partial f}{\partial y}(x^*, y^*) \\
  \frac{\partial g}{\partial x}(x^*, y^*) & \frac{\partial g}{\partial y}(x^*, y^*)
\end{pmatrix}
\]

We saw that linear systems are (up to degenerate cases) of one of the four types : (stable or unstable) node and saddle, spiral or center. We use this same terminology for non-linear equilibrium. Please observe that, whereas a linear center is build up with closed (periodic) solutions, in the non-linear case the solutions may be non-closed and could “spiral weakly” in or outwards. The type of the behaviour of the linearized system is sometimes called the nature of the equilibrium : it gives useful information on the qualitative behaviour of the system near the equilibrium which is easy to compute.

As an example, let us look at two following non-linear differential systems :

\[
\begin{align*}
x' &= (2 - x - 2y/3)x \\
y' &= (2 - 2x/3 - y)y
\end{align*}
\]
Real eigenvalues $\lambda$ and $\mu$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Diagram</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; \lambda &lt; \mu$</td>
<td><img src="image1.png" alt="Diagram" /></td>
<td>Unstable node</td>
</tr>
<tr>
<td>$0 &lt; \lambda = \mu$, $A$ diagonalizable</td>
<td><img src="image2.png" alt="Diagram" /></td>
<td>Degenerate unstable node</td>
</tr>
<tr>
<td>$0 &lt; \lambda = \mu$, $A$ non-diagonalizable</td>
<td><img src="image3.png" alt="Diagram" /></td>
<td>Unstable node</td>
</tr>
<tr>
<td>$\lambda &lt; 0 &lt; \mu$</td>
<td><img src="image4.png" alt="Diagram" /></td>
<td>Saddle</td>
</tr>
<tr>
<td>$\lambda = \mu &lt; 0$, $A$ diagonalizable</td>
<td><img src="image5.png" alt="Diagram" /></td>
<td>Degenerate stable node</td>
</tr>
<tr>
<td>$\lambda = \mu &lt; 0$, $A$ non-diagonalizable</td>
<td><img src="image6.png" alt="Diagram" /></td>
<td>Stable node</td>
</tr>
<tr>
<td>$\mu &lt; \lambda &lt; 0$</td>
<td><img src="image7.png" alt="Diagram" /></td>
<td>Stable node</td>
</tr>
</tbody>
</table>

Complex eigenvalues ($\alpha \pm i\omega$, $\omega \neq 0$)

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</tr>
</thead>
<tbody>
<tr>
<td>$\alpha &gt; 0$</td>
<td><img src="image8.png" alt="Diagram" /></td>
<td>Unstable spiral</td>
</tr>
<tr>
<td>$\alpha = 0$</td>
<td><img src="image9.png" alt="Diagram" /></td>
<td>Center</td>
</tr>
<tr>
<td>$\alpha &lt; 0$</td>
<td><img src="image10.png" alt="Diagram" /></td>
<td>Stable spiral</td>
</tr>
</tbody>
</table>

Fig. 1 – Poincaré classification of linear systems
\[
\begin{aligned}
    x' &= (1 - x - 2y)x \\
y' &= (1 - 2x - y)y \\
\end{aligned}
\]  
(4)

Above the picture of the corresponding field of directions. In both of these examples there are three equilibria located at the coordinate axes, namely (0,0), (0,2), (2,0) and a fourth equilibrium with non-zero coordinates : \((\frac{6}{5}, \frac{6}{5})\) for the first example and \((\frac{1}{3}, \frac{1}{3})\) for the second one. One gets the nature of equilibria by linearizing the system at each equilibrium – actually, compute the jacobian matrix at these points– and compute the eigenvalues. Here we see that the equilibrium “with non-zero coordinates” is a stable node in the first case and a saddle point in the second one. Having a picture available, we can easily see this too.