

Numerical analysis of a non-singular boundary integral method: Part I. The circular case

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SUMMARY

In order to numerically solve the interior and the exterior Dirichlet problems for the Laplacian operator, we present here a method which consists in inverting, on a finite element space, a non-singular integral operator. This operator is a geometrical perturbation of the Steklov operator, and we precisely define the relation between the geometrical perturbation and the dimension of the finite element space, in order to obtain a stable and convergent scheme. Furthermore, this numerical scheme does not give rise to any singular integral.

The scheme can also be considered as a special quadrature formula method for the standard piecewise linear Galerkin approximation of the weakly singular single layer potential, the special quadrature formula being defined by the introduction of a neighbouring curve.

In the present paper, we prove stability and we give error estimates of our numerical scheme when the Laplace problem is set on a disk. We will extend our results to any domains by using compact perturbation arguments, in a second paper. Copyright © 2001 John Wiley & Sons, Ltd.

1. INTRODUCTION

Let Ω be a simply connected, bounded, open domain in \mathbb{R}^2 , the boundary Γ of which is assumed regular. For the sake of simplicity, we will suppose in the following that Γ is \mathcal{C}^∞ . The spaces $H^1(\Omega), H^{1/2}(\Gamma), H^{-1/2}(\Gamma)$ will denote the classical Sobolev spaces of functions on Ω or Γ . The duality $H^{-1/2}(\Gamma), H^{1/2}(\Gamma)$ will be denoted by $\langle \cdot, \cdot \rangle_{-1/2, 1/2}$. If u_0 is given in $H^{1/2}(\Gamma)$, we are looking for a function u in $H^1(\Omega)$ satisfying

$$\begin{aligned}\Delta u &= 0 && \text{in } \Omega \\ u &= u_0 && \text{on } \Gamma\end{aligned}\tag{1}$$

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It is well known that problem (1) has a unique solution u and it is easy to show that if x does not belong to $\bar{\Omega} = \Omega \cup \Gamma$, then we have

$$\forall x \notin \bar{\Omega}: \int_{\Gamma} G(x, y) \frac{\partial u}{\partial n_y}(y) ds_y = \int_{\Gamma} \frac{\partial G}{\partial n_y}(x, y) u_0(y) ds_y \quad (2)$$

where G denotes the Green kernel and $\partial G / \partial n_y$ its external normal derivative with respect to the variable y ; i.e. if $x \in \mathbb{R}^2$, $y \in \Gamma$:

$$G(x, y) = -\frac{1}{2\pi} \log |x - y| \quad (3)$$

$$\frac{\partial G}{\partial n_y}(x, y) = -\frac{(x - y, n_y)}{2\pi |x - y|^2} \quad (4)$$

where (\cdot, \cdot) is the scalar product, $|\cdot|$ the Euclidian norm in \mathbb{R}^2 and n_y the outward unit normal vector at point y .

Let us assume that we are only interested in knowing the normal derivative $\partial u / \partial n_y$ on Γ instead of the whole u in Ω . Then if $\tilde{\Gamma}$ is the boundary of a regular, simply connected, bounded, open domain $\tilde{\Omega}$ containing $\bar{\Omega}$, and if we call ζ the unknown $\partial u / \partial n_y$ on Γ , the integral of which is vanishing, we are in the position to give a variational formulation of (2).

Find $\zeta \in H_0^{-1/2}(\Gamma)$ satisfying

$$\forall \mu \in H_0^{-1/2}(\tilde{\Gamma}): \int_{\tilde{\Gamma}} ds_x \int_{\Gamma} ds_y G(x, y) \zeta(y) \mu(x) = \int_{\tilde{\Gamma}} ds_x \int_{\Gamma} ds_y \frac{\partial G}{\partial n_y}(x, y) u_0(y) \mu(x) \quad (5)$$

where $H_0^{-1/2}(\Gamma) = \{\gamma \in H^{-1/2}(\Gamma): \langle \gamma, 1 \rangle_{-1/2, 1/2} = 0\}$. Clearly speaking, problem (5) possesses at least one solution ζ because it suffices to take $\zeta = \partial u / \partial n_y$ when u is the solution of problem (1). The uniqueness is a delicate thing (see Reference [1] for instance).

Our numerical method is built on the Galerkin approximation of integral formulation (5) called ‘Kupradze integral equation’. Historically, this formulation was introduced by Kupradze (cf. Reference [2]). Since this equation is ill-posed, classical numerical methods to solve it are very unstable. This fact was shown by Christiansen (cf. References [3, 4]). The Tikhonov regularization can be used in order to put right this situation but we obtain a method which converges slowly (cf. Reference [5, p.327]). Nevertheless, engineers have empirically found how to construct stable and effective numerical methods (cf. Reference [6]), which are sometimes used nowadays. In order to explain it, we define the Hilbert spaces X and Y by $X = H_0^{-1/2}(\Gamma)$ and $Y = H_0^{-1/2}(\tilde{\Gamma})$. In view of numerically solving problem (5), we define the continuous bilinear form $\tilde{a}: X \times Y \rightarrow \mathbb{R}$ by

$$\zeta \in X, \mu \in Y: \tilde{a}(\zeta, \mu) = \int_{\tilde{\Gamma}} ds_x \int_{\Gamma} ds_y G(x, y) \zeta(y) \mu(x) \quad (6)$$

Unfortunately, the bilinear form $\tilde{a}(\cdot, \cdot)$ does not satisfy the ‘inf-sup’ conditions of Babuska–Ladyzenskaja (see Reference [7] for instance) because the problem of finding $\zeta \in X$ satisfying $\tilde{a}(\zeta, \mu) = \langle \mu, g \rangle_{-1/2, 1/2}$ for all $\mu \in Y$, where g is given in $H^{1/2}(\tilde{\Gamma})$, has in principle no solution, except if g is a trace of an harmonic function on $\tilde{\Gamma}$. It follows that if we define the operator

$\tilde{R}: H^{1/2}(\tilde{\Gamma}) \rightarrow H^{1/2}(\tilde{\Gamma})$ by $(\tilde{R}u_0)(x) = \int_{\tilde{\Gamma}} (\partial G / \partial n_y)(x, y) u_0(y) ds_y$, $\forall x \in \tilde{\Gamma}$, then the problem of finding $\zeta \in X$ such that

$$\forall \mu \in Y: \tilde{a}(\zeta, \mu) = \langle \mu, \tilde{R}u_0 \rangle_{-1/2, 1/2} \quad (7)$$

