2. Interval of existence of maximal solutions

Exercise 2.1. Proof of the theorem about compact subsets and maximal solutions. We fix a compact subset $A \subset U$, a solution $\varphi : (\alpha, \beta) \to \mathbb{R}^d$ of (S) and $t_0 \in (\alpha, \beta)$. We suppose that

$$\forall t \in (t_0, \beta),\ (t, \varphi(t)) \in A.$$

1. Prove that $\beta < +\infty$ and that there exists $l \in \mathbb{R}^d$ and a sequence $(t_n)_{n\in\mathbb{N}}$ in $(t_0, \beta)^{\mathbb{N}}$ such that $\lim_{n\to+\infty} t_n = \beta$ and
$$\lim_{n\to+\infty} \varphi(t_n) = l.$$
2. Let $\tilde{\varphi} : (\alpha, \beta] \to \mathbb{R}^d$ be the map whose restriction to (α, β) is φ and such that $\tilde{\varphi}(\beta) = l$. Prove that $\tilde{\varphi}$ is a solution of (S).
3. Complete the proof of the theorem.

Exercise 2.2 What can be said about the interval of existence of maximal solutions of the following differential equations ?

$$\begin{array}{ll}(E_1) & y'(t) = \cos(e^t y(t)) \\ (E_2) & y'(t) = \cos(y(t))t^2 + \arctan(y(t)).\end{array}$$

Exercise 2.3 Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a C^1 map. Suppose there exists $R > 0$ suxh that, for any point $x \in \mathbb{R}^d$ with $\|x\| \geq R$, $f(x)$ vanishes. Prove that any solution of the differential equation $y' = f(y)$ is bounded. What can be deduced about the interval of definition of maximal solutions ?

Exercise 2.4

1. Prove that maximal solutions of the following differential equations are defined on $\mathbb{R}$.
$$\begin{cases} x' = xy^2 \\ y' = -yx^2 \end{cases}$$
2. Prove that maximal solutions of the following differential equations are defined on a neighbourhood of $+\infty$.
$$\begin{cases} x' = -xy^2 \\ y' = -yx^2 \end{cases}$$

Exercise 2.5 We denote by (S) the following differential system

$$\begin{cases} x' = x^3 + y^2 x \\ y' = y^3 + yx^2 \end{cases}.$$

Let

$$\begin{array}{lcl} I & \to & \mathbb{R} \\ t & \mapsto & (x(t), y(t)) \end{array}$$

be a maximal solution of (S) and let $f(t) = x(t)^2 + y(t)^2$.

1. Find a differential equation (E) satisfied by f.
2. What can we deduce about the interval I ?

Exercise 2.6. Proof of the linear Gronwall lemma.

1. Let $f : I \to \mathbb{R}$ a differentiable function defined on an open interval I. Let $t_0 \in I$. Assume there exists $A > 0$ and $B \geq 0$ such that
$$\forall t \in [t_0, +\infty) \cap I,\ f'(t) \leq Af(t) + B.$$
Prove that
$$\forall t \in [t_0, +\infty) \cap I,\ f(t) \leq f(t_0)e^{(t-t_0)A} + \frac{B}{A}(e^{(t-t_0)A} - 1).$$

2. Prove the linear Gronwall lemma in the case where, for any $t \in [t_0, +\infty) \cap I$, $\|\varphi(t)\| > 0$. Hint : take $\psi(t) = \|\varphi(t)\|$ and use the first question.
3. Prove the linear Gronwall lemma in the general case.

Exercise 2.7 What can be said about the interval of existence of maximal solutions of the following differential systems ?

$$(S_1)\ \begin{cases} x' = \arctan(y)x + 2y + \cos(t) \\ y' = \sin(y)x \end{cases}$$

$$(S_2)\ \begin{cases} x' = -3x + 2e^{-x^2}y + \frac{1}{1+y^2} \\ y' = -\cos(ty)x + \frac{y}{1+x+x^2} + \arctan(t) \end{cases}.$$

Exercise 2.8. Linear differential systems.

1. Prove that linear differential systems satisfy the hypothesis of the existence and uniqueness theorem.
2. Prove the theorem about the interval of definition of maximal solutions of linear differential systems.

Exercise 2.9. Continuity with respect to the initial solution For $\epsilon > 0$, let

$$K_\epsilon = \left\{(t,p) \in [t_-, t_+] \times \mathbb{R}^d,\ \left\|p - \varphi_{t_0}^t(x_0)\right\| \le \epsilon\right\}.$$

1. Prove that there exists $\epsilon > 0$ such that $K_\epsilon \subset U$.
2. In what follows, we fix $\epsilon > 0$ such that the above property is satisfied. Prove that there exists $k > 0$ such that, for any points $(t,p) \in K_\epsilon$ and $(t,p') \in K_\epsilon$,

$$\|f(t,p) - f(t,p')\| \le k\,\|p - p'\|.$$

3. Fix $\delta > 0$ such that $\delta e^{k(t_+ - t_-)} < \epsilon$. Let

$$V = \left\{y_0 \in \mathbb{R}^d,\ \|y_0 - x_0\| < \delta\right\}.$$

   Prove the theorem of continuity with respect to the initial condition with $k > 0$ and V defined as above.

Exercise 2.10 What can be said about the interval of existence of maximal solutions y_{max} of the differential equation $y' = (\cos(ty) + 2)y^2$ which are defined at 0 in terms of $y_0 = y_{max}(0)$.

Exercise 2.11 Let U be an open set of $\mathbb{R}^2$ and $f : U \to \mathbb{R}^2$ be a continuous function which is locally lipschitzian with respect to the phase variable. Denote by (E) the differential equation $x' = f(t,x)$. Let $\varphi : (\alpha, \beta) \to \mathbb{R}$ be a subsolution of (E) and $\rho : (\alpha, \beta) \to \mathbb{R}$ be a solution of (E). Let $t_0 \in (\alpha, \beta)$.

1. Suppose that $\varphi(t_0) \le \rho(t_0)$ and that, for any $t \in [t_0, \beta)$, $\varphi'(t) < f(t, \varphi(t))$. Suppose for a contradiction that there exists $t_1 > t_0$ such that $\varphi(t_1) > \rho(t_1)$. We let

$$\tau = \inf\{t \in [t_0, \beta),\ \varphi(t) > \rho(t)\}.$$

   (a) Prove that $\varphi'(\tau) < \rho'(\tau)$.
   (b) Find a contradiction. What can we conclude ?
2. Now, we just suppose that $\varphi(t_0) \le \rho(t_0)$ and that φ is a subsolution of (E). Deduce from the first part of the exercise that, for any $t \in [t_0, \beta)$, $\varphi(t) \le \rho(t)$ by using, for $\epsilon > 0$, $f_\epsilon(t,x) = f(t,x) + \epsilon$.

Exercise 2.12 Adapt Exercise 2.11 to provide a proof of the nonlinear higher-dimensional Gronwall lemma.