Nonlinear ordinary differential equations

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Chapitre 1

Existence and uniqueness

Let $d \geq 1$ and U be an open subset of $\mathbb{R} \times \mathbb{R}^d$. Let f be a map

$$f : \begin{array}{ccc} U & \to & \mathbb{R}^d \\ (t,x) & \mapsto & f(t,x) \end{array}.$$

We call t the time variable and x the phase variable. We want to study the differential system

$$(S)\ x' = f(t,x).$$

Observe that an order $k \geq 2$ differential system $x^{(k)} = g(t, x, x', \dots, x^{(k-1)})$ is equivalent to an order 1 system as it is equivalent to the order 1 system

$$\begin{pmatrix} x_0' \\ x_1' \\ \vdots \\ x_{k-2}' \\ x_{k-1}' \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k-1} \\ g(t, x_0, x_1, \dots, x_{k-1}) \end{pmatrix}.$$

Hence there is no loss of generality when we only consider order 1 differential systems.

1.1 Local existence and uniqueness theorem

Definition 1.1. *We call* solution *of (S) any differentiable map $\varphi : I \to \mathbb{R}^d$, where $I \subset \mathbb{R}$ is an interval with nonempty interior, such that, for any $t \in I$*

1. $(t, x(t)) \in U$.
2. $x'(t) = f(t, x(t))$.

The existence and uniqueness theorem we will state relies on a hypothesis on the map f which is given by the following definition. We denote by $\|.\|$ the Euclidean norm on $\mathbb{R}^d$

Definition 1.2. *The map f is* locally Lipschitz with respect to the phase variable *if, for any point (t_0, x_0) in U, there exists $K > 0$, a neighbourhood I of t_0 in $\mathbb{R}$ and a neighbourhood V of x_0 in $\mathbb{R}^d$ such that*

$$\begin{cases} I \times V \subset U \\ \forall t \in I, \forall x \in V, \forall y \in V, \|f(t,x) - f(t,y)\| \leq K \|x - y\| \end{cases}.$$

Notice that, if f is a C^1 function, then it is locally Lipschitz with respect to the phase variable by the mean value inequality.

It is now possible to state the existence and uniqueness theorem. We fix a point (t_0, x_0) in U.

Theorem 1.3. *Suppose the map f is continuous and locally Lipschitz with respect to the phase variable. Then there exists $\alpha > 0$ and a unique solution φ of (S) defined on $(t_0 - \alpha, t_0 + \alpha)$ such that $\varphi(t_0) = x_0$. Moreover, for any solution $\psi : J \to \mathbb{R}^d$ such that $t_0 \in J \subset (t_0 - \alpha, t_0 + \alpha)$ and $\psi(t_0) = x_0$, we have $\psi = \varphi_{|J}$.*

Exercise 1.1 provides a proof of this theorem.

Observe that the second statement implies the uniqueness of the solution on $]t_0 - \alpha, t_0 + \alpha[$.

The first statement can be rephrased in the following way. There exists a unique solution defined on $]t_0 - \alpha, t_0 + \alpha[$ to the *Cauchy problem*

$$(PC) \begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}.$$

1.2 Maximal solutions-Global existence and uniqueness theorem

In order to have a global existence and uniqueness theorem about solutions of (S), we consider the solutions which are defined on the biggest intervals possible. We call them the maximal solutions of (S). To define this notion properly, we need the following partial order relation.

Definition 1.4 (The restriction order relation). *Let $\varphi_1 : I_1 \to \mathbb{R}^d$ and $\varphi_2 : I_2 \to \mathbb{R}^d$ be maps defined on intervals I_1 and I_2 of $\mathbb{R}$. We say that φ_1 is a restriction of φ_1 and we write $\varphi_1 \preceq \varphi_2$ if $I_1 \subset I_2$ and, for any t in I_1, $\varphi_1(t) = \varphi_2(t)$.*

We can now define the notion of maximal solution of (S).