has a solution ζ , but it is ill-posed. In the following, we attack the numerical approximation of problem (7) by using a Galerkin method. By splitting up Γ and $\tilde{\Gamma}$ into $n+1$ simple arcs of curve $\Gamma_1, \Gamma_2, \dots, \Gamma_{n+1}$ and $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \dots, \tilde{\Gamma}_{n+1}$, respectively, we can look for a piecewise constant function ζ on Γ_j , $1 \leq j \leq n+1$, satisfying (7) for all piecewise constant function μ on $\tilde{\Gamma}_j$, $1 \leq j \leq n+1$. Actually, this method does not work when n is very large except if we choose $\tilde{\Gamma}$ very close to Γ . In practice, the Galerkin method is efficient when the distance between Γ and $\tilde{\Gamma}$ is of the same order as the smallest arc of curve splitting up Γ and $\tilde{\Gamma}$. For this reason it is necessary to choose $\tilde{\Gamma}$ by taking into account the degree of accuracy of the approximation. To do this we replace $\tilde{\Gamma}$ by $(\Gamma^{(n)})_{n=1}^{\infty}$ which will be a family of curves $\Gamma^{(n)}$ surrounding Ω and converging to Γ when n tends to infinity. We split up Γ and $\Gamma^{(n)}$ into $n+1$ simple arcs of curve denoted by $\Gamma_1, \Gamma_2, \dots, \Gamma_{n+1}$ and $\Gamma_1^{(n)}, \Gamma_2^{(n)}, \dots, \Gamma_{n+1}^{(n)}$, respectively, and we define

$$X_n = \{\lambda \in H_0^{-1/2}(\Gamma): \lambda|_{\Gamma_i} \text{ is constant, } i = 1, 2, \dots, n+1\}$$

$$Y_n = \{\mu \in H_0^{-1/2}(\Gamma^{(n)}): \lambda|_{\Gamma_i^{(n)}} \text{ is constant, } i = 1, 2, \dots, n+1\}$$

The discrete problem corresponding to (7) consists in finding $\hat{\zeta}_n \in X_n$ such that

$$\forall \mu_n \in Y_n: a_n(\hat{\zeta}_n, \mu_n) = \langle \mu_n, R_n u_0 \rangle_{-1/2, 1/2} \quad (8)$$

where $a_n(\cdot, \cdot)$ is defined by $\tilde{a}(\cdot, \cdot)$ in which $\tilde{\Gamma}$ is replaced by $\Gamma^{(n)}$ and where $R_n: H^{1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma^{(n)})$ is defined by

$$\forall x \in \Gamma^{(n)}: (R_n u_0)(x) = \int_{\Gamma} \frac{\partial G}{\partial n_y}(x, y) u_0(y) ds_y \quad (9)$$

Remark that $\dim X_n = \dim Y_n = n$ and problem (8) is a linear system of n equations for n unknowns. A consequence of the fact that (7) is ill-posed is that often the approximate problem (8) is unstable. The goal of this paper is to establish how we have to choose $\Gamma^{(n)}$ and its splitting (together with the splitting of Γ) in order to obtain the stability of the numerical scheme (8) and the convergence of $\hat{\zeta}_n$ to $\partial u / \partial n_y$ when n tends to infinity. We also establish error estimates between $\hat{\zeta}_n$ and $\partial u / \partial n_y$. Moreover, we give the rules for the choice of a quadrature formula when we want to numerically compute $\hat{\zeta}_n$. This quadrature rule does not affect the stability and convergence properties of the scheme. In particular, if ζ denotes the normal derivative of u on Γ then we have

$$\|\hat{\zeta}_n - \zeta\|_{-1/2, \Gamma} = \mathcal{O}\left(\frac{1}{n^{3/2}}\right) \quad (10)$$

Note that instead of (8) one can also consider the problem of finding $\zeta_n \in Y_n$ such that

$$\forall \mu_n \in X_n: \int_{\Gamma} ds_x \int_{\Gamma^{(n)}} ds_y G(x, y) \zeta_n(y) \mu_n(x) = \int_{\Gamma} u_0(x) \mu_n(x) ds_x \quad (11)$$

In fact, problem (11) is a linear system in which appears the same matrix as in (8), and thus it is stable under the same conditions. Moreover, by using the arguments developed in this paper, one can prove that the same estimate (10) holds, where in this case ζ is the solution of the weakly singular single layer equation

$$\forall \mu \in H_0^{-1/2}(\Gamma): \int_{\Gamma} ds_x \int_{\Gamma} ds_y G(x, y) \zeta(y) \mu(x) = \int_{\Gamma} u_0(x) \mu(x) ds_x$$

Thus, the proposed scheme can also be considered as a special quadrature formula method for the standard piecewise linear Galerkin approximation of the weakly singular single layer potential. Under this point of view, the method is related to a class that has been studied for many years, from the early papers [8, 9] up to recent papers (Reference [10] for instance). Nevertheless, the special quadrature obtained here by introducing a neighbouring curve does not fall in a category which has been analysed in a previous work.

The numerical analysis we present here show that the efficiency of our scheme is asymptotically of the same order than the method currently used, but its major feature resides in the simplicity of the ideas used for its construction. It can be also used (but without rigorous justification) in the case when the curve has corners, and the same simple ideas can be applied in the 3D case (cf. Reference [11]). It may be the main reason which has motivated engineers to employ it. We point out that similar but unstable methods seem to have a certain success with engineers (see, for instance, References [12, 13]).

In addition to the papers [8–10], for a review of classical boundary element methods, we refer to References [5, 14–16]. In practice, the boundary element methods are often used in combination with finite element method (cf. Reference [17] for a reference article). We can see in References [18] or [19] how our method can be used with finite element methods in order to simulate a two-dimensional induction heating problem. Another interesting non-singular method has been presented in Reference [20].

In the present paper, we analyse the only circular case where the Laplace problem is set on a disk. We obtain similar results when the Laplace problem is set outside the disk. In a second paper we use compact perturbation arguments to treat the general case.

2. THE CIRCULAR CASE

In this section, we assume that Γ is a circle centred at the origin with radius c . The curves $\Gamma^{(n)}$ will also be some circles centred at the origin with radius $c_n > c$. We will use the complex notation to describe Γ and $\Gamma^{(n)}$, i.e.

$$\Gamma = \{z(t) \in \mathbb{C}: z(t) = ce^{it}, t \in [0, 2\pi]\} \quad (12)$$

$$\Gamma^{(n)} = \{z^{(n)}(t) \in \mathbb{C}: z^{(n)}(t) = c_n e^{it}, t \in [0, 2\pi]\} \quad (13)$$

The main factor which will appear in the following is the ratio γ_n between c_n and c , i.e. $\gamma_n = c_n/c$. In order to discretize Γ and $\Gamma^{(n)}$, we set $h = 2\pi/(n+1)$, $t_j = jh$, $j = 0, 1, \dots, n+1$, $t_{j+1/2} = (j+1/2)h$, $j = 0, 1, \dots, n$. To these points we associate the corresponding points on

Γ and $\Gamma^{(n)}$, that is to say

$$z_j = ce^{it_j}, \quad 0 \leq j \leq n+1, \quad z_{j+1/2} = ce^{it_{j+1/2}}, \quad 0 \leq j \leq n \quad (14)$$

$$z_j^{(n)} = c_n e^{it_j}, \quad 0 \leq j \leq n+1, \quad z_{j+1/2}^{(n)} = c_n e^{it_{j+1/2}}, \quad 0 \leq j \leq n \quad (15)$$

