Numerical Modelling of induction heating for two dimensional geometries.

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Summary

We present both a mathematical model and a numerical method for simulating induction heating processes. The aim is to determine the temperature field when the inductor total current is known. We assume that all the conductor (the inductor and the work piece to be heated by eddy currents) are cylindrical and infinite in one direction. Thus the induction heating problem can be studied in a plane perpendicular to the invariance direction. By considering the temperature θ in the work-piece and the complex potential φ in the whole plane as unknowns, we derive a model which involves the coupling of a non linear diffusion problem for θ with a non linear Helmholtz problem for φ . The numerical scheme we propose consists in a semi-explicit Euler scheme for the time discretization and a coupling between a finite element method with a regular boundary element method for the space discretization.

1 Introduction.

Induction heating in heat generation uses metal conductors with the Joule power. Among various applications of such process in industry there are metal melting, preheating for forging operations, hardening and brasing. The reader may refer for instance to the book [5] for a more detailed description of the process and its technological implications.

The numerical modelling of induction heating has been for many years a research field at the department of mathematics of the Swiss Federal Institut of Technology of Lausanne, with a closed industrial collaboration. Corresponding to various physical configuration of heating, different models have been studied: a one-dimensional model in [3], a 2D cartesian model in [3], [2] and [10], an axisymmetric model in [13] and a three-dimensional model in [12]. The configuration studied here cannot be simulated with the precedent models (except the 3D model) and so a new 2D model has been established in [6] and [9].

In section 2, we present this mathematical model, which involves the coupling of Maxwell's equations with a thermal diffusion problem. We obtain a non linear partial differential formulation which describes the phenomena with two variables: the temperature θ in the work piece and the complex magnetic potential φ in the whole plane. The formulation is non linear since:

- (1) Maxwell's equations lead to a PDE formulation for the variable φ , and for a ferromagnetic conductor, this PDE is non linear.
- (2) The physical properties of the work piece to be heated depends on the temperature. This implies that the diffusion equation is non linear and there appears a non linear coupling with temperature in the electromagnetics phenomena.

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(3) The heat source is the Joule power density which is related to φ through a non linear relation.

We conclude Section 2 with some theoretical results concerning existence, uniqueness and regularity of a solution (φ, θ) . Because we are essentially interested in practical considerations here, the proofs are omitted.

In Section 3, we describe with some details a numerical scheme to approximatively solve the problem. In particular we use a coupling between a finite element method and a Regular Boundary Element Method (RBEM). The RBEM we have used, has recently been analysed in [7], [8] and presents some advantages with respect to classical BEM. The numerical analysis of the scheme is the scope of a next paper.

In Section 4, we present a serie of numerical experiments we have performed on the code developed according to the material presented in this paper. We present also a comparison with the three-dimensional code Thelma which has been validated with in situ measurements (cf [12]).

2 The mathematical model.

In the sequel we shall assume that the work piece Λ_0 is cylindrical and infinite in the x3direction. The inductor Λ is a loop also cylindrical which surrounds the work piece. In figure 1 we have represented a physical situation wich approximately satisfies this assumptions.



FIG. 1 - 3D geometry of the induction process.

Let us assume that a sinusoidal current with a frequency $\frac{\omega}{2\pi}$ and an intensity \mathcal{J} is applied to the inductor. This current induces an eddy current in the work piece and a magnetic field **H** in all the space \mathbb{R}^3 . It follows that the eddy current creates via the Joule effect a heat generation in the work piece. The inductor is water-cooled and we assume its temperature is constant. Assume moreover that the total current is known in the inductor, the main goal is to determine the temperature field in the work piece. Following [11], the main hypothesis in our model is:

 $(\mathcal{H}) \quad \left\{ \begin{array}{l} \text{The current density } \mathbf{j} \text{ does not depend on } x_3 \text{ and remains parallel to } \mathbf{e_3} \\ \text{ in all the conductor domains. Moreover we assume that the total current} \\ \text{ in the work piece is equal to zero.} \end{array} \right.$

We denote by $\Omega_0, \Omega_1, \Omega_2$ the intersection of the work piece and inductors respectively with the plane Ox_1x_2 (see figure 2). We also define $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ and $\Omega' = \mathbb{R}^2 \setminus \overline{\Omega}$.



FIG. 2 - 2D geometry.

The frequency $\frac{\omega}{2\pi}$ is assumed to be sufficiently small with respect to the size of the conductors making it possible to neglect the currents displacement in the conductors. However, this frequency is assumed to be sufficiently large in order to consider the electromagnetic time scale insignificant with respect to the heating time scale. It follows that we will use a quasi-stationary modelling for treating the electromagnetic problem. The commonly used physical fields for such problems are the magnetic induction **b**, the current density **j**, the electric field **e**, the temperature θ . Due to the geometry, we consider x_3 -invariant solutions, i.e all the fields **b**, **j**, **e**, θ are spacially depending only on $x = (x_1, x_2) \in \mathbb{R}^2$.

In the quasi-stationary model, we seek all electromagnetic fields of the form :

$$\mathbf{b}(x,t) = Re(e^{i\omega t}\mathbf{b}(x)), \qquad (2.1)$$

$$\mathbf{j}(x,t) = Re(e^{i\omega t}\mathbf{j}(x)), \qquad (2.2)$$

$$\mathbf{e}(x,t) = Re(e^{i\omega t}\mathbf{e}(x)), \qquad (2.3)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, t denotes the time variable, and $\mathbf{b}(x), \mathbf{j}(x), \mathbf{e}(x)$ are complex-valued functions.

The modelling of induction heating involves a coupling of Maxwell's equations and a thermal diffusion problem.

2.1 The electromagnetic problem.

In this section, following [11], we describe the phenomena with the use of a magnetic potential $\varphi : \mathbb{R}^2 \to \mathbb{C}$.

Let \mathbf{e}_3 denote the unit vector in the x_3 direction. Because of the hypothesis (\mathcal{H}) , we can write $\mathbf{j}(x) = j(x)\mathbf{e}_3$, $\mathbf{b}(x) = (b_1(x), b_2(x))$, $x = (x_1, x_2) \in \mathbb{R}^2$.