Definition 1.5. *A map φ is a maximal solution of (S) if it is a solution of (S) and if, for any solution ψ of (S) such that $\varphi \preceq \psi$, we have $\varphi = \psi$.*

Theorem 1.6 (Global existence and uniqueness theorem). *Suppose the maps f is continuous and locally Lipschitz with respect to the phase variable. Then there exists a unique solution to the Cauchy problem (PC). Its interval of definition is open and any solution of (PC) is a restriction of this maximal solution.*

In Exercises 1.2 and 1.3, there are examples of computations of maximal solutions.

In Exercise 1.4, we see that this theorem is not true without the locally Lipschitz hypothesis. However, if we just assume that f is continuous, a theorem by Peano asserts that there always exist a solution to a Cauchy problem. However, this solution might not be unique, as Exercise 1.4 shows.

Exercise 1.5 provides a proof of this global existence and uniqueness theorem.

Chapitre 2

Interval of existence

Let U be an open set of $\mathbb{R} \times \mathbb{R}^d$. Let $f : U \to \mathbb{R}^d$ be a map which is continuous and locally Lipschitz with respect to the phase variable, so that the existence and uniqueness theorem applies. In this section, we want to obtain information about the intervals of definition of maximal solutions of

$$(S)\ x' = f(t,x).$$

2.1 Maximal solutions and compact subsets

Theorem 2.1. *Let A be a compact subset of U. For any maximal solution $\varphi : I \to \mathbb{R}^d$ of (S) and any time $t_0 \in I$, there exists $t_1 \in I$ with $t_1 > t_0$ such that $(t_1, \varphi(t_1)) \notin A$ and there exists $t_2 \in I$ with $t_2 < t_0$ such that $(t_2, \varphi(t_2)) \notin A$.*

In particular, a solution $\psi : J \to \mathbb{R}^d$ of (S) such that, for any $t \in (t_0, +\infty) \cap J$, $(t, \psi(t)) \in A$ is not a maximal solution. If $U = \mathbb{R} \times \mathbb{R}^d$ and $\varphi : (a,b) \to \mathbb{R}^d$ is a maximal solution of (S) with $b < +\infty$, then $\varphi([a,b))$ is not contained in any compact subset of $\mathbb{R}^d$.

Exercise 2.1 provides a proof of this theorem. Exercises 2.2, 2.3, 2.4 and 2.5 are examples of situations where this theorem can be used.

2.2 Linear Gronwall lemma

In most books, the following statement is simply called the "Gronwall lemma". However, we will see later a more general version of this lemma which we will call the "Gronwall lemma". That is why we call this statement the "linear Gronwall lemma". It is a very useful statement : in combination with the theorem of the first section, it enables us to obtain information about the interval of existence of a maximal solution.

Lemma 2.2 (The linear Gronwall lemma). *Let $\varphi : I \to \mathbb{R}^d$ be a differentiable function on an open interval I and let $t_0 \in I$. Suppose there exists $A > 0$ and $B \geq 0$ such that*

$$\forall t \in [t_0, +\infty) \cap I, \|\varphi'(t)\| \leq A\|\varphi(t)\| + B.$$

Denote by $\rho : \mathbb{R} \to \mathbb{R}$ any solution of the differential equation $y' = Ay + B$ such that $\rho(t_0) \geq \|\varphi(t_0)\|$. Then

$$\forall t \in [t_0, +\infty) \cap I, \|\varphi(t)\| \leq \rho(t).$$

Remarks :

1. Of course, it is possible to compute ρ explicitly. Indeed, the function $\rho : \mathbb{R} \to \mathbb{R}$ is solution of $y' = Ay + B$ if and only if

   $$\forall t \in \mathbb{R},\ (e^{-tA}\rho(t))' = Be^{-tA}.$$

   Integrating between t_0 and t, this is equivalent to

   $$\forall t \in \mathbb{R},\ e^{-tA}\rho(t) - e^{-t_0A}\rho(t_0) = \frac{B}{A}e^{-t_0A} - \frac{B}{A}e^{-tA}.$$

   This can be rewritten

   $$\forall t \in \mathbb{R},\ \rho(t) = \rho(t_0)e^{(t-t_0)A} + \frac{B}{A}(e^{(t-t_0)A} - 1).$$

   Hence the conclusion of the linear Gronwall lemma can be written

   $$\|\varphi(t)\| \leq \|\varphi(t_0)\| e^{(t-t_0)A} + \frac{B}{A}(e^{(t-t_0)A} - 1).$$

2. This lemma gives estimates only in the case $t \geq t_0$. To obtain estimates in the case $t \leq t_0$, apply the lemma to $t \mapsto \varphi(-t)$.