Now we can define

$$\Gamma_j = \{z(t) \in \mathbb{C}: z(t) = ce^{it}, \quad t \in [t_{j-1/2}, t_{j+1/2}]\}, \quad 1 \leq j \leq n \quad (16)$$

$$\Gamma_{n+1} = \{z(t) \in \mathbb{C}: z(t) = ce^{it}, \quad t \in [t_{n+1/2}, 2\pi] \cup [0, t_{1/2}]\} \quad (17)$$

$$\Gamma_j^{(n)} = \{z^{(n)}(t) \in \mathbb{C}: z^{(n)}(t) = c_n e^{it}, \quad t \in [t_{j-1/2}, t_{j+1/2}]\}, \quad 1 \leq j \leq n \quad (18)$$

$$\Gamma_{n+1}^{(n)} = \{z^{(n)}(t) \in \mathbb{C}: z^{(n)}(t) = c_n e^{it}, \quad t \in [t_{n+1/2}, 2\pi] \cup [0, t_{1/2}]\} \quad (19)$$

and we recall that

$$X_n = \{\lambda \in H_0^{-1/2}(\Gamma): \lambda|_{\Gamma_i} \text{ is constant, } i = 1, 2, \dots, n+1\} \quad (20)$$

$$Y_n = \{\mu \in H_0^{-1/2}(\Gamma^{(n)}): \mu|_{\Gamma_i^{(n)}} \text{ is constant, } i = 1, 2, \dots, n+1\} \quad (21)$$

Our main result is the following:

Theorem 2.1. Assume that $\gamma_n = 1 + \delta/n$ where δ is a positive number. Then there exists $\delta_0 > 0$ such that for all $\delta \in]0, \delta_0[$, problem (8) has a unique solution $\hat{\zeta}_n \in X_n$ for any integer n and we have $\lim_{n \rightarrow \infty} \|\partial u / \partial n - \hat{\zeta}_n\|_{-1/2, \Gamma} = 0$, where u is the solution of (1).

Moreover, if $\partial u / \partial n \in H^1(\Gamma)$, there exists a constant C independent of n such that

$$\left\| \frac{\partial u}{\partial n} - \hat{\zeta}_n \right\|_{-1/2, \Gamma} \leq \frac{C}{n^{3/2}}$$

Before proving this theorem, we establish three technical lemmata. In the following we will use Fourier technical arguments and for this reason we will work in the complex spaces $H_0^{-1/2}(\Gamma), H_0^{-1/2}(\Gamma^{(n)})$. We define the sesquilinear form $a: H_0^{-1/2}(\Gamma) \times H_0^{-1/2}(\Gamma) \rightarrow \mathbb{C}$ by

$$\hat{\lambda}, \hat{\mu} \in H_0^{-1/2}(\Gamma): a(\hat{\lambda}, \hat{\mu}) = \int_{\Gamma} ds_x \int_{\Gamma} ds_y G(x, y) \hat{\lambda}(y) \overline{\hat{\mu}(x)} \quad (22)$$

where $\bar{\mu}$ is the complex conjugate of μ . When $\mu \in H_0^{-1/2}(\Gamma^{(n)})$ we set $\hat{\mu}(x) = \mu(\gamma_n x)$ for $x \in \Gamma$ and we have $\hat{\mu} \in H_0^{-1/2}(\Gamma)$. In the following $\tilde{a}(\cdot, \cdot)$ and $a_n(\cdot, \cdot)$ are also considered as sesquilinear forms on the complexified Sobolev spaces $H^{-1/2}$.

Now we are in the position to set the eigenproblem which consists in looking for $\hat{\lambda} \in X_n$, $\hat{\lambda} \neq 0$ and $\omega \in \mathbb{C}$ satisfying

$$\forall \mu \in Y_n: a_n(\hat{\lambda}, \mu) = \omega a(\hat{\lambda}, \hat{\mu}) \quad (23)$$

By setting $\hat{v}_m(z) = (z/c)^m$, $z \in \Gamma \subset \mathbb{C}$ and by defining $\hat{\Pi}_n : H^1(\Gamma) \rightarrow L^2(\Gamma)$ by

$$\begin{aligned}\hat{\Pi}_n \hat{v}(z) &= \hat{v}(z_j), \quad z \in \Gamma_j, \quad j = 1, 2, \dots, n \\ \hat{\Pi}_n \hat{v}(z) &= \hat{v}(z_{n+1}), \quad z \in \Gamma_{n+1}\end{aligned}$$

we obtain the following main result:

Lemma 2.1. The eigenproblem (23) possesses the solutions

$$\omega_m = \gamma_n \frac{\sum_{p \in \mathbb{Z}} \gamma_n^{-|m+p(n+1)|} |m+p(n+1)|^{-3}}{\sum_{p \in \mathbb{Z}} |m+p(n+1)|^{-3}}, \quad \hat{\lambda}_m = \hat{\Pi}_n \hat{v}_m, \quad m = 1, 2, \dots, n \quad (24)$$

Proof. We begin by setting $v_k(z) = (z/c_n)^k$, $z \in \Gamma^{(n)}$, and we calculate for any integer m and $k \in \mathbb{Z}$:

$$\begin{aligned}a_n(\hat{v}_m, v_k) &= \int_{\Gamma^{(n)}} ds_x \int_{\Gamma} ds_y G(x, y) \hat{v}_m(y) \overline{v_k(x)} \\ &= -\frac{cc_n}{2\pi} \int_0^{2\pi} ds \int_0^{2\pi} dt \log |ce^{it} - c_n e^{is}| e^{imt} e^{-iks} dt \\ &= -\frac{cc_n}{2\pi} \int_0^{2\pi} e^{-iks} ds \int_0^{2\pi} dt \log |ce^{i(t-s)} - c_n| e^{imt} dt\end{aligned}$$

By using a change of variable we obtain

$$\begin{aligned}\int_0^{2\pi} dt \log |ce^{i(t-s)} - c_n| e^{imt} dt &= \int_{-s}^{2\pi-s} dt \log |ce^{it} - c_n| e^{im(s+t)} dt \\ &= e^{ims} \int_{-\pi}^{\pi} dt \log |ce^{it} - c_n| e^{imt} dt \\ &= e^{ims} \int_{-\pi}^{\pi} dt \log |c^2 + c_n^2 - 2cc_n \cos(t)|^{1/2} e^{imt} dt\end{aligned}$$

By using an elementary result (see Reference [3]) we conclude that

$$\int_0^{2\pi} dt \log |ce^{i(t-s)} - c_n| e^{imt} dt = \begin{cases} -\frac{\pi}{|m|} \gamma_n^{-|m|} e^{ims} & \text{if } m \neq 0 \\ 2\pi \log c_n & \text{if } m = 0 \end{cases}$$

It follows that

$$a_n(\hat{v}_m, v_k) = \frac{\pi}{|m|} \gamma_n^{-|m|} \delta_{km} cc_n \quad \text{when } m \neq 0 \quad (25)$$

By an analogous calculation we obtain with $\hat{\mu}(x) = \mu(\gamma_n x)$, $x \in \Gamma$:

$$a(\hat{v}_m, \hat{v}_k) = \frac{\pi}{|m|} \delta_{km} c^2 \quad \text{when } m \neq 0 \quad (26)$$

