3. Qualitative study of differential systems

Exercise 3.1. Flow of a vector field. Prove Proposition 3.1 about properties of the flow of a vector field.

Exercise 3.2. Orbits of a vector field We denote by φ the flow of a complete vector field $X : U \to \mathbb{R}^d$, where $U \subset \mathbb{R}^d$ is an open set.

1. Prove that two different orbits of the vector field X are disjoint.
2. Fix a point $p \in U$. Let $G_p = \{t \in \mathbb{R},\ \varphi^t(x) = x\}$. Prove that the group G_p is a closed subgroup of $\mathbb{R}$.
3. Prove that any subgroup of $\mathbb{R}$ is either $\{0\}$, or of the form $a\mathbb{Z}$ with $a > 0$ or dense in $\mathbb{R}$.
4. Prove the proposition about the different shapes of an orbit.

Exercise 3.3. Qualitative study in dimension 1 and 2

1. Let $c \in \mathbb{R}$. Without computation, describe the qualitative behaviour of the solutions of the differential equation
$$y' = y(1-y) - c.$$
2. Draw the phase portraits of the following differential systems.
$$\begin{cases} x' = y - \sin(x) \\ y' = \frac{x}{4} - y \end{cases}$$
$$\begin{cases} x' = x - \frac{1}{4}x^2 - xy \\ y' = 2y - y^2 - xy \end{cases} .$$
$$\begin{cases} x' = x - y^2 \\ y' = (y-1-x)^2 \end{cases}$$

Exercise 3.4. Lotka-Volterra differential system. Let α, β, γ and δ be positive real numbers. We want to study the following differential system (which represents the evolution of populations of predators and preys) :

$$(LV) \quad \begin{cases} x' = \alpha x - \beta xy \\ y' = -\gamma y + \delta xy \end{cases} .$$

1. For any real number x and y, compute $\varphi^t(x, 0)$ and $\varphi^t(0, y)$, where φ is the flow associated with (LV).
2. Prove that any orbit which contains a point of
$$C = \{(x, y), x > 0, y > 0\}$$
is contained in C.
3. Draw the restriction to $\overline{C}$ of the phase portrait of the differential system (LV).
4. Let
$$C_1 = \left\{x < \tfrac{\gamma}{\delta}, y > \tfrac{\alpha}{\beta}\right\} \quad C_2 = \left\{x < \tfrac{\gamma}{\delta}, y < \tfrac{\alpha}{\beta}\right\} \quad C_3 = \left\{x > \tfrac{\gamma}{\delta}, y < \tfrac{\alpha}{\beta}\right\} \quad C_4 = \left\{x > \tfrac{\gamma}{\delta}, y > \tfrac{\alpha}{\beta}\right\}$$
$$D_1 = \left\{x = \tfrac{\gamma}{\delta}, y > \tfrac{\alpha}{\beta}\right\}.$$
Prove that, for any solution ψ of (LV) such that $\psi(0) \in D_1$, $\psi(t)$ visits successively the domains C_1, C_2, C_3, C_4 and comes back to D_1.
5. Find a non-constant first integral of (LV) defined on C of the form $I(x, y) = F(x) + G(y)$. Deduce that any solution of (LV) which is contained in C is defined on $\mathbb{R}$.

6. Prove that the orbits of (LV) which are contained in C are periodic. Hint : Prove that the first integral I is one-to-one on D_1.

Exercise 3.5. Change of coordinates. Let $X : U \to \mathbb{R}^d$ be a locally lipschitzian vector field defined on an open set $U \subset \mathbb{R}^d$. Let $h : U \to h(U) = V \subset \mathbb{R}^d$ be a C^2-diffeomorphism.

1. Prove that the vector field h_*X is locally lipschitzian on V.
2. Denote by ψ the flow of the vector field h_*X. Prove that, at any point (t,x) where this relation makes sense
$$\psi^t(x) = h \circ \varphi^t \circ h^{-1}(x).$$

Exercise 3.6. Proof of the flow box theorem Let x_0 be a zero of the C^2 vector field $X : U \to \mathbb{R}^d$. Suppose that $X(x_0) \neq 0$ and let us denote by φ the flow of the vector field X.