This lemma is proved in Exercise 2.6 and Exercise 2.7 gives basic examples of situations where this lemma is useful.

2.3 Consequences of the linear Gronwall lemma

2.3.1 Linear differential sytems

We denote by $M_d(\mathbb{R})$ the space of matrices $d \times d$ with real coefficients. Let I be an open interval of $\mathbb{R}$. Let $A : I \to M_d(\mathbb{R})$ and $B : I \to \mathbb{R}^d$ be continuous maps. Let

$$(S)\ x'(t) = A(t)x(t) + B(t).$$

Such a differential system is called a linear differential system. The following theorem tells us that maximal solutions of (S) are defined on the biggest possible interval : I.

Theorem 2.3 (Interval of existence of solutions of linear systems). *Maximal solutions of (S) are defined on I.*

2.3.2 Continuity with respect to the initial condition

In this subsection, U denotes an open set of $\mathbb{R} \times \mathbb{R}^d$ and $f : U \to \mathbb{R}^d$ denotes a continuous map which is locally Lipschitzian with respect to the phase variable. For a point (t_0, x_0) in U, we denote by

$$\begin{array}{rcl} I_{t_0,x_0} & \to & \mathbb{R}^d \\ t & \mapsto & \varphi_{t_0}^t(x_0) \end{array}$$

the maximal solution of the Cauchy problem

$$\left\{ \begin{array}{rcl} x' & = & f(t,x) \\ x(t_0) & = & x_0 \end{array} \right. .$$

Proposition 2.4 (Dependence with respect to the intial condition). *Fix a point (t_0, x_0) in U. Let $[t_-, t_+]$ be an interval contained in I_{t_0,x_0} which contains t_0. Then there exists a neighbourhood V of x_0 and $k > 0$ such that, for any $y_0 \in V$, $[t_-, t_+] \subset I_{t_0,y_0}$ and*

$$\forall t \in [t_-, t_+],\ \left\|\varphi_{t_0}^t(y_0) - \varphi_{t_0}^t(x_0)\right\| \leq \|x_0 - y_0\|\, e^{k|t-t_0|}.$$

The first part of this proposition indicates that, in a sense, the interval of definition of a maximal solution depends continuously on the initial condition. The second part implies that the map $(t, x_0) \mapsto \varphi_{t_0}^t(x_0)$ is continuous where it is defined. To prove it, use the above inequality and the mean value inequality.

Another way to see this inequality is the following : if you have an uncertainty about the initial condition, this uncertainty grows at most exponentially with time. Unfortunately, most of the time, this uncertainty will indeed grow exponentially. In a sense, this explains why it is not possible to predict the weather for many days.

This proposition is proved during Exercise 2.9.

2.4 Dependence of solutions on parameters

As usual, U denotes an open set of $\mathbb{R} \times \mathbb{R}^d$. Let $d' > 0$ be an integer and Λ be an open set (of parameters) in $\mathbb{R}^{d'}$.

Let

$$\begin{array}{rrcl} f: & U \times \Lambda & \to & \mathbb{R}^d \\ & ((t,x),\lambda) & \mapsto & f_\lambda(t,x) \end{array}$$

be a continuous function which is loccaly Lipschitzian with respect to the phase variable x. The latter condition means that, for any point $((t_0, x_0), \lambda_0)$ in $U \times \Lambda$, there exist a neighbourhood I of t_0 in $\mathbb{R}$, V of x_0 in $\mathbb{R}^d$ and W of λ_0 in Λ as well as $k > 0$ such that $I \times V \subset U$ and

$$\forall \lambda \in W,\ \forall t \in I,\ \forall x, x' \in V,\ \|f(t,x) - f(t,x')\| \leq k \|x - x'\|.$$

Observe that the Lipschitz constant k does not depend on the parameter λ in W.