Let now χ_j be the characteristic function of $[t_{j-1/2}, t_{j+1/2}]$ if $0 \leq j \leq n$ and χ_{n+1} be the characteristic function of $[t_0, t_{1/2}] \cup [t_{n+1/2}, t_{n+1}]$. We verify that the complex Fourier series of χ_j is given by

$$\chi_j(t) = \sum_{k=-\infty}^{\infty} c_{jk} e^{ikt} \quad (27)$$

$$c_{jk} = \frac{\sin(k\pi/(n+1))}{k\pi} e^{-ikt_j}, \quad j = 0, 1, \dots, n+1, \quad k \neq 0 \quad (28)$$

$$c_{j0} = \frac{1}{n+1}, \quad j = 0, 1, \dots, n+1 \quad (29)$$

Now we define $\Pi_n e^{imt}$ by

$$\Pi_n e^{imt} = \sum_{j=1}^{n+1} e^{im t_j} \chi_j(t) \quad (30)$$

By replacing (27)–(29) into (30) we obtain from a standard calculation

$$\Pi_n e^{imt} = \frac{n+1}{\pi} \sin\left(\frac{m\pi}{n+1}\right) \sum_{p=-\infty}^{\infty} \frac{(-1)^p}{m+p(n+1)} e^{i(m+p(n+1))t}, \quad 1 \leq m \leq n \quad (31)$$

Now let m be equal to $1, 2, \dots, n$. By using (31) and the definitions of $\hat{\Pi}_n$ and \hat{v}_m we have

$$\hat{\Pi}_n \hat{v}_m(z) = \frac{n+1}{\pi} \sin\left(\frac{m\pi}{n+1}\right) \sum_{p=-\infty}^{\infty} \frac{(-1)^p}{m+p(n+1)} \hat{v}_{m+p(n+1)}(z) \quad (32)$$

for all $z \in \Gamma$.

It suffices to use (25) and (32) to show that if $1 \leq m, k \leq n$, we have

$$a_n(\hat{\Pi}_n \hat{v}_m, \Pi_n v_k) = \left(\frac{n+1}{\pi}\right)^2 \sin^2\left(\frac{m\pi}{n+1}\right) \pi c c_n \sum_{p=-\infty}^{\infty} \frac{\gamma_n^{-|m+p(n+1)|}}{|m+p(n+1)|^3} \delta_{mk} \quad (33)$$

By using (26) instead of (25), together with (32), we obtain

$$a(\hat{\Pi}_n \hat{v}_m, \hat{\Pi}_n \hat{v}_k) = \left(\frac{n+1}{\pi}\right)^2 \sin^2\left(\frac{m\pi}{n+1}\right) \pi c^2 \sum_{p=-\infty}^{\infty} \frac{1}{|m+p(n+1)|^3} \delta_{mk} \quad (34)$$

By setting $\omega_m = \gamma_n \sum_{p \in \mathbb{Z}} \gamma_n^{-|m+p(n+1)|} |m+p(n+1)|^{-3} / \sum_{p \in \mathbb{Z}} |m+p(n+1)|^{-3}$ we obtain for all $1 \leq k, m \leq n$:

$$a_n(\hat{\Pi}_n \hat{v}_m, \Pi_n v_k) = \omega_n a(\hat{\Pi}_n \hat{v}_m, \hat{\Pi}_n \hat{v}_k) \quad (35)$$

The fact that $\hat{\Pi}_n \hat{v}_k$ for $k = 1, 2, \dots, n$ is a basis of X_n which is orthogonal with respect to the scalar product $a(\cdot, \cdot)$ together with (35) proves our lemma. \square

Remark 2.1. It is well known (see Reference [21] for instance) that $a(\cdot, \cdot)$ is a scalar product on $H_0^{-1/2}(\Gamma)$ and the norm $\|\hat{\lambda}\| = a(\hat{\lambda}, \hat{\lambda})^{1/2}$ is equivalent to $\|\hat{\lambda}\|_{-1/2, \Gamma}$. Moreover, from relations (25) and (26) we obtain that $a_n(\hat{v}_m, \hat{v}_k) = \gamma_n^{1-|m|} a(\hat{v}_m, \hat{v}_k)$ for all k and for all $m \neq 0$. Because $\gamma_n > 1$ and $(\hat{v}_m)_{m \in \mathbb{Z}^*}$ is an Hilbertian basis of $H_0^{-1/2}(\Gamma)$, this last equality proves that $a_n(\cdot, \cdot)$ cannot satisfy a inf-sup condition on $X \times Y$.

In order to have a family of curves $(\Gamma^{(n)})_{n=1}^\infty$ which converges to Γ , we choose from now $c_n = \gamma_n c$ with

$$\gamma_n = 1 + \frac{\delta}{n} \quad (36)$$

where δ is a positive number. With this choice we have the following result:

Lemma 2.2. For all $\varepsilon > 0$ there exists $\delta_0 > 0$ such that if $0 < \delta < \delta_0$, then

$$\sup_{\hat{\lambda} \in X_n} \frac{|a_n(\hat{\lambda}, \lambda) - a(\hat{\lambda}, \hat{\lambda})|}{a(\hat{\lambda}, \hat{\lambda})} \leq \varepsilon \quad \text{for all } n \geq 1 \quad (37)$$

where we denote by $\hat{\lambda}$ the function belonging to Y_n and defined by $\hat{\lambda}(z) = \hat{\lambda}(\gamma_n^{-1} z)$, $z \in \Gamma^{(n)}$.

Proof. In Lemma 2.1 we have seen that $\hat{\lambda}_m = \hat{\Pi}_n \hat{v}_m$ are the eigenvectors of (23) corresponding to the eigenvalues ω_m defined in (24). We set $\hat{\lambda}_m(\cdot) = \hat{\lambda}_m(\gamma_n^{-1} \cdot)$. When $\hat{\lambda} \in X_n$, we can write $\hat{\lambda} = \sum_{m=1}^n \alpha_m \hat{\lambda}_m$ because $\hat{\lambda}_m$ is an orthogonal basis of X_n with respect to the scalar product $a(\cdot, \cdot)$. Moreover, we have by using Lemma 2.1 and relation (25):

$$\begin{aligned} |a_n(\hat{\lambda}, \lambda) - a(\hat{\lambda}, \hat{\lambda})| &= \left| \sum_{l,m=1}^n \alpha_m \alpha_l (a_n(\hat{\lambda}_m, \hat{\lambda}_l) - a(\hat{\lambda}_m, \hat{\lambda}_l)) \right| \\ &= \left| \sum_{l,m=1}^n (1 - \omega_m) \alpha_m \alpha_l a(\hat{\lambda}_m, \hat{\lambda}_l) \right| = \left| \sum_{m=1}^n (1 - \omega_m) \alpha_m^2 a(\hat{\lambda}_m, \hat{\lambda}_m) \right| \\ &\leq \max_{1 \leq m \leq n} |1 - \omega_m| a(\hat{\lambda}, \hat{\lambda}) \end{aligned}$$

In order to prove Lemma 2.2, it remains to show that for all $\varepsilon > 0$ there exists $\delta_0 > 0$ such that if $0 < \delta \leq \delta_0$, then we have

$$\max_{1 \leq m \leq n} |1 - \omega_m| \leq \varepsilon, \quad \forall n \geq 1 \quad (38)$$

where ω_m is explicitly given by (24).