From the Maxwell's equation: div $\mathbf{b} = 0$ in \mathbb{R}^3 , we deduce the existence of a scalar field $\varphi : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\mathbf{b}(x) = \mathbf{curl}\varphi(x), \ x \in \mathbb{R}^2, \tag{2.4}$$

where the two dimensional vector curling operator is defined by $\operatorname{curl} \varphi := \left(\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1}\right)$. We also introduce the two dimensional scalar curling operator defined by $\operatorname{curl} \mathbf{A} := \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$. From the Maxwell's equation

$$\operatorname{curl} \frac{\mathbf{b}}{\mu} = \mathbf{j} \quad \text{in } \mathbb{R}^3,$$
 (2.5)

in which we have neglected current displacements we deduce that

$$-\operatorname{div}(\frac{1}{\mu}\nabla\varphi) = \operatorname{curl}(\frac{1}{\mu}\operatorname{curl}\varphi) = j, \qquad (2.6)$$

where the first equality is a trivial identity. The function μ which appears in (2.6) is the magnetic permeability. We assume that μ is a constant equal to μ_0 in Ω' . At the opposite, in the conductor domain, μ is in general a more complicated function. In the most complicated case, when the material is ferromagnetic, then μ can depend on the temperature θ and the magnetic field $|\mathbf{h}|$. We will treat this aspect later.

Carrying on with the study of equation (2.6), we observe that if $x \notin \overline{\Omega}$, then j(x) = 0 and the function φ is harmonic outside the domain $\overline{\Omega}$ since $\mu = \mu_0$. It remains to establish a relationship between j and φ inside the domain Ω . This can be obtained with the Maxwell's equation :

$$\mathbf{curl} \ \mathbf{e} + i\omega \mathbf{b} = 0 \quad in \ \Lambda,$$

which can be written here with the two dimensional curl operator as:

$$\operatorname{curl}(e + i\omega\varphi) = 0 \quad in \ \Omega.$$

This last relationship implies that there are complex constants C_k depending on k, k = 0, 1, 2, such that

$$e + i\omega\varphi = C_k \quad in \ \Omega_k, \ k = 0, 1, 2. \tag{2.7}$$

Moreover, by using the Ohm law $j = \sigma e$, relation (2.7) implies

$$j = \sigma(C_k - i\omega\varphi) \quad in \ \Omega. \tag{2.8}$$

From (\mathcal{H}) we have

$$\int_{\Omega_0} j(x)dx = 0, \quad \mathcal{J} = \int_{\Omega_1} j(x)dx = -\int_{\Omega_2} j(x)dx,$$

and then we deduce from (2.8) that

$$C_0 = C_0(\varphi) = \frac{i\omega \int_{\Omega_0} \sigma(x)\varphi(x)dx}{\int_{\Omega_0} \sigma(x)dx},$$
(2.9)

$$C_1 = C_1(\varphi) = \frac{\mathcal{J} + i\omega \int_{\Omega_1} \sigma(x)\varphi(x)dx}{\int_{\Omega_1} \sigma(x)dx},$$
(2.10)

$$C_2 = C_2(\varphi) = \frac{-\mathcal{J} + i\omega \int_{\Omega_2} \sigma(x)\varphi(x)dx}{\int_{\Omega_2} \sigma(x)dx}.$$
(2.11)

Assuming that the electric conductivity is a constant equal to σ in the inductor and a known function of temperature θ in the work piece, the relation (2.8) can be written:

$$\begin{aligned} j(x) &= J_k + i\omega\sigma(I_k(\varphi) - \varphi) & in \ \Omega_k, \ k = 1, 2, \\ j(x) &= i\omega\sigma(\theta)(I_0(\theta, \varphi) - \varphi) & in \ \Omega_0, \end{aligned}$$

where $J_k := \frac{(-1)^{k+1}}{|\Omega_k|} \mathcal{J}$, $I_0(\theta, \varphi) := \frac{\int_{\Omega_0} \sigma(\theta)\varphi(x)dx}{\int_{\Omega_0} \sigma(\theta)dx}$, $I_k(\varphi) := \frac{\int_{\Omega_k} \varphi}{|\Omega_k|}$, and $|\Omega_k|$ is the measure of Ω_k .

Hence at this point, by using (2.6) to (2.11), we obtain

$$-\Delta\varphi = 0 \qquad \qquad \text{in } \Omega', \qquad (2.12)$$

$$- div(\frac{1}{\mu}\nabla\varphi) + i\omega\sigma(\theta)(\varphi - I_0(\theta,\varphi)) = 0 \quad \text{in } \Omega_0,$$
(2.13)

$$-div(\frac{1}{\mu}\nabla\varphi) + i\omega\sigma(\varphi - I_k(\varphi)) = J_k \qquad \text{in } \Omega_k, \ k = 1, 2.$$
(2.14)

It remains to achieve the electromagnetic model to give boundary conditions. The interface conditions are deduced by the Maxwell's equations considered here in the sense of distributions and by assuming the non existence of surface currents. Namely

$$[\varphi] = \left[\frac{1}{\mu}\frac{\partial\varphi}{\partial n}\right] = 0 \quad on \ \Gamma, \tag{2.15}$$

where $[\psi] := \psi_{ext} - \psi_{int}$ denotes the jump of the function ψ on Γ and $\frac{\partial}{\partial n}$ stands for the outward normal derivative, the vector **n** being the outward unit normal to Γ . Following [4] we have the behaviour at infinity:

$$\varphi(x) = \alpha \log |x| + \beta + O(\frac{1}{|x|}) \quad \text{when } |x| \to \infty,$$
(2.16)

$$\nabla \varphi(x) = \alpha \frac{x}{|x|^2} + O(\frac{1}{|x|^2}) \quad \text{when } |x| \to \infty.$$
(2.17)

Let r > 0, such that the ball B(0, r) contains $\overline{\Omega}$. By integrating equations (2.12)-(2.14) over B(0, r), and using relation (2.15) together with Green formula, we obtain :

$$\int_{B(0,r)} div(\frac{1}{\mu}\nabla\varphi) = 0 = \int_{\partial B(0,r)} \frac{1}{\mu_0} \frac{\partial\varphi}{\partial n}.$$