For a point $((t_0, x_0), \lambda)$ of $U \times \Lambda$, we denote by

$$\begin{array}{rcl} I_{t_0,x_0,\lambda} & \to & \mathbb{R}^d \\ t & \mapsto & \varphi_{t_0,\lambda}^t(x_0) \end{array}$$

the maximal solution of the Cauchy problem

$$\left\{ \begin{array}{rcl} x' & = & f_\lambda(t,x) \\ x(t_0) & = & x_0 \end{array} \right. .$$

Theorem 2.5 (Dependence of the solution on parameters)**.** *Suppose the map f is a C^k map, where $k \geq 0$ is an integer. Fix a point $((t_0, x_0), \lambda_0)$ of $U \times \Lambda$ and a closed interval $[t_-, t_+] \subset I_{t_0,x_0,\lambda_0}$ which contains t_0. There exists a neighbourhood $W \subset U \times \Lambda$ of the point $((t_0, x_0), \lambda_0)$ such that, for any point $((t_1, x_1), \lambda)$ in W, $[t_-, t_+] \subset I_{t_1,x_1,\lambda}$ and the map*

$$\begin{array}{rcl} [t_-, t_+] \times W & \to & \mathbb{R}^d \\ (t, ((x_1, t_1), \lambda)) & \mapsto & \varphi^t_{t_1,\lambda}(x_1) \end{array}$$

is a C^k map.

We will not prove this theorem. In the case $k = 0$, this theorem can be proved using the Gronwall lemma, by a method which is similar to the section about the continuity with respect to the initial condition.

2.5 Subsolutions and supersolutions. Nonlinear Gronwall lemmas.

We come back to a setting without parameter. In the beginning of this section, we restrict ourself to a one-dimensional setting. Hence U denotes an open set of $\mathbb{R} \times \mathbb{R}$ and $f : U \to \mathbb{R}$ is a continuous function which is locally Lipschitzian with respect to the phase variable. We are interested in the following differential equation

$$(E)\ x' = f(t, x).$$

Definition 2.6 (Sub and supersolutions)**.** *Let $\varphi : I \to \mathbb{R}$ be a differentiable function which is defined on an interval I with nonempty interior. The function φ is said to be a* subsolution *of (E) if, for any $t \in I$, $(t, \varphi(t)) \in U$ and*

$$\varphi'(t) \leq f(t, \varphi(t)).$$

The function φ is said to be a supersolution *of (E) if, for any $t \in I$, $(t, \varphi(t)) \in U$ and*

$$\varphi'(t) \geq f(t, \varphi(t)).$$

The following theorem is a first version of a nonlinear Gronwall lemma.

Theorem 2.7. *Let* $\varphi : (\alpha, \beta) \to \mathbb{R}$ *be a function and* $\rho : (\alpha, \beta) \to \mathbb{R}$ *be a solution of* (E) *and fix* $t_0 \in I$.

1. *Suppose the function* φ *is a subsolution of* (E).
   (a) *Suppose* $\varphi(t_0) \leq \rho(t_0)$. *Then*
   $$\forall t \in [t_0, \beta), \varphi(t) \leq \rho(t).$$
   (b) *Suppose* $\varphi(t_0) \geq \rho(t_0)$. *Then*
   $$\forall t \in (\alpha, t_0], \varphi(t) \geq \rho(t).$$
2. *Suppose the function* φ *is a supersolution of* (E).
   (a) *Suppose* $\varphi(t_0) \geq \rho(t_0)$. *Then*
   $$\forall t \in [t_0, \beta), \varphi(t) \geq \rho(t).$$
   (b) *Suppose* $\varphi(t_0) \leq \rho(t_0)$. *Then*
   $$\forall t \in (\alpha, t_0], \varphi(t) \leq \rho(t).$$

Intuition : Suppose that φ is a subsolution and that $\varphi(t_0) = \rho(t_0)$. Suppose further that $\varphi'(t_0) < f(t_0, \varphi(t_0))$. Then $\varphi'(t_0) < f(t_0, \rho(t_0)) = \rho'(t_0)$. Hence, for $t > t_0$ close to t_0, $\rho(t) > \varphi(t)$, and, for $t < t_0$ close to t_0, $\rho(t) < \varphi(t)$. Thsi remark can help to remember the good inequalities in the above proposition. This remark is also central to the proof of this theorem.