Let m be an integer in $[1, n]$ and define $\beta_p = |m + p(n+1)|$ with $p \in \mathbb{Z}$. Because we have the development $(1+x)^{-\beta_p} = 1 - x\beta_p(1+\tilde{x})^{-(1+\beta_p)}$, where $\tilde{x} \in (0, x)$, $x > 0$, we have the existence

of a number $\eta_p \in (0, \delta)$ satisfying

$$\left(1 + \frac{\delta}{n}\right)^{-\beta_p} = 1 - \frac{\delta}{n} \beta_p \left(1 + \frac{\eta_p}{n}\right)^{-(1+\beta_p)} \quad (39)$$

Since $\gamma_n = 1 + \delta/n$, we have $1 - \gamma_n^{-\beta_p} = \delta/n \beta_p (1 + \eta_p/n)^{-(1+\beta_p)}$ and consequently

$$\begin{aligned} \gamma_n - \omega_m &= \gamma_n - \gamma_n \frac{\sum_{p=-\infty}^{\infty} \gamma_n^{-\beta_p} \beta_p^{-3}}{\sum_{p=-\infty}^{\infty} \beta_p^{-3}} = \gamma_n \left(1 - \frac{\sum_{p=-\infty}^{\infty} \gamma_n^{-\beta_p} \beta_p^{-3}}{\sum_{p=-\infty}^{\infty} \beta_p^{-3}}\right) \\ &= \gamma_n \frac{\delta \sum_{p=-\infty}^{\infty} (1 + \frac{\eta_p}{n})^{-(1+\beta_p)} \beta_p^{-2}}{\sum_{p=-\infty}^{\infty} \beta_p^{-3}} \end{aligned}$$

It follows that

$$|\gamma_n - \omega_m| \leq \gamma_n \frac{\delta \sum_{p=-\infty}^{\infty} \beta_p^{-2}}{\sum_{p=-\infty}^{\infty} \beta_p^{-3}} \quad (40)$$

Now we evaluate an upper bound for $\sum_{p=-\infty}^{\infty} \beta_p^{-2}$ and a lower bound for $\sum_{p=-\infty}^{\infty} \beta_p^{-3}$.

• We have

$$\begin{aligned} \sum_{p=-\infty}^{\infty} \beta_p^{-2} &= \sum_{p \geq 0} \frac{1}{(m + p(n+1))^2} + \sum_{p \geq 1} \frac{1}{(p(n+1) - m)^2} \\ &= \frac{1}{(n+1)^2} \left(\sum_{p \geq 0} \frac{1}{(p + m/(n+1))^2} + \sum_{p \geq 1} \frac{1}{(p - m/(n+1))^2} \right) \\ &= \frac{1}{(n+1)^2} \left(\frac{(n+1)^2}{m^2} + \frac{(n+1)^2}{(n+1-m)^2} + \sum_{p \geq 1} \frac{1}{(p + m/(n+1))^2} \right. \\ &\quad \left. + \sum_{p \geq 2} \frac{1}{(p - m/(n+1))^2} \right) \end{aligned}$$

By using the following inequalities:

$$\sum_{p \geq 1} \frac{1}{(p + m/(n+1))^2} \leq \int_{m/(n+1)}^{\infty} \frac{dx}{x^2} \quad \text{and} \quad \sum_{p \geq 2} \frac{1}{(p - m/(n+1))^2} \leq \int_{1-m/(n+1)}^{\infty} \frac{dx}{x^2}$$

we obtain

$$\sum_{p=-\infty}^{\infty} \beta_p^{-2} \leq \frac{1}{m^2} + \frac{1}{(n+1-m)^2} + \frac{1}{n+1} \left(\frac{1}{m} + \frac{1}{n+1-m} \right) \quad (41)$$

- In the same way, we have

$$\sum_{p=-\infty}^{\infty} \beta_p^{-3} = \frac{1}{(n+1)^3} \left(\sum_{p \geq 0} \frac{1}{(p + m/(n+1))^3} + \sum_{p \geq 1} \frac{1}{(p - m/(n+1))^3} \right)$$

and by using the following inequalities:

$$\sum_{p \geq 0} \frac{1}{(p + m/(n+1))^3} \geq \int_{m/(n+1)}^{\infty} \frac{dx}{x^3} \quad \text{and} \quad \sum_{p \geq 1} \frac{1}{(p - m/(n+1))^3} \geq \int_{1-m/(n+1)}^{\infty} \frac{dx}{x^3}$$

we obtain

$$\sum_{p=-\infty}^{\infty} \beta_p^{-3} \geq \frac{1}{2(n+1)} \left(\frac{1}{m^2} + \frac{1}{(n+1-m)^2} \right) \quad (42)$$

Due to (41) and (42) we obtain

$$\frac{\sum_{p=-\infty}^{\infty} \beta_p^{-2}}{\sum_{p=-\infty}^{\infty} \beta_p^{-3}} \leq 2(n+1) \left(1 + \frac{m(n+1-m)}{m^2 + (n+1-m)^2} \right) \quad (43)$$

By setting

$$g(x) = \frac{x(n+1-x)}{x^2 + (n+1-x)^2} = \frac{1}{2} \left(\frac{(n+1)^2}{x^2 + (n+1-x)^2} - 1 \right)$$

we easily show that $\max_{x \in [1, n]} g(x) = g((n+1)/2) \leq \frac{1}{2}$. Consequently, relation (43) implies

$$\frac{\sum_{p=-\infty}^{\infty} \beta_p^{-2}}{\sum_{p=-\infty}^{\infty} \beta_p^{-3}} \leq 3(n+1) \quad (44)$$

By using (40), we finally obtain

$$|\gamma_n - \omega_m| \leq 3\gamma_n \delta \frac{n+1}{n} \leq 6\delta\gamma_n \quad \text{for all } n \geq 1 \quad (45)$$

It suffices to write $\gamma_n = 1 + \delta/n$ to conclude that (38) is true. \square

A corollary of Lemma 2.2 is

Lemma 2.3. By setting $\gamma_n = 1 + \delta/n$, there exists δ_0 and $\alpha > 0$ independent of n such that for $0 < \delta < \delta_0$ we have

$$a_n(\hat{\lambda}, \lambda) \geq \alpha \|\hat{\lambda}\|_{-1/2, \Gamma}^2, \quad \forall \hat{\lambda} \in X_n, \quad \forall n \geq 1 \quad (46)$$

Proof. This is a direct consequence of Lemma 2.2 together with Remark 2.1. \square

Proof of Theorem 2.1. Lemma 2.3 implies an uniform ‘inf-sup’ condition for $\delta \in]0, \delta_0[$, i.e.

$$\inf_{\hat{\lambda} \in X_n} \sup_{\mu \in Y_n} a_n(\hat{\lambda}, \mu) \geq \alpha, \quad n \geq 1 \quad (47)$$

$$\|\hat{\lambda}\|_{-1/2, \Gamma} = 1 \quad \|\mu\|_{-1/2, \Gamma(n)} = 1$$

where α is a positive constant independent of n . It follows (see Reference [7] for instance) that problem (8) has a unique solution $\hat{\zeta}_n \in X_n$. By setting $\hat{\zeta} = \partial u / \partial n$ on Γ , where u is solution of (1), we have seen that for all $\mu \in H_0^{-1/2}(\Gamma^{(n)})$ we obtain $a_n(\hat{\zeta}, \mu) = \langle \mu, R_n u_0 \rangle$, and consequently for all $\mu_n \in Y_n$ we have

$$a_n(\hat{\zeta} - \hat{\zeta}_n, \mu_n) = 0 \quad (48)$$

Let now $\hat{\phi}_n$ be any function of X_n . By using Lemma 2.3 and (48), we have

$$\|\hat{\zeta}_n - \hat{\phi}_n\|_{-1/2, \Gamma} \leq \frac{1}{\alpha} a_n(\hat{\zeta}_n - \hat{\phi}_n, \zeta_n - \phi_n) = \frac{1}{\alpha} a_n(\hat{\zeta} - \hat{\phi}_n, \zeta_n - \phi_n) \quad (49)$$

where we denote by $\zeta_n(z) = \hat{\zeta}(\gamma_n^{-1}z)$ and $\phi_n(z) = \hat{\phi}_n(\gamma_n^{-1}z)$, $\forall z \in \Gamma^{(n)}$.

