1. Existence and uniqueness

Exercise 1.1. Proof of the local existence and uniqueness theorem Suppose $f : U \to \mathbb{R}^d$ is continuous and suppose that there exists $K > 0$ such that, for any $(t, x) \in U$ and $(t, y) \in U$

$$\|f(t,x) - f(t,y)\| \leq K \|x - y\|.$$

1. Let $\varphi : (t_0 - \alpha, t_0 + \alpha) \to \mathbb{R}^d$ be a continuous function. Prove that φ is a solution of (PC) if and only if, for any $t \in (t_0 - \alpha, t_0 + \alpha)$, we have $(t, \varphi(t)) \in U$ and

$$\varphi(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s))ds.$$

2. For $r > 0$, we denote by $\overline{B}(x_0, R)$ the closed ball of center x_0 and of radius $R > 0$. We denote by $C^0([t_0-\alpha, t_0+\alpha], \overline{B}(x_0, R))$ the space of continuous function $[t_0-\alpha, t_0+\alpha] \to \overline{B}(x_0, R)$ which is endowed with the norm defined by $\|\varphi\|_\infty = \sup_{t\in[t_0-\alpha,t_0+\alpha]} \|\varphi(t)\|$. Recall that the space $(C^0([t_0 - \alpha, t_0 + \alpha], \overline{B}(x_0, R)), \|.\|_\infty)$ is complete. Find $\alpha > 0$ and $R > 0$ such that the following map is well defined and contracting for $\|.\|_\infty$.

$$\begin{array}{rccl} T: & C^0([t_0 - \alpha, t_0 + \alpha], \overline{B}(x_0, R)) & \to & C^0([t_0 - \alpha, t_0 + \alpha], \overline{B}(x_0, R)) \\ & \varphi & \mapsto & (t \mapsto x_0 + \int_{t_0}^{t} f(s, \varphi(s))ds) \end{array}$$

3. Prove the local existence and uniqueness theorem using the Banach fixed point theorem. We denote by φ this solution which is defined on $[t_0 - \alpha, t_0 + \alpha]$.
4. Prove that, for any solution ψ of (PC) such that $t_0 \in J \subset [t_0 - \alpha, t_0 + \alpha]$, we have $\psi = \varphi_{|J}$.

Exercise 1.2. Example of computation of maximal solutions In this exercise, we want to compute the maximal solutions of the following differential equation.

$$(E)\ y' = y^2 x.$$

1. Find the constant solutions of (E).
2. Prove that, if φ is a nonconstant solution of (E), then the function φ has either positive or negative values.
3. Find the maximal solutions of (E).

Exercise 1.3. Another example of computation of maximal solutions In this exercise, we want to find the maximal solutions of the following differential equation.

$$(E)\ y' + xy = x^3 y^3 e^{x^2}.$$

1. Prove that, if $\varphi : I \to \mathbb{R}$ is a solution of (E) which is not the zero function, then

$$\forall x \in I, \varphi(x) \neq 0.$$

2. Find the maximal solutions of (E).

Exercise 1.4. Necessity of the locally Lipschitz hypothesis We are interested in the following differential equation

$$(E)\ y' = \sqrt{|y|}.$$

1. Can we apply the existence and uniqueness theorem to this equation ? Why ?

2. Prove that there are multiple solutions to the following Cauchy problem.

$$\begin{cases} y' = \sqrt{|y|} \\ y(0) = 0 \end{cases}.$$

Hint : use the function φ defined on $\mathbb{R}$ by

$$\begin{array}{ll} \forall t > 0, & \varphi(t) = \frac{1}{4}t^2 \\ \forall t \leq 0, & \varphi(t) = 0 \end{array}.$$

Exercise 1.5. Proof of the global existence and uniqueness theorem Denote by $\mathcal{F} = (\varphi_k)_{k \in K}$ a family of maps φ_k which are each defined on an open interval I_k of $\mathbb{R}$ with values in $\mathbb{R}^d$. We say that the family $\mathcal{F}$ is compatible if

$$\begin{cases} \bigcap_{k \in K} I_k \neq \emptyset \\ \forall k,\ k' \in K, \varphi_{k|I_k \cap I_{k'}} = \varphi_{k'|I_k \cap I_{k'}}. \end{cases}$$

1. Prove that any compatible family $\mathcal{F}$ has a unique upper bound $\varphi : I \to \mathbb{R}^d$, where I is an open interval, for the order relation $\preceq$, *i.e.*
   — For any member ψ of the family $\mathcal{F}$, $\psi \preceq \varphi$.
   — For any map $\varphi' : I' \to \mathbb{R}^d$, where I' is an open interval, such that

   $$\forall \psi \in \mathcal{F}, \psi \preceq \varphi',$$

   we have $\varphi \preceq \varphi'$.
   Prove that this upper bound is locally the restriction of a member of $\mathcal{F}$
2. Prove that the family $\mathcal{F}$ of solutions of (PC) defined on an open interval is a compatible family.
3. Prove the global existence and uniqueness theorem.