We use this theorem in Exercise 2.10 and we prove it in Exercise 2.11.

There also exists a higher-dimensional version of the nonlinear Gronwall lemma which we state now.

Theorem 2.8. *Let* $x : (\alpha, \beta) \to \mathbb{R}^d$ *be a differentiable map and* $\rho : (\alpha, \beta) \to \mathbb{R}$ *be a solution of the differential equation* (E). *Let* $t_0 \in I$. *Suppose that*

$$\forall t \in I, \begin{cases} (t, \|x(t)\|) \in U \\ \|x'(t)\| \leq f(t, \|x(t)\|) \end{cases}.$$

If $\|x(t_0)\| \leq \rho(t_0)$, *then, for any* $t \in [t_0, \beta)$, $\|x(t)\| \leq \rho(t)$.

Here also, if we want estimates for $t \leq t_0$, we need to apply the theorem to $t \mapsto x(-t)$. The proof of this theorem is quite similar to the proof of the theorem about sub and supersolutions (see Exercise 2.12).

Chapitre 3

Qualitative study of differential systems

In this chapter, we will only look at *autonomous* differential systems, *i.e.* systems which do not depend on the time t. Notice that the study of a nonautonomous differential system $x' = f(t, x)$ is equivalent to the following autonomous system

$$\left\{ \begin{array}{rcl} x' & = & f(\tau, x) \\ \tau' & = & 1 \end{array} \right. .$$

Denote by U an open subset of $\mathbb{R}^d$ and $X : U \to \mathbb{R}^d$ be a locally lipschitzian (hence continuous) map. Such a map is called a vector field on U. Such a map can be visualized as a vector $X(p)$ attached to each point p of U. We are interested in the differential system

$$(S)\ x' = X(x).$$

To simplify the exposition, we suppose that all the maximal solutions of (S) are defined on the whole real line $\mathbb{R}$: we say that the vector field X is *complete*. Nevertheless, the definitions and theorems of this section can be adapted to the case where a vector field is not complete.

3.1 The flow of a vector field

Let $x_0 \in U$. We denote by

$$\begin{array}{rcl} \mathbb{R} & \to & \mathbb{R}^d \\ t & \mapsto & \varphi^t(x_0) \end{array}$$

the unique maximal solution of (S) which is equal to 0 when $t = 0$. The map φ is called the flow of the vector field X.

Proposition 3.1 (Properties of the flow)**.**

1. *Let $t_0 \in \mathbb{R}$ and $x_0 \in U$. The map*

   $$\begin{array}{rcl} \mathbb{R} & \to & \mathbb{R}^d \\ t & \mapsto & \varphi^{t-t_0}(x_0) \end{array}$$

   is the maximal solution of (S) which is equal to x_0 when $t = t_0$.
2. $\varphi^0 = Id_U$.
3. *For any t_1, t_2 in $\mathbb{R}$, $\varphi^{t_1} \circ \varphi^{t_2} = \varphi^{t_1+t_2}$.*
4. *Suppose X is a C^k map, with $k \geq 0$. Then, for any $t \in \mathbb{R}$, the map $\varphi^t : U \to U$ is a C^k-diffeomorphism (a homeomorphism if $k = 0$).*

3.2 Orbits of a vector field

Definition 3.2. *We call orbit of the vector field X the image of $\mathbb{R}$ under a map of the form $t \mapsto \varphi^t(x_0)$, where $x_0 \in U$.*

Let us start with a direct application of the existence and uniqueness theorem.

Proposition 3.3. *Two distinct orbits of X are disjoint.*

Topologically, there exist only three topological types of orbits.

Proposition 3.4 (Possible shapes of orbits)**.** *An orbit of a vector field X is*

1. *either reduced to a point $\{p\}$. In this case, $X(p) = 0$ and the point p is called an equilibrium point of the system.*
2. *or it is a simple closed loop. In this case, we say that the orbit is periodic and there exists $T > 0$ such that, for any point p of the orbit, $\varphi^T(p) = p$.*
3. *or it is the image of $\mathbb{R}$ under a continuous one-to-one map.*

Those two propositions are proved in Exercise 3.2.