From inequality (49) and from the uniform continuity of a_n with respect to n (see Reference [18, pp. 34–35]), we deduce there exists a constant C , independent of n , such that

$$\|\hat{\zeta}_n - \hat{\phi}_n\|_{-1/2, \Gamma} \leq C \|\hat{\zeta} - \hat{\phi}_n\|_{-1/2, \Gamma}$$

By using the triangular inequality we finally obtain

$$\|\hat{\zeta} - \hat{\zeta}_n\|_{-1/2, \Gamma} \leq (1 + C) \|\hat{\zeta} - \hat{\phi}_n\|_{-1/2, \Gamma}$$

and consequently

$$\|\hat{\zeta} - \hat{\zeta}_n\|_{-1/2, \Gamma} \leq (1 + C) \min_{\hat{\phi}_n \in X_n} \|\hat{\zeta} - \hat{\phi}_n\|_{-1/2, \Gamma} \quad (50)$$

The proof of Theorem 2.1 is a consequence of the fact that $\overline{\bigcup_{n=1}^{\infty} X_n} = H_0^{-1/2}(\Gamma)$ and from the following estimate (see Reference [15] for instance):

$$\|\hat{\zeta} - P_n \hat{\zeta}\|_{-1/2, \Gamma} \leq \frac{C}{n^{3/2}} \|\hat{\zeta}\|_{1, \Gamma} \quad (51)$$

when P_n is the orthogonal projector of $L^2(\Gamma)$ onto X_n . \square

3. ERRORS ESTIMATES WITH NUMERICAL INTEGRATION

In order to solve the discrete problem (8) in the circular case situation, we will have to build the matrix of coefficients $A_{ij} = a_n(\hat{\phi}_i, \phi_j)$, $1 \leq i, j \leq n$, where $(\hat{\phi}_i)_{i=1}^n$ is a finite element basis of X_n . It follows that we will have to calculate expressions of the form

$$\int_{\Gamma_k^{(n)}} ds_x \int_{\Gamma_l} ds_y G(x, y) = -\frac{cc_n}{2\pi} \int_{t_k-1/2}^{t_k+1/2} ds \int_{t_l-1/2}^{t_l+1/2} dt \log |ce^{it} - c_n e^{is}| \quad (52)$$

since $(\hat{\phi}_i)_{i=1}^n$ are constant on Γ_k . Clearly speaking we take $c_n = \gamma_n c$ with $\gamma_n = 1 + \delta/n$ and for n sufficiently large, it is difficult to numerically evaluate the quantity $\int_{t_k-1/2}^{t_k+1/2} ds \int_{t_l-1/2}^{t_l+1/2} dt \log |ce^{it} - c_n e^{is}|$ when $k = l, l+1$ or $l-1$, because the integrand is almost singular. Remark that we have the same problem when we want to compute the right-hand side of (8). In the following, we give a rule in order to numerically perform this type of integrals by keeping the rate

of accuracy $\mathcal{O}(1/n^{3/2})$ in $H_0^{-1/2}(\Gamma)$. Recall that the length of the interval $[t_{k-1/2}, t_{k+1/2}]$ is h , with $h = 2\pi/(n+1)$, and consider, for a positive integer m , the points in $[0, 2\pi]$ defined by $t_{k,j} = t_{k-1/2} + j(h/m) \bmod 2\pi$, $0 \leq j \leq m$, $1 \leq k \leq n+1$.

For a continuous function $f(s, t)$ on $[0, 2\pi] \times [0, 2\pi]$ (for instance $f(s, t) = \log |ce^{it} - c_n e^{is}|$) we have

$$\int_{t_{k-1/2}}^{t_{k+1/2}} ds \int_{t_{l-1/2}}^{t_{l+1/2}} dt f(s, t) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \int_{t_{ki}}^{t_{k,i+1}} ds \int_{t_{lj}}^{t_{l,j+1}} dt f(s, t)$$

and by changing the variables s and t :

$$\int_{t_{k-1/2}}^{t_{k+1/2}} ds \int_{t_{l-1/2}}^{t_{l+1/2}} f(s, t) = \frac{h^2}{4m^2} \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \int_{-1}^1 d\zeta \int_{-1}^1 d\eta f^{k_i, l_j}(\zeta, \eta) \quad (53)$$

with $f^{k_i, l_j}(\zeta, \eta) = f(t_{k,i} + (\zeta + 1)/2(h/m), t_{l,j} + (\eta + 1)/2(h/m))$.

Let now $J(g) = \sum_{p=1}^r w_p g(\xi_p)$ be a numerical quadrature formula with r integration points $-1 \leq \xi_1 < \xi_2 < \dots < \xi_r \leq 1$ and weights w_1, w_2, \dots, w_r . We assume that this formula is exact for polynomials \mathbb{P}_M of degree M , that is to say for all $g \in \mathbb{P}_M$ we have

$$J(g) = \int_{-1}^1 g(\zeta) d\zeta \quad (54)$$

An approximation of (53) will be

$$\int_{t_{k-1/2}}^{t_{k+1/2}} ds \int_{t_{l-1/2}}^{t_{l+1/2}} dt f(s, t) \approx \frac{h^2}{4m^2} \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \sum_{p=1}^r \sum_{q=1}^r w_p w_q f^{k_i, l_j}(\xi_p, \xi_q) \quad (55)$$

We set $I_{nkl}(f)$ as the left-hand side of (55) and $J_{nkl}(f)$ as its right-hand side. Then it is well known that if $f \in \mathcal{C}^{M+1}(\bar{\Omega}_{kl})$ with $\Omega_{kl} = (t_{k-1/2}, t_{k+1/2}) \times (t_{l-1/2}, t_{l+1/2})$, then we have the following estimate:

$$|I_{nkl}(f) - J_{nkl}(f)| \leq Ch^2 \left(\frac{h}{m}\right)^{M+1} \max_{\alpha_1 + \alpha_2 = M+1} \max_{(s,t) \in \Omega_{kl}} \left| \frac{\partial^{M+1} f}{\partial s^{\alpha_1} \partial t^{\alpha_2}}(s, t) \right| \quad (56)$$

where C is a constant independent of n, m and f but is depending on M .