Then by taking the limit when r tends to infinity and using relation (2.17) we obtain $\alpha = 0$. Moreover, because φ is defined (cf (2.4)) only up to an additive constant, we can choose $\beta = 0$. Thus (2.16) and (2.17) become :

$$\varphi(x) = O(\frac{1}{|x|}) \quad \text{when } |x| \to \infty,$$
(2.18)

$$\nabla \varphi(x) = O(\frac{1}{|x|^2}) \quad \text{when } |x| \to \infty.$$
 (2.19)

Reporting equations (2.12),(2.13),(2.14),(2.15) and (2.18), the electromagnetic problem consists in seeking $\varphi : \mathbb{R}^2 \to \mathbb{C}$ such that:

$$-\Delta \varphi = 0 \qquad \qquad \text{in } \Omega', \qquad (2.20)$$

$$-div(\frac{1}{\mu}\nabla\varphi) + i\omega\sigma(\theta)(\varphi - I_0(\theta,\varphi)) = 0 \quad \text{in } \Omega_0,$$
(2.21)

$$-div(\frac{1}{\mu}\nabla\varphi) + i\omega\sigma(\varphi - I_k(\varphi)) = J_k \qquad \text{in } \Omega_k, \ k = 1, 2,$$
(2.22)

$$[\varphi] = \left[\frac{1}{\mu}\frac{\partial\varphi}{\partial n}\right] = 0 \qquad \text{on } \Gamma, \qquad (2.23)$$

$$\varphi(x) = O(\frac{1}{|x|})$$
 when $|x| \to \infty$. (2.24)

2.2 The thermal problem

The thermal problem consists in seeking the temperature field resulting from the Joule effect. Here we are only interested in the temperature distribution in the work piece Ω_0 during a time T, and we assume that the inductor does not thermally interact with the piece. This assumption can be justified by the fact that, in practical applications the inductor is cooled so that simulating the cooling process would be a complicated task which is out of the scope of this paper. The heat equation is given by

$$C_p(\theta)\frac{\partial\theta}{\partial t} - \operatorname{div}(\lambda(\theta)\nabla\theta) = P, \qquad (2.25)$$

where C_p is the specific heat, λ is the thermal conductivity and P is the Joule power. By using the Ohm law, we obtain

$$P(x,t) = \sigma^{-1}(\theta(x,t))|j(x)e^{i\omega t}|^2$$

Because the frequency $\frac{\omega}{2\pi}$ is relatively high, we can consider that the term $\sigma^{-1}(\theta(x,t))$ is constant with respect to t during a period $\frac{2\pi}{\omega}$. So we approximate P(x,t) by $\frac{\omega}{2\pi}\sigma^{-1}(\theta(x,t))\int_{0}^{\frac{2\pi}{\omega}}|j(x)e^{i\omega t}|^{2}dt$, and with a simple calculation we obtain

$$P(x,t) \simeq \frac{\omega^2}{2} \sigma(\theta(x,t)) |I_0(\theta,\varphi) - \varphi|^2.$$

Hence equation (2.25) becomes

$$C_p(\theta)\frac{\partial\theta}{\partial t} - \operatorname{div}(\lambda(\theta)\nabla\theta) = \frac{\omega^2}{2}\sigma(\theta)|I_0(\theta,\varphi) - \varphi|^2 \quad in \ \Omega_0 \times]0, T[.$$
(2.26)

As far as the boundary conditions are concerned, we shall prescribe a radiation condition on the boundary Ω_0 . Notice that this condition can be simply expressed if the domain is convex, which is assumed in the present application. Assuming a large difference between the temperature on Γ_0 and the temperature in the neighbourhood of the conductor, we have the boundary condition:

$$\lambda(\theta)\frac{\partial\theta}{\partial n} + \alpha\theta^4 = 0 \quad \text{on } \Gamma_0, \qquad (2.27)$$

where α is a radiation coefficient. More precisely, α is the product of the Stephan-Boltzmann constant by the material emissivity coefficient.

2.3 The complete model

The final model consists in seeking the magnetic potential $\varphi : \mathbb{R}^2 \times]0, T[\to \mathbb{C}$ and the temperature field $\theta : \Omega_0 \times]0, T[\to \mathbb{R}$ such that :

$$-\Delta \varphi = 0 \qquad \qquad \text{in } \Omega' \times]0, T[, \qquad (2.28)$$

$$-div(\frac{1}{\mu}\nabla\varphi) + i\omega\sigma(\theta)(\varphi - I_0(\theta,\varphi)) = 0 \qquad \text{in } \Omega_0 \times]0, T[, \qquad (2.29)$$

$$-div(\frac{1}{\mu}\nabla\varphi) + i\omega\sigma(\varphi - I_k(\varphi)) = J_k \qquad \text{in } \Omega_k \times]0, T[, \ k = 1, 2 \qquad (2.30)$$

$$[\varphi] = \left[\frac{1}{\mu}\frac{\partial\varphi}{\partial n}\right] = 0 \qquad \text{on } \Gamma \times]0, T[, \qquad (2.31)$$
$$\varphi(x) = O(\frac{1}{1-1}) \qquad \text{when } |x| \to \infty, \qquad (2.32)$$

$$x) = O(\frac{-}{|x|}) \qquad \qquad \text{when } |x| \to \infty, \tag{2.32}$$

$$C_p(\theta)\frac{\partial\theta}{\partial t} - \operatorname{div}(\lambda(\theta)\nabla\theta) = \frac{\omega^2}{2}\sigma(\theta)|I_0(\theta,\varphi) - \varphi|^2 \quad \text{in } \Omega_0 \times]0, T[,$$
(2.33)

$$\lambda(\theta)\frac{\partial\theta}{\partial n} + \alpha\theta^4 = 0 \qquad \qquad \text{on } \Gamma_0 \times]0, T[, \qquad (2.34)$$

$$\theta(.,0) = \theta_0 \qquad \qquad \text{in } \Omega_0. \tag{2.35}$$

2.4 Some theoretical results.

We begin by recalling some basic notations. If D is an open subset of \mathbb{R}^2 and if $v : D \to \mathbb{C}$ is a complex-valued function, we denote by v^* its complex conjugate and by |v| its modulus. The Sobolev spaces $W^{m,p}(D), H^m(D), L^p(D)$ and the corresponding norms $\|.\|_{m,p,D}, \|.\|_{m,D}, \|.\|_{0,p,D}$ have their usual meanings for real or complex-valued functions. If X is a Banach space, and T is a positif number, we also consider the Banach spaces $W^{m,p}(0,T;X)$ defined by