3.3 Phase portraits

We call phase portrait of the differential system (S) the partition of U into orbits. In the case where the dimension d is equal to 2, as two distinct orbits are disjoint, we represent those orbits in the same plane.

There can be many different methods to draw a phase portrait : we introduce now a standard method which can be efficient to draw most phase portraits.

Method to draw a phase portrait in the plane

1. Draw the isocline $x' = 0$, *i.e.* the set of points of the plane where the vector field X is vertical (or where the first component of the vector field X vanishes).
2. Draw the isocline $y' = 0$, *i.e.* the set of points of the plane where the vector field X is horizontal (or where the second component of the vector field X vanishes).
3. Find the equilibrium points of the system which lie at the intersection of the two above isoclines.
4. Draw the vector field on each isocline and on each component of the complement of the isoclines.
5. Draw orbits of the differential system.

In Exercise 3.3, phase portraits are drawn according to this method.

The following notion is useful to draw a phase portrait more accurately (see Exercise 3.4).

Definition 3.5. *We call* first integral *of the differential system* (S) *any continuous function* $\mathcal{I} : V \to \mathbb{R}$*, where* V *is an open subset of* U*, such that, for any point* $x \in V$*, the map* $t \mapsto \mathcal{I}(\varphi^t(x))$ *is constant on each interval where it is defined.*

This means that each orbit of the system is contained in one of the level sets of $\mathcal{I}$. Hence drawing the level sets of $\mathcal{I}$ can help to draw the phase portrait.

When $\mathcal{I}$ is dfferentiable, to prove that $\mathcal{I}$ is a first integral of the system (S), it suffices to prove that the derivative of $t \mapsto \mathcal{I}(\varphi^t(x))$ vanishes at each point where it is defined.

3.4 Change of coordinates

There are many situations in mathematics where we want to change coordinates. For instance, in linear algebra, when we try to diagonalize an endomorphism, we look for new coordinates where our endomorphism has the simplest possible form.

Hence it is natural to try to do the same thing for vector fields and look for new coordinates where the vector field is simpler. In this section, we will try to understand how vector fields behave with respect to coordinate changes. In the next section, we will classify vector fields at a regular point (they all look like a constant vector field).

Let $X : U \to \mathbb{R}^d$ be a locally lipschtzian vector field on an open set U of $\mathbb{R}^d$. Let $h : U \to h(U) = V \subset \mathbb{R}^d$ be a diffeomorphism (our coordinate change). The vector field X defines a flow φ^t on U.

Let us be informal in this paragraph to explain a forthcoming definition. For any time t and any point x in U, define ψ^t on V by $\psi^t(h(x)) = h(\varphi^t(x))$. Suppose that ψ^t is the flow of a vector field Y. If we differentiate this relation at $t = 0$, we obtain

$$\forall x \in U,\ Y(h(x)) = dh(x).X(x).$$

In other words,

$$\forall x \in V,\ Y(x) = dh(h^{-1}(x)).X(h^{-1}(x)).$$

Definition 3.6. *We call* image of the vector field X under the diffeomorphism h *the vector field h_*X defined on V by*

$$\forall x \in V,\ (h_*X)(x) = dh(h^{-1}(x)).X(h^{-1}(x)).$$

Proposition 3.7. *Suppose h is a C^2 diffeomorphism. Let us denote by ψ the flow of the vector field h_*X. For any point $(t, x) \in \mathbb{R} \times V$ for which the relation makes sense*

$$\psi^t(x) = h \circ \varphi^t \circ h^{-1}(x).$$

Démonstration. See Exercise 3.5. □

3.5 Flow box

We use the same notation as in the preceding section.

Theorem 3.8 (Flow box theorem). *Suppose the vector field X is C^1. Let $x_0 \in U$. Suppose that $X(x_0) \neq 0$. Then there exists a neighbourhood $U' \subset U$ of the point x_0 and a diffeomorphism $h : U' \to h(U') = V'$ such that, for any point x in V',*

$$h_* X(x) = e_1,$$

where e_1 is the first vector of the canonical base of $\mathbb{R}^d$.