Now we are able to establish an approximation of the discrete problem (8) by using above quadrature formulas. To this end, we define in a similar way as (2)–(7), for $\hat{\zeta} \in X_n$, $\mu \in Y_n$:

$$N(\hat{\zeta}, \mu)(s, t) = -\frac{cc_n}{2\pi} \log |ce^{it} - c_n e^{is}| \hat{\zeta}(ce^{it}) \mu(c_n e^{is}) \quad (57)$$

$$\tilde{a}_n(\hat{\zeta}, \mu) = \sum_{k,l=1}^{n+1} J_{nkl}(N(\hat{\zeta}, \mu)) \quad (58)$$

$$S(\mu)(s, t) = -\frac{cc_n}{2\pi} \frac{(c_n e^{is} - ce^{it}, e^{it})}{|c_n e^{is} - ce^{it}|^2} u_0(ce^{it}) \mu(c_n e^{is}) \quad (59)$$

$$\tilde{b}_n(\mu) = \sum_{k,l=1}^{n+1} J_{nkl}(S(\mu)) \quad (60)$$

where we denote by (\cdot, \cdot) the scalar product in $\mathbb{R}^2 = \mathbb{C}$, i.e.

$$(c_n e^{is} - c e^{it}, e^{it}) = c_n (\cos t \cos s + \sin t \sin s) - c = c_n \cos(t - s) - c \quad (61)$$

With these notations, the approximate problem with quadrature formula consists in finding $\tilde{\zeta}_n \in X_n$ satisfying

$$\tilde{a}_n(\tilde{\zeta}_n, \mu_n) = \tilde{b}_n(\mu_n), \quad \forall \mu_n \in Y_n \quad (62)$$

Now we are able to establish our main result.

Theorem 3.1. Assume that $\gamma_n = 1 + \delta/n$ where δ is a positive number and $u_0 \in \mathcal{C}^{M+1}(\Gamma)$. Moreover assume that $m = n^\beta$ with $\beta \geq 3/(M+1)$. Then, there exists $\delta_0 > 0$ and $N_0 > 0$ such that for all $\delta \in]0, \delta_0[$, problem (62) has a unique solution $\tilde{\zeta}_n \in X_n$ for any integer $n \geq N_0$. Moreover, if $\partial u / \partial n \in H^1(\Gamma)$, then there exists a constant C independent of n such that

$$\left\| \frac{\partial u}{\partial n} - \tilde{\zeta}_n \right\|_{-1/2, \Gamma} \leq \frac{C}{n^{3/2}}$$

Before proving this result, we first establish:

Lemma 3.1. We assume that $c_n = c(1 + \delta/n)$ with $\delta > 0$ and that the quadrature formula is exact for polynomials of degree M . Then, there exists a constant C independent of n and m such that for all $\hat{\lambda} \in X_n$, $\mu \in Y_n$ we have

$$|a_n(\hat{\lambda}, \mu) - \tilde{a}_n(\hat{\lambda}, \mu)| \leq C m^{-(M+1)} \|\hat{\lambda}\|_{L^2(\Gamma)} \|\mu\|_{L^2(\Gamma^{(n)})} \quad (63)$$

$$|b_n(\mu) - \tilde{b}_n(\mu)| \leq C n m^{-(M+1)} \|\mu\|_{L^2(\Gamma^{(n)})} \quad (64)$$

where $b_n(\mu) = \int_{\Gamma^{(n)}} ds_x \int_{\Gamma} ds_y (\partial G / \partial n_y)(x, y) u_0(y) \mu(x)$.

Proof. In order to prove (62), we begin by remarking that

$$a_n(\hat{\lambda}, \mu) = \sum_{k,l=1}^{n+1} I_{nkl}(N(\hat{\lambda}, \mu))$$

By subtraction with (58) we obtain

$$|a_n(\hat{\lambda}, \mu) - \tilde{a}_n(\hat{\lambda}, \mu)| \leq \sum_{k,l=1}^{n+1} |I_{nkl}(N(\hat{\lambda}, \mu)) - J_{nkl}(N(\hat{\lambda}, \mu))| \quad (65)$$

Now we use the error estimate (56) and we have

$$|a_n(\hat{\lambda}, \mu) - \tilde{a}_n(\hat{\lambda}, \mu)| \leq C h^2 \left(\frac{h}{m} \right)^{M+1} \sum_{k,l=1}^{n+1} \max_{\alpha_1 + \alpha_2 = M+1} \max_{(s,t) \in \Omega_{kl}} \left| \frac{\partial^{M+1} N(\hat{\lambda}, \mu)}{\partial s^{\alpha_1} \partial t^{\alpha_2}}(s, t) \right| \quad (66)$$

Since $\hat{\lambda}(ce^{it})\mu(c_n e^{is})$ is constant on Ω_{kl} , we verify, by derivating $(M+1)$ times the function $\log |ce^{it} - c_n e^{is}|$, that there exists C such that

$$\max_{\alpha_1+\alpha_2=M+1} \max_{(s,t) \in \Omega_{kl}} \left| \frac{\partial^{M+1} N(\hat{\lambda}, \mu)}{\partial s^{\alpha_1} \partial t^{\alpha_2}}(s, t) \right| \leq C n^{M+1} |\hat{\lambda}(ce^{it_l})| |\mu(c_n e^{it_k})| \quad (67)$$

By using a Cauchy–Schwartz inequality together with the fact that the measure of $\Gamma_l^{(n)}$ and Γ_k is of order $1/n$, we finally obtain

$$\begin{aligned} |a_n(\hat{\lambda}, \mu) - \tilde{a}_n(\hat{\lambda}, \mu)| &\leq Ch^2 \left(\frac{h}{m} \right)^{M+1} n^{M+2} \left(\sum_{l=1}^{n+1} |\hat{\lambda}(ce^{it_l})|^2 \right)^{1/2} \left(\sum_{k=1}^{n+1} |\mu(c_n e^{it_k})|^2 \right)^{1/2} \\ &\leq Ch^2 \left(\frac{h}{m} \right)^{M+1} n^{M+3} \|\hat{\lambda}\|_{L^2(\Gamma)} \cdot \|\mu\|_{L^2(\Gamma^{(n)})} \end{aligned} \quad (68)$$

It suffices to see that $h = 2\pi/(n+1)$ in order to prove (63).