1. Denote by Σ the hyperplane $\Sigma = X(x_0)^{\perp} = \{u \in \mathbb{R}^d,\ < u, X(x_0) >= 0\}$, where $< .,. >$ is the standard scalar product on $\mathbb{R}^d$. Prove that the map
$$\begin{array}{cccc} g: & \mathbb{R}\times\Sigma & \to & \mathbb{R}^d \\ & (t,u) & \mapsto & \varphi^t(x_0+u) \end{array}$$
defines a diffeomorphism from an open neighbourhood W_1 of $(0,0)$ in $\mathbb{R}\times\Sigma$ to an open neighbourhood W_2 of x_0 in $\mathbb{R}^d$.
2. Let $h = g^{-1} : W_2 \to W_1$. Prove that, for any point (t_0,u) in W_2 and for t sufficiently close to 0, we have
$$h \circ \varphi^t \circ h^{-1}(t_0,u) = (t_0+t,u).$$
3. Deduce that h_*X is a constant vector field.

Exercise 3.7. Stability for linear systems. Let A be a $d\times d$ matrix. Let us denote by (S) the differential system $x' = Ax$.

1. Prove that the origin is not stable for (S) if A has an eigenvalue with positive real part.
2. By decomposing a solution in reduction base for A, prove that the origin is stable for (S) if and only if all the eigenvalues of A have a negative real part.
3. In which case is the origin stable ?

Exercise 3.8. Let $g : [0,+\infty[\to \mathbb{R}$ be the function defined by
$$\left\{\begin{array}{ccl} g(r) & = & r^3\sin(\frac{1}{r}) \text{ si } r>0 \\ g(0) & = & 0 \end{array}\right. .$$
We consider the differential system given in polar coordinates by
$$\left\{\begin{array}{ccl} \theta' & = & 1 \\ r' & = & g(r) \end{array}\right. .$$

1. Why can we apply the existence and uniqueness theorem to this system ?
2. Discuss the stability at the origin.

Exercise 3.9. Discuss the stability at the origin for each of those differential systems.
$$\left\{\begin{array}{ccl} x' & = & -2x-3xy \\ y' & = & 2x^2-y \end{array}\right. .\left\{\begin{array}{ccl} x' & = & 2y(z-1) \\ y' & = & -x(z-1) \\ z' & = & -z^3 \end{array}\right. .$$
Hint : Look for a Lyapunov function of the form ax^2+by^2 or $ax^2+by^2+cz^2$.

Exercise 3.10. Let $V : \mathbb{R}^3 \to \mathbb{R}$ be a C^∞-function which has a strict minimum at a point $x_0 \in \mathbb{R}^3$ and which has no critical point on a neighbourhood of the point x_0 (the gradient of V does not vanish on a neighbourhood of the point x_0). Discuss the stability of the equilibrium point x_0 for each of the following differential systems (where $f > 0$).
$$x' = -\nabla V(x).$$

$$x'' + fx' + \nabla V(x) = 0.$$
$$x'' + \nabla V(x) = 0.$$

Exercise 3.11. Lyapunov functions and stability. Let $X : U \to \mathbb{R}^d$ be a vector field on an open set U of $\mathbb{R}^d$. Let p_0 be an equilibrium point for the differential system $x' = X(x)$ and suppose there exists a weak Lyapunov function $L : V \to \mathbb{R}$ for this equilibrium point. Let $m = L(p_0)$. Take $\epsilon_0 > 0$ such that the closed ball $\overline{B}(p_0, \epsilon_0)$ is a subset of V.