 $W^{m,p}(0,T;X) = \left\{ \psi : [0,T[\to X, \| \frac{\partial^{\alpha}\psi(t)}{\partial t^{\alpha}} \|_X \in L^p]0, T[, \forall \alpha = 0,..,m. \right\},$ and equipped with their usual norm $\| . \|_{W^{m,p}(0,T;X)}.$

Hereafter, we recall the existence, unicity and regularity results for problem (2.28)-(2.35) (with some modified assumptions) proved in [9] chap. 3 and [6] chap. 2.

The modified assumptions that they have considered are: all the physical functions which appear in (2.28)-(2.35) are constants, except the electrical conductivity σ which depends on temperature and a Dirichlet boundary condition $\theta = \theta_1$ is precribed on the boundary instead of (2.34). In [9] the study is about the evolutive case, when [6] is about the stationary case. We have the following results:

Theorem 2.1 Assume that the conductivity σ is a continuous, bounded and strictly positive function of the temperature. Moreover assume that the initial temperature field θ_0 is in $H^1(\Omega_0)$ and the imposed value on the boundary θ_1 is in $L^2(0,T; L^2(\Omega_0))$. Then there exists at least one solution $(\varphi, \theta) \in L^{\infty}(0,T; H^2_{loc}(\mathbb{R}^2)) \times H^1(0,T; L^2(\Omega_0)) \cap L^2(0,T; H^2(\Omega_0))$ for the evolutive problem. Furthermore there exists at least one stationary solution $(\varphi, \theta) \in$ $H^2_{loc}(\mathbb{R}^2) \times H^2(\Omega_0)$.

Proof. The proofs use the Schauder fixed point theorem, cf. [9] page 59 and [6] page 83.

Theorem 2.2 In addition to the assumptions of theorem 2.1, if we suppose that σ is Lipschitz continuous, then the solution (φ, θ) of the evolutive problem is unique. Moreover, as-

sume that σ is continuously differentiable and σ' is bounded and Lipchitz continuous. Then there exists a positive value \mathcal{J}_0 such that if $\mathcal{J} \leq \mathcal{J}_0$, then the stationary solution is unique.

Proof. cf [9] page 65 and [6] page 87. ■

Remark 2.1 Assume that we have the following symmetries with respect to the axis Ox_2 :

- i) Ω_0 is self-symmetrical,
- ii) Ω_2 is the symmetric of Ω_1 ,

iii) The initial temperature field θ_0 is symmetric, i.e. $\theta_0(x_1, x_2) = \theta_0(-x_1, x_2) \quad \forall x \in \Omega_0$. Moreover assume that the hypotheses of theorem 2.2 are fulfilled. Then we have:

- (1) φ is antisymmetric, i.e. $\varphi(x_1, x_2, t) = -\varphi(-x_1, x_2, t) \quad \forall x \in \mathbb{R}^2, \forall t \in]0, T[,$
- (2) $I_0(\theta,\varphi) \equiv 0,$
- (3) $I_1(\varphi) \equiv -I_2(\varphi).$

This implies that problem (2.28)-(2.35) can be studied only in the half plane $x_1 \ge 0$. Note that the hypotheses i)-iii) correspond to a particular case of physical importance.

3 Numerical solution.

3.1 New formulation of the problem.

In order to give an approximation of problem (2.28)-(2.35), we introduce a new formulation of it. We start with the electromagnetic problem (2.28)-(2.32).

Let $q = \frac{\partial \varphi}{\partial n}|_{\Gamma}^{e}$ denote the exterior trace of the normal derivative of φ on Γ . Following [7],[8], we consider an auxiliary curve $\tilde{\Gamma}$ such that:

$$\begin{cases} \tilde{\Gamma} = \tilde{\Gamma}_0 \cup \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2, \\ \text{for } i=0,1,2, \ \tilde{\Gamma}_i \subset \Omega_i \text{ is a simple, regular and closed curve.} \end{cases}$$
(3.1)

So equation (2.28) with (2.32) can be replaced by the integral system :

$$\int_{\Gamma} q(y)K(x,y)ds_y - \int_{\Gamma} \varphi(y)K_n(x,y)ds_y = 0, \quad \forall x \in \tilde{\Gamma}$$
(3.2)

$$\int_{\Gamma} q(y)ds_y = 0, \tag{3.3}$$

where $K(x,y) = -\frac{1}{2\pi} \log |x-y|$ and $K_{n_y}(x,y) = \frac{\partial}{\partial n_y} K(x,y) = -\frac{1}{2\pi} \frac{(n_y,x-y)}{|x-y|^2}$. Note that the integral equation (3.2) is posed on a finite domain and remark that the integrands which appear are well defined and non singular because Γ and $\tilde{\Gamma}$ are disconnected. This ultimate property is not true with the classical integral formulation in simple and double layer potential which is:

$$\frac{\varphi(x)}{2} + \int_{\Gamma} q(y)K(x,y)ds_y - \int_{\Gamma} \varphi(y)K_n(x,y)ds_y = 0, \quad \forall x \in \Gamma.$$
(3.4)

In the meantime, in the regular formulation (3.2), the auxiliary curve $\tilde{\Gamma}$ may be choosen carrefully in order to obtain the stability of the numerical scheme. This point was be first schown by S. Christiansen (cf [1]) and more later some ingeneers have presented empirical rules given satisfactory results (cf [14], [15] for instance). Recently the autors P. Dreyfuss and J. Rappaz have confirmed this empirical rules by a mathematical proof (cf [7] and [8]). It appears that considering formulation (3.2) instead of (3.4) in order to construct a numerical scheme, can be a satisfactory choice (perhaps more particulary when the curve Γ as corners, see for instance [15]). We will precise how we have to choose the auxiliary curve, in relation with the discretization, in the next section.