In order to prove (64), we remark that for $\mu \in Y_n$:

$$b_n(\mu) - \tilde{b}_n(\mu) = \sum_{k,l=1}^{n+1} (I_{nkl}(S(\mu)) - J_{nkl}(S(\mu)))$$

and we use the same arguments as above after showing that $\max_{(s,t) \in \Omega_{kl}} |(\partial^{M+1}/\partial s^{\alpha_1} \partial t^{\alpha_2})S(\mu)|$ is bounded by $Cn^{M+2}|\mu(c_n e^{it_k})|$. \square

Proof of Theorem 3.1. By using Lemma 2.3 we can conclude there exists δ_0 and $\alpha > 0$ such that for $0 < \delta < \delta_0$ we have

$$a_n(\hat{\lambda}, \lambda) \geq \alpha \|\hat{\lambda}\|_{-1/2, \Gamma}^2, \quad \forall \hat{\lambda} \in X_n, \quad \forall n \geq 1 \quad (69)$$

By using Lemma 3.1 with $m = n^\beta$ and $\beta \geq 3/(M+1)$ we obtain

$$|a_n(\hat{\lambda}, \lambda) - \tilde{a}_n(\hat{\lambda}, \lambda)| \leq \frac{C}{n^3} \|\hat{\lambda}\|_{L^2(\Gamma)}^2, \quad \forall \hat{\lambda} \in X_n, \quad \forall n \geq 1$$

By considering the inverse inequality (see Reference [15] for instance)

$$\|\hat{\lambda}\|_{L^2(\Gamma)} \leq C\sqrt{n} \|\hat{\lambda}\|_{-1/2, \Gamma} \quad (70)$$

we have

$$|a_n(\hat{\lambda}, \lambda) - \tilde{a}_n(\hat{\lambda}, \lambda)| \leq \frac{C}{n^2} \|\hat{\lambda}\|_{-1/2, \Gamma}^2, \quad \forall \hat{\lambda} \in X_n, \quad \forall n \geq 1 \quad (71)$$

By using (69) and (71), we obtain the existence of N_0 such that the following relation is satisfied:

$$\tilde{a}_n(\hat{\lambda}, \lambda) \geq \frac{\alpha}{2} \|\hat{\lambda}\|_{-1/2, \Gamma}^2, \quad \forall \hat{\lambda} \in X_n, \quad \forall n \geq N_0 \quad (72)$$

It follows that for $n \geq N_0$, the sesquilinear form $\tilde{a}_n(., .)$ satisfies an ‘inf–sup’ condition on $X_n \times Y_n$ and, consequently, the existence and uniqueness of solution of (62) is proven.

By a similar calculation we obtain from (64) and (70):

$$|\tilde{b}_n(\mu) - b_n(\mu)| \leq \frac{C}{n^{3/2}} \|\mu\|_{-1/2, \Gamma^{(n)}}, \quad \forall \mu \in Y_n, \quad \forall n \geq 1 \quad (73)$$

Now we are able to prove the error estimate of Theorem 3.1. Actually, we have

$$a_n(\hat{\zeta}_n, \mu_n) = b_n(\mu_n), \quad \forall \mu_n \in Y_n$$

and

$$\tilde{a}_n(\tilde{\zeta}_n, \mu_n) = \tilde{b}_n(\mu_n), \quad \forall \mu_n \in Y_n$$

It follows that for all $\mu_n \in Y_n$:

$$\tilde{a}_n(\hat{\zeta}_n - \tilde{\zeta}_n, \mu_n) = -\tilde{b}_n(\mu_n) + b_n(\mu_n) - a_n(\hat{\zeta}_n, \mu_n) + \tilde{a}_n(\hat{\zeta}_n, \mu_n) \quad (74)$$

By using (72) together with (63), (70), (73) and (74) in which we take $\mu_n = \zeta_n - \tilde{\zeta}_n$, $(\tilde{\zeta}_n'(\cdot) = \tilde{\zeta}_n(\gamma_n^{-1} \cdot))$, we obtain the existence of a constant C such that for $n \geq N_0$:

$$\|\hat{\zeta}_n - \tilde{\zeta}_n\|_{-1/2, \Gamma} \leq C \left(\frac{1}{n^{3/2}} + \frac{1}{n^2} \|\hat{\zeta}_n\|_{-1/2, \Gamma} \right) \quad (75)$$

It remains to use Theorem 2.1 in order to obtain the error estimate of Theorem 3.1. \square

Remark 3.1. By choosing $\beta=1$, that is to say we divide each interval $[t_{k-1/2}, t_{k+1/2}]$ by n parts, it suffices to choose a quadrature formula exact for polynomials of degree $M=2$ in order to keep the rate of accuracy $1/n^{3/2}$ in the norm $H_0^{-1/2}(\Gamma)$. It follows that Simpson quadrature formula is sufficient.

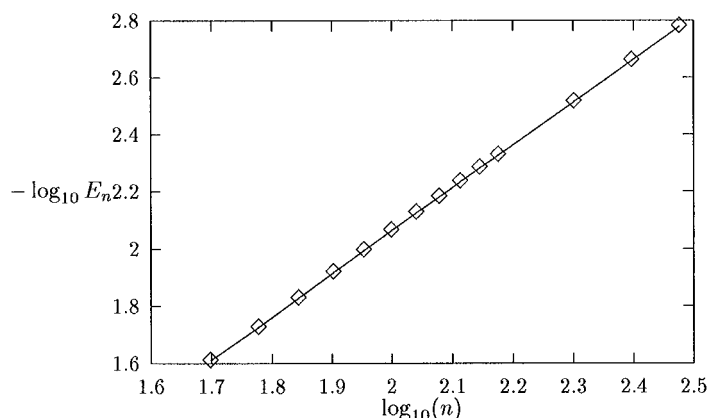
Remark 3.2. If we restrict the quadrature to be a Gauss–Legendre formula (with r points), then we have $M+1=2r$. In this case, the numerical evaluations of all integrals require $(n^{\beta+1}r)^2$ computations. Since Theorem 3.1 says that it suffices to take $\beta \geq 3/2r$ to keep the rate of accuracy $\mathcal{O}(1/n^{3/2})$, we have interest to take β small enough. By choosing $\beta=0.1$ and $r=3/2\beta=15$ we obtain $m \approx 2$ if $n=10^3$ and $m \approx 3$ if $n=10^5$. This result shows that in practice, it suffices to split each intervals $[t_{k-1/2}, t_{k+1/2}]$ into three parts and to use a Gauss–Legendre formula with 15 points.

Remark 3.3. In practice, it suffices to split only the ‘critical’ intervals $[t_{k-1/2}, t_{k+1/2}]$ and $[t_{l-1/2}, t_{l+1/2}]$ for $l=k, k-1, k+1$ when we numerically compute (52).

4. NUMERICAL RESULTS

In the circular case, the practical interest of our method is limited, but we conclude this article by a brief example.

Let Ω be the disk of radius two centred at the origin, and u the function defined on Ω by $u(x_1, x_2) = e^{x_1} \cos(x_2)$, $(x_1, x_2) \in \Omega$. Then u is harmonic on Ω and $\partial u / \partial n|_{\Gamma}$ is the exact solution of (2) when we put $u_0 = u|_{\Gamma}$. On the other hand, setting $u_0 = u|_{\Gamma}$, we can calculate, for any integer n , the function $\tilde{\zeta}_n$ which is the solution of (62) when we split each interval into three

Figure 1. Relative L^2 error.

parts and we use a Gauss–Legendre formula with 15 points. For an integer n , we denote by $E_n = \|\partial u / \partial n - \tilde{\zeta}_n\|_{L^2} / \|\partial u / \partial n\|_{L^2}$, the relative L^2 error between the known exact solution and its approximation by our method.

In Figure 1, we have represented $-\log_{10} E_n$ in function of $\log_{10} n$. We can see that the curve obtained is a straight line with a slope approximatively equal to 1.5. By using the inverse inequalities (cf. 70) in X_n , we can see that this experimental result is a little better than expected (slope equal one) by Theorem 3.1.

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