1. In this first part of the exercise, we want to prove that the equilibrium point p_0 is stable.
   (a) Let $\epsilon \in (0, \epsilon_0)$. Prove that there exists $c > 0$ such that
   $$N_c = \left\{x \in \overline{B}(p_0, \epsilon_0),\ L(x) < m + c\right\} \subset B(p_0, \epsilon).$$
   (b) Prove that there exists $\eta > 0$ such that $B(p_0, \eta) \subset N_c \cap B(p_0, \epsilon)$.
   (c) Let $\psi : I \to \mathbb{R}^d$ be a maximal solution of the system $x' = X(x)$ such that $0 \in I$ and $\psi(0) \in B(p_0, \eta)$. Prove that, for any $t \in I \cap [0, +\infty)$,
   $$\psi(t) \in B(p_0, \epsilon_0) \cap N_c.$$
   What can we say about the interval I ?
   (d) Prove that the equilibrium point p_0 is stable.
2. In this question, suppose that L is a Lyapunov function. We want to prove that the equilibrium point p_0 is asymptotically stable. By the first part of the exercise, we can choose $\eta > 0$ and $c > 0$ such that
   $$B(p_0, \eta) \subset N_c \cap B(p_0, \epsilon_0) \subset B(p_0, \frac{\epsilon_0}{2}).$$
   Let $\psi : I \to \mathbb{R}^d$ be a maximal solution of $x' = X(x)$ with $0 \in I$ and $\psi(0) \in B(p_0, \eta) \setminus \{p_0\}$. By the first part of the exercise $[0, +\infty) \subset I$.
   (a) Let $(t_n)_n$ be a sequence of real numbers such that $\lim_{n\to+\infty} t_n = +\infty$. Prove that there exists a subsequence $(t_{\varphi(n)})_{n\in\mathbb{N}}$ of this sequence a point $x_\infty \in \mathbb{R}^d$ such that $\lim_{n\to+\infty} \psi(t_{\varphi(n)}) = x_\infty$.
   (b) Prove that $x_\infty = p_0$ and that $\lim_{t\to+\infty} L(\psi(t)) = L(p_0)$.
   (c) Conclude.

Exercise 3.12. In those two cases, compute the singular points of the vector field (*i.e.* the points where those vector fields vanish). What can we say about their stability by computing the differential of those vector fields ?

1. $$X_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -x^2 - y \\ -x + y^2 \end{pmatrix}.$$
2. $$X_2\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -1 + x^2 + y^2 \\ -x \end{pmatrix}.$$

Exercise 3.13. Stability by linearization.

1. Let A be a $d \times d$ matrix with real entries. Suppose the eigenvalues of A have negative real parts. For any $x \in \mathbb{R}^d$ we let
   $$L(x) = \int_0^{+\infty} \left\|e^{sA}x\right\|^2 ds.$$
   Prove that L is well-defined and is a Lyapunov function for the differential system $x' = Ax$.

2. Let $X : U \to \mathbb{R}^d$ be a vector field and p_0 be an equilibrium point of the differential system $x' = X(x)$. Suppose A is the jacobian matrix of X at p_0 and let L' be the map defined on $\mathbb{R}^d$ by
$$L(x) = \int_0^{+\infty} \left\| e^{sA}(x - x_0) \right\|^2 ds.$$
Prove that the restriction of L to a small neighbourhood of the point p_0 is a Lyapunov function for the equilibrium point p_0 (which is hence asymptotically stable).

Exercise 3.14. Consider the following differential system.
$$\left\{ \begin{array}{ccc} x' & = & -y - x(x^2 + y^2) \\ y' & = & x - y(x^2 + y^2) \end{array} \right. .$$
Find the equilibrium points and study their stablity. Without solving the system : what can we say about the interval of existence of the solutions ?

Exercise 3.15. We want to study the differential equation (of the pendulum) $\theta'' = -\sin(\theta)$.

1. Find a (non-constant) first integral of this equation.
2. Study the interval of existence of maximal solutions.
3. Draw the phase portrait associated to this equation.
4. Study the stability of the equilibrium points.
5. Same questions for the equation $\theta'' = -f\theta' - \sin(\theta)$, avec $f > 0$. Look for a Lyapunov function instead of a first integral.

Exercise 3.16. We want to study the following differential system (which describes the evolution of two populations in interaction).
$$\left\{ \begin{array}{ccc} x' & = & -xy - 2x^2 + 2x \\ y' & = & -y^2 - \frac{1}{2}xy + y \end{array} \right. .$$

1. Find the equilibrium points and study their stability.
2. Prove that the set $C = \{x \geq 0, y \geq 0\}$ is invariant under the flow.
3. Draw the restriction to C of this phase portrait.
4. Discuss the time of existence of the solutions contained in C.