Remark next that by integrating equations (2.29),(2.30), and using relation (2.31), we obtain:

$$0 = \int_{\Omega} div(\frac{1}{\mu}\nabla\varphi)dx = \int_{\Gamma} \frac{1}{\mu} \frac{\partial\varphi^{i}}{\partial n} ds = \int_{\Gamma} \frac{1}{\mu_{0}} \frac{\partial\varphi^{e}}{\partial n} ds = \frac{1}{\mu_{0}} \int_{\Gamma} q ds,$$

where φ^i and φ^e denote the internal and external value of φ . Thus, the relation (3.3) is a consequence of equations (2.29)-(2.31) and so it can be removed.

We further consider classical variational formulations of the remaining equations. Consider the test functions $\psi : \Omega \to \mathbb{C}$ and $u : \Omega_0 \to \mathbb{R}$. We multiply equations (2.29),(2.30) by ψ^* and we integrate them over the definition domain. Then, with a simple calculation using Green formula together with relation (2.31) we obtain:

$$\mu_0 \int_{\Omega} \frac{1}{\mu} \nabla \varphi \nabla \psi^* - \int_{\Gamma} q \psi^* + i \omega \mu_0 \Big(\int_{\Omega_0} \sigma(\theta) \varphi \psi^* - I_0 \int_{\Omega_0} \sigma(\theta) \psi^* + \sum_{k=1}^2 \sigma \Big(\int_{\Omega_k} \varphi \psi^* - I_k \int_{\Omega_k} \psi^* \Big) \Big) = \int_{\Omega} f \psi^*,$$
(3.5)

where
$$I_0 = \frac{\int_{\Omega_0} \sigma(\theta)\varphi}{\int_{\Omega_0} \sigma(\theta)},$$
 (3.6)

$$I_k = \frac{\int_{\Omega_k} \varphi}{|\Omega_k|}, \ k = 1, 2, \tag{3.7}$$

and
$$f(x) = \begin{cases} 0 & \text{if } x \in \Omega_0, \\ J_k & \text{if } x \in \Omega_k, \ k = 1, 2. \end{cases}$$

Similarly, equations (2.33),(2.34) are replaced by:

$$\int_{\Omega_0} C_p(\theta) \frac{\partial \theta}{\partial t} u + \int_{\Omega_0} \lambda(\theta) \nabla \theta \nabla u + \int_{\Gamma} \alpha \theta^4 u = \frac{\omega^2}{2} \int_{\Omega_0} \sigma(\theta) |I_0 - \varphi|^2 u$$
(3.8)

At this point we are in position to give a new formulation (\mathcal{P}) of problem (2.28)-(2.35):

 $(\mathcal{P}) \left\{ \begin{array}{l} \text{Let } \tilde{\Gamma} \text{ be an auxiliary curve such as (3.1). We are seeking for a function } q:\\ \Gamma \times]0, T[\to \mathbb{C}, \text{ a function } \varphi: \Omega \times]0, T[\to \mathbb{C}, \text{ a function } \theta: \Omega_0 \times]0, T[\to \mathbb{R}, \text{ and three functions } I_0, I_1, I_2 :]0, T[\to \mathbb{C} \text{ such that } (3.2) \text{ and } (3.5) - (3.8) \text{ are fulfilled.} \end{array} \right.$

Remark 3.1 The new unknowns I_0, I_1, I_2 are being introduced in order to simplify the numerical approximation of the problem.

3.2 Numerical scheme.

In this section, we present a numerical method in order to obtain a good approximation of the solution to be found. The numerical method is obtained after considering three aspects :

- The decupling of equations (3.5)-(3.7) and equation (3.8).
- The time discretization for the unknown θ .
- The space discretization for the unknown φ , q and θ .

Before this, we turn back to the modelisation of the magnetic permeability which we have evocated previously. For the numerical simulation, we consider magnetic permeability of the form:

$$\mu(x) = \begin{cases} & \mu_0 \quad \text{if } x \in \Omega', \\ & \mu(\theta, |\mathbf{h}|) \quad \text{if } x \in \Omega_0, \\ & \mu_1 \quad \text{if } x \in \Omega_k, \ k = 1, 2. \end{cases}$$

So, in order to compute numerically μ in the conductor domain Ω , we must know an approximation of the field $|\mathbf{h}|$. Using equation (2.4), we deduce that μ , φ and $|\mathbf{h}|$ are linked through the following relation:

$$|\mathbf{h}| = \frac{1}{\mu} |\nabla \varphi|. \tag{3.9}$$

We are now interested in replacing the formulation (\mathcal{P}) which is non linear in φ, θ , by an iterative linear formulation $(\mathcal{P}_i)_{i \in \mathbb{N}}$. The idea is to use an semi-explicit Euler scheme for the time discretization and to decouple equations (3.5)-(3.7) and equation (3.8) for each time step. More precisely, we consider the following iterative process:

- 1. At i=0, we start with the data of:
 - an initial field of temperature $\theta^{(0)}$: $\Omega_0 \to \mathbb{R}$.
 - an initial value $\mu^{(0)}$ of the magnetic permeability in Ω_0 .
 - a time step τ .
- 2. For i = 1, 2, ... we construct and resolve the linear problem (\mathcal{P}_i) defined by the following two points a),b) and then we follow the point c):
 - a) seek $q^{(i)}: \Gamma \to \mathbb{C}, \ \varphi^{(i)}: \Omega \to \mathbb{C}, \ I_0^{(i)}, I_1^{(i)}, I_2^{(i)} \in \mathbb{C}, \text{ such that :}$

$$\int_{\tilde{\Gamma}} ds_x \mu^*(x) \int_{\Gamma} q^{(i)}(y) K(x,y) ds_y - \int_{\tilde{\Gamma}} ds_x \mu^*(x) \int_{\Gamma} \varphi^{(i)}(y) K_n(x,y) ds_y = 0, \quad (3.10)$$

$$\mu_0 \int_{\Omega} \frac{1}{\mu^{(i-1)}} \nabla \varphi^{(i)} \nabla \psi^* - \int_{\Gamma} q^{(i)} \psi^* + i\omega \mu_0 \Big(\int_{\Omega_0} \sigma(\theta^{(i-1)}) \varphi^{(i)} \psi^* - I_0^{(i)} \int_{\Omega_0} \sigma(\theta^{(i-1)}) \psi^*$$

$$\sum_{k=0}^{2} e^{-\int_{\Gamma} \sigma(\theta^{(i)}) \psi^*} \int_{\Omega} \sigma(\theta^{(i-1)}) \varphi^{(i)} \psi^* - I_0^{(i)} \int_{\Omega_0} \sigma(\theta^{(i-1)}) \psi^*$$

$$+\sum_{k=1}^{2}\sigma\left(\int_{\Omega_{k}}\varphi^{(i)}\psi^{*}-I_{k}^{(i)}\int_{\Omega_{k}}\psi^{*}\right)\right)=\int_{\Omega}f\psi^{*},$$
(3.11)

$$I_0^{(i)} = \frac{\int_{\Omega_0} \sigma(\theta^{(i-1)})\varphi^{(i)}}{\int_{\Omega_0} \sigma(\theta)},\tag{3.12}$$

$$I_{k}^{(i)} = \frac{\int_{\Omega_{k}} \varphi^{(i)}}{|\Omega_{k}|}, \ k = 1, 2,$$
(3.13)

where $\psi: \Omega \to \mathbb{C}$ and $\mu: \tilde{\Gamma} \to \mathbb{C}$ are any test functions.

b) seek $\theta^{(i)}: \Omega_0 \to \mathbb{R}$ such that:

$$\int_{\Omega_0} C_p(\theta^{(i-1)}) \frac{\theta^{(i)} - \theta^{(i-1)}}{\tau} u + \int_{\Omega_0} \lambda(\theta^{(i-1)}) \nabla \theta^{(i)} \nabla u$$
$$+ \int_{\Gamma} \alpha \theta^{(i-1)^4} u = \frac{\omega^2}{2} \int_{\Omega_0} \sigma(\theta^{(i-1)}) |I_0^{(i)} - \varphi^{(i)}|^2 u, \qquad (3.14)$$

where $u: \Omega_0 \to \mathbb{R}$ is a any test function.

c) compute $\mu^{(i)}: \Omega_0 \to \mathbb{R}$ with the formula:

$$\mu^{(i)} = \mu\left(\theta^{(i)}, \frac{1}{\mu^{(i-1)}} |\nabla\varphi^{(i)}|\right).$$
(3.15)

Hereafter, in order to discretize in space the iterate problems $(\mathcal{P}_i)_{i=1,..}$, we considere a polygonal approximation Ω_h of Ω and a triangular mesh τ_h of $\overline{\Omega}_h$. We next describe a method combining the finite element method and the boundary element method.

Let $K_1, ..., K_{n_t}; e_1, ..., e_{n_e}; x_1, ..., x_{n_v}$ be respectively the triangles, the boundary edges and the vertex of the mesh. For k = 0, 1, 2, we denote by $\Omega_{k,h} = \Omega_k \cap \Omega_h$, the three connected components of the domain Ω_h .

Let V_h and W_h be the \mathbb{P}_1 -finite element spaces defined by:

$$V_h = \left\{ \psi : \overline{\Omega}_h \to \mathbb{C}, \ \psi \text{ continuous on } \overline{\Omega}_h \text{ and } \psi |_K \in \mathbb{P}_1 \text{ for all triangle } K \subset \overline{\Omega}_h \right\},$$

$$W_h = \left\{ u : \overline{\Omega}_{0,h} \to \mathbb{R}, \ u \text{ continuous on } \overline{\Omega}_{0,h} \text{ and } u |_K \in \mathbb{P}_1 \text{ for all triangle } K \subset \overline{\Omega}_{0,h} \right\}.$$

For each $i, 1 < i < n_v$, we denote by ψ_i the function defined on $\overline{\Omega}_h$ such that $\psi_i|_K \in \mathbb{P}_1$, for all triangle K and $\psi_i(x_j) = \delta_{ij}$ for all vertex x_j . The sets $\{\psi_i, i = 1, ..., n_v\}$ and $\{\psi_i, i$ such that $x_i \in \overline{\Omega}_{0,h}\}$ will respectively define a basis of V_h and a basis of W_h .

Then, for the numerical computation of the iterate problems (\mathcal{P}_i) we use test functions $\psi \in V_h$, $u \in W_h$ and we seek $\varphi^{(i)} \in V_h$, $\theta^{(i)} \in W_h$, i = 1, 2... This implies that the functions $\mu^{(i)}$ are constant in each triangle $K \subset \overline{\Omega}_h$.

It remains to discretize the integral equation (3.10). To do this we adopt the regular boundary integral method described and analysed in [6],[7],[8]. We consider the space X_h defined by

$$X_h = \{q : \Gamma_h \to \mathbb{C}, q|_e \in \mathbb{P}_0 \text{ for all edge e of } \Gamma_h.\}$$

where Γ_h is the boundary of Ω_h , and we have to contruct an auxiliary curve Γ_h and a discrete space \tilde{X}_h of functions defined on $\tilde{\Gamma}_h$. We next describe an admissible construction which preserves the stability of the scheme. Let $n_{k,e}$, k = 0, 1, 2, be the number of boundary edges in $\Gamma_{k,h} = \partial \Omega_{k,h}$. For convenience we assume that the edges of Γ_h are oriented and numerated such that $\Gamma_{0,h}$ is defined by covering the $n_{0,e}$ first edges $e_1, ..., e_{n_{0,e}}$, when $\Gamma_{1,h}$ is obtained by covering the next $n_{1,e}$ edges and $\Gamma_{2,h}$ is obtained by covering the last $n_{2,e}$ edges. Let γ be a positive number such that $0 < \gamma < 1$ (for instance choose $\gamma = 0.5$). Remark that each edge $e_i = [x_1^i, x_2^i]$ of Γ_h , is also an edge for an unique triangle $K = (x_1^i, x_2^i, x_3^i)$ of the mesh τ_h and consequently we can define the points $\tilde{x}_i \in \overline{\Omega}_h$ by setting $\tilde{x}_i = (1 - \gamma)x_1^i + \gamma x_3^i$, $i = 1, ..., n_e$. Next we create $n_{0,e}$ auxiliary segments $\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_{n_{0,e}}$ defined by:

$$\tilde{e}_i = \begin{cases} & [\tilde{x}_{n_{0,e}}, \tilde{x}_1] & \text{if } i = n_{0,e}, \\ & [\tilde{x}_i, \tilde{x}_{i+1}] & \text{in the other cases,} \end{cases}$$

and we define the first connected component $\tilde{\Gamma}_{0,h}$ of the auxiliary curve $\tilde{\Gamma}_h$ by setting $\tilde{\Gamma}_h = \bigcup_{i=1}^{n_{0,e}} \tilde{e}_i$ (see figure 3). In the same manner we define the two other connected component $\tilde{\Gamma}_{1,h}$ and $\tilde{\Gamma}_{2,h}$.

Let X_h be the vectorial space defined by

$$\tilde{X}_h = \left\{ \tilde{q} : \tilde{\Gamma}_h \to \mathbb{C}, \ \tilde{q}|_{\tilde{e}} \in \mathbb{P}_0 \text{ for all edge } \tilde{e} \text{ of } \tilde{\Gamma}_h \right\}.$$

Then, in the iterate problems (\mathcal{P}_i) , we will seek $q^{(i)} \in X_h$, $\forall i = 1, 2, ...,$ and we will approximate equation (3.10) by:

$$\int_{\tilde{\Gamma}_h} \tilde{q}(x) ds_x \int_{\Gamma_h} q^{(i)}(y) K(x,y) ds_y - \int_{\tilde{\Gamma}_h} \tilde{q}(x) ds_x \int_{\Gamma_h} \varphi^{(i)}(y) K_n(x,y) ds_y ds_x = 0, \quad \forall \tilde{q} \in \tilde{X}_h.$$
(3.16)

At this point, we have replaced each problem (\mathcal{P}_i) by a linear system which is easy to construct and resolve. Moreover this scheme is stable an convergent (i.e in particular it can



FIG. 3 – Construction of the first connected component $\tilde{\Gamma}_{0,h}$ of the auxiliary curve $\tilde{\Gamma}_{h}$.

be proved that the approximate temperature $\theta^{(i)} \in W_h$ calculated, converge to $\theta(T)$ when τ and h tend to zero). Note however that we need to take some precaution for the numerical evaluation of the integrals appearing in equation (3.16). In fact, the integrands are always regular but they can be almost singular. In the following, we explain how a Gauss-Legendre quadrature formula with a changeable number of integration points can be used.

3.3 Numerical integration for the equation 3.16.

Let $k, l = 1, ..., n_e$, the integrals terms appearing in equation (3.16) are on the form:

$$I_{k,l} = \int_{\tilde{e}_k} \int_{e_l} f(x,y) ds_y ds_x, \qquad (3.17)$$

with two possibilities for f: a) f(x, y) = K(x, y) or b) $f(x, y) = K_n(x, y)\psi_j(y)$, $j = 1, ..., n_v$. It's easy to show that in the two cases $f \in C^{\infty}(\tilde{e}_k \times e_l)$ but its maximum (and the maximum of its derivatives) can tends to infinity when n tends to infinity. However, if we denote by $\delta_{k,l}$ the distance between the edges \tilde{e}_k and e_l , and if m is an integer, then we have

$$\max_{(x,y)\in\tilde{e}_k\times e_l} |f^{(m)}(x,y)| \le C_m \delta_{k,l}^{-\alpha(m)},$$
(3.18)

where $\alpha(m) = m$ in the case a), $\alpha(m) = m + 1$ in the case b), and C_m is a constante depending only on m.

In practice, we have to replace $I_{k,l}$ by a numerical approximation $I_{k,l}$. Moreover, in order to not deterior the stability and the consistency of the scheme by the numerical integration, we need have (cf. [7]):

$$\left|I_{k,l} - \tilde{I}_{k,l}\right| \le C n_e^{-4}.$$
(3.19)

A possible way to construct a formula $I_{k,l}$ satisfying (3.19) is to consider a composite Gauss-Legendre quadrature formula with an adaptative subdivision of the edges. We can see in [7] how a formula of this kind can be built using the property (3.18). However, it is more advantageous to use a formula with a changeable number of integration points without any subdivision of the edges.

For instance, the numerical evaluation $\tilde{I}_{k,l}$ of $I_{k,l}$ can be performed by the use of a Gauss-Legendre quadrature formula with $m \times m$ points of integration, if we take

$$m \ge \frac{3}{2} \frac{\log n_e}{\log(1 + \tilde{\delta}_{k,l})},$$

where $\tilde{\delta}_{k,l}$ is the scaled distance between the edges \tilde{e}_k and e_l defined by $\tilde{\delta}_{k,l} = \frac{2\delta_{k,l}|\Gamma|}{n_e}$. The justification is based on the properties of analicity of the kernel f (see [8]).

4 Numerical experiment.

In this section we present a series of numerical experiments we have performed on the code developed according to the material presented in this paper. We present also a comparison with the three dimensional induction code Thelma which has been validated by industrial experiments (cf. [12]).

4.1 Geometrical and physical data.

We have treated the induction heating of a cylindical work piece with radius 20 mm, surrounded by a cylindrical inductor with radius 2 mm (cf fig. 4). For the three dimensional comparison, we have constructed an elevated mesh, with a height of 40 mm.

We have considered different types of materials for the inductor and the work piece. More precisely, the physical characteristics (in MKSA unit system) of the inductor were the following: $\sigma = 3.10^7$, $\mu = \mu_0 = 4\pi .10^{-7}$, $\omega = 2\pi .10^4$, $\mathcal{J} = 3.10^3$, when for the work piece we have considered the same stainless steel (non magnetic case) and magnetic steel (magnetic case) as in [2]. In this cases, the electrical conductivity σ , the thermal conductivity λ and the specific heat C_p are depending on the temperature θ . We have:

$$\sigma(\theta) = (a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3)^{-1}, \text{ with } a_0 = 4.9659.10^{-7}, a_1 = 8.4121.10^{-10}, \\ a_2 = -3.7246.10^{-13}, a_3 = -6.1960.10^{-17}, \\ \lambda(\theta) = b_0 + b_1\theta, \text{ with } b_0 = 11.215, b_1 = 0.014087, \\ C_p(\theta) = c_0 + c_1\theta, \text{ with } c_0 = 2.81398.10^6, c_1 = 7.8052.10^2.$$

The radiation coefficient α appearing in (2.27) and (3.14) is set equal to $0.8 \times 5.67 \times 10^{-8}$. The steels differ in magnetic property. For the stainless steel the magnetic permeability μ is assumed to be μ_0 , when for the magnetic steel we have the following dependency on the temperature θ and on the intensity of magnetic field $|\mathbf{h}|$:

$$\mu = \begin{cases} & \mu_0 \left(1 + \sqrt{(1033 - \theta)/740} \cdot \frac{2000}{1 + |\mathbf{h}|/200} \right) & \text{if } \theta < 1033, \\ & \mu_0 & \text{if } \theta \ge 1033. \end{cases}$$

4.2 Numerical data.

It is well known that in electromagnetic computation the mesh has to take into account the skin effect. This effect can be estimated with the skin depth $\delta = \sqrt{\frac{1}{\mu\sigma\omega}}$, which is the

depth at which the value of the magnetic field decays by a factor e^{-1} .

In the application we have treated, the skin depth is of order 1mm for the inductor and 5 mm for the work piece in the case of stainless steel. The situation is even more intricated in the case of magnetic steel, since the skin depth strongly varies with the temperature and with the intensity of magnetic field, ranging from about 0.015 mm up to almost 1 mm.

We can state that in the work piece the whole eddy current (and consequently the whole heating source) is concentrated in a thin layer along the boundary near the inductor, and heat is then carried by diffusion inside the piece.

Consequently, for the domain Ω_0 , we have constructed a mesh more refined along the boundary near the inductor but not too coarse in the other parts (cf fig 4). In the ferromagnetic case, as the skin depth varies considerably during the heating, it is difficult to choose a priori a good mesh. In this case it would be adequate to use an adaptative mesh. This remains a possible direction for future research.



FIG. 4 – Mesh of the section. The points $P_1, ..., P_5$ are used as control points for the comparison with the 3D code.

4.3 Numerical Comparisons.

We consider here the case of stainless steel. The comparison between the temperature simulated with 2D and 3D code is showed in figure 5.



FIG. 5 – Evolution of the temperature in the control points (P_1 is at the top and left, and P_4 at the bottom and right). The temperature (in degree K) is plotted in function of time (in second).

5 Conclusions.

We have described, in detail a mathematical model to simulate induction heating processes in industry. The described numerical methods were tested with realistic data given by the industrial environment. The comparisons between the produced code and a threedimensionnal code, proves the efficiency of our two-dimensionnal approach, even when the work piece is not very long in the x_3 direction. Moreover the computations are quite fast. In consequence we think that our numerical model is of practical importance. However, some improvements remain to be done in order to apply the method efficiently to large scale problems involving ferromagnetic behaviour. Thus, a possible direction for research would be the study of an adaptative refinement mesh procedure.

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Références

 S. Christiansen, Condition number of matrices derived from two classes of integral equations, Math. Meth. in the Appl. Sci. 1981; 3: 364-392.

- [2] S. Clain, J. Rappaz, M. Swierkosz, R. Touzani, Numerical modeling of induction heating for two-dimensional geometries, Math. Models and Meth. in Appl. Sci. 1993; 3(6):805-822.
- [3] S. Clain, Analyse mathématique et numérique d'un modèle de chauffage par induction, Ph.D. thesis nº1240, Ecole Polytechnique Fédérale de Lausanne, 1994.
- [4] M. Crouzeix, J. Descloux, A Bidimensional electromagnetic problem, SIAM J. Math. Anal. 1990; 21(3):577-592.
- [5] E. J. Davies, Conduction and induction heating, P. Peregrinus Etd., London, 1990.
- [6] P. Dreyfuss, Analyse numérique d'une méthode intégrale frontière sans singularité -Application à l'électromagnétisme, Ph.D. thesis n°2049, Ecole Polytechnique Fédérale de Lausanne, 1999.
- [7] P. Dreyfuss, J. Rappaz, Numerical analysis of a non singular boundary integral method. Part I: the circular case, Math. Meth. in the Appl. Sci. 2001; 24:847-863.
- [8] P. Dreyfuss, J. Rappaz, Numerical analysis of a non singular boundary integral method. Part II: the general case, to appear in Math. Meth. in the Appl. Sci., 2001.
- C. Parietti, Modélisation mathématique et analyse numérique d'un problème de chauffage électromagnétique, Ph.D. thesis n°1838, Ecole Polytechnique Fédérale de Lausanne, 1998.
- [10] C. Parietti, J. Rappaz, A quasi-static two-dimensional induction problem, I: Modelling and analysis, Math. Models and Meth. in Appl. Sciences, 1999; 9(9):1333-1350.
- [11] J. Rappaz, R. Touzani, On a two-dimensional magnetohydrodynamic problem. Part I. Modelling and analysis, in Math. Modeling and Num. Anal., 1991; 26(2):347-364.
- [12] J. Rappaz, M. Swierkosz, C. Trophime, Un modèle mathématique et numérique pour un logiciel de simulation tridimentionnelle d'induction électromagnétique, internal report, Ecole Polytechnique Fédérale de Lausanne, 1999.
- [13] J. Rappaz, M. Swierkosz, Mathematical modelling and numerical simulation of induction heating processes, Appl. Math. and Comp. Sci., 1996; 6(2):207-221.
- [14] J.L Wearing, M.A. Sheikh, A regular indirect boundary element method for thermal analysis, Int. Jour. of Num. Methods in Engineering, 1988; 25:495-515.
- [15] J.L Wearing, O. Bettahar, The analysis of plate bending problems using the regular direct boundary element method, Eng. Anal. with Boundary Elements, 1996; 16:261-271.