Hence, at any regular point x_0, *i.e.* a point x_0 such that $X(x_0) \neq \emptyset$, the vector field X looks like a constant vector field.

See Exercise 3.6 for a proof of this theorem.

Understanding how a vector field can behave near an equilibrium point is much harder.

3.6 Stability and Lyapunov functions

As we understand well the behaviour of a vector field near a regular point, we will try to give some notions near an equilibrium point.

For a point p in $\mathbb{R}^d$ and $r > 0$ we denote by $B(p, r)$ the ball of center p and of radius r for the Euclidean norm :

$$B(p, r) = \left\{ x \in \mathbb{R}^d, \ \|x - p\| < r \right\}.$$

Let us denote by p_0 an equilibrium point of the differential system (S) $x' = X(x)$, *i.e.* a point p_0 such that $X(p_0) = 0$. Recall that we denoted by φ the flow of the vector field X.

Definition 3.9 (Stability and asymptotic stability). *The equilibrium point p_0 is* stable *if, for any $\epsilon > 0$, there exists $\eta > 0$ such that, for any point x in $B(p_0, \eta)$,*

1. *The map $t \mapsto \varphi^t(x)$ is defined on $[0, +\infty)$.*
2. *$\forall t \geq 0, \ \varphi^t(x) \in B(p_0, \epsilon)$.*

The equilibrium point p_0 is said to be asymptotically stable if it is stable and there exists $\eta > 0$ such that, for any point x in the ball $B(p_0, \eta)$,

$$\lim_{t \to +\infty} \varphi^t(x) = p_0.$$

Depending on the books, the word "unstable" can have two different meanings :

1. In some books, it means that the equilibrium point is not stable.
2. In other books, it means that the equilibrium point is "stable when $t \to -\infty$".

Hence, one has to be careful when he uses this word.

In the case of linear systems with constant coefficients, we are able to compute the solutions of a differential system. Hence it is not difficult to prove the following proposition.

Proposition 3.10 (Stability in linear systems). *Let A a $d \times d$ matrix with real coefficients. The origin is asymptotically stable for the differential system $x' = Ax$ if and only if all the eigenvalues of A have a negative real part. Moreover, if one of the eigenvalues of A has a positive real part, then the origin is not stable for this differential system.*

This proposition is proved in Exercise 3.7. In Exercise 3.8, we study the stability of an equilibrium point for a nonlinear differential system.

The following notion is fundamental when we want to study the stability of an equilibrium point.

Definition 3.11 (Lyapunov function). *Let $L : V \to \mathbb{R}$ be a continuous function defined on an open neighbourhood V of the point p_0. We say that L is a* Lyapunov function *(respectively a* weak Lyapunov function*) for the equilibrium point p_0 if*

1. *L has a strict minimum at the point p_0 :*

$$\forall x \in V \setminus \{p_0\}\,,\ L(x) > L(p_0).$$

2. *For any point x in $V \setminus \{p_0\}$, the function $t \mapsto L(\varphi^t(x))$ is strictly decreasing (respctively decreasing) on each interval where it is defined.*

In practice, it suffices to compute a derivative to check the second condition. The terminology "weak Lyapunov function" is not standard but, to my knowledge, there is no standard word for this concept.

The following theorem is main reason why Lyapunov functions are important.

Theorem 3.12 (Stabilty and Lyapunov functions)**.** *If there exists a Lyapunov function for the equilibrium point p_0, then the equilibrium point p_0 is asymptotically stable.*

If there exists a weak Lyapunov function for the equilibrium point p_0, then the equilibrium point p_0 is stable.

We give examples of applications of this theorem in Exercises 3.9 and 3.10 and we prove it in Exercise 3.11.

3.7 Stability by linearization

Let A be a $d \times d$ matrix with real entries. Suppose that all the eigenvalues of A have a negative real part. Then the map L defined on $\mathbb{R}^d$ by

$$L(x) = \int_0^{+\infty} \left\| e^{sA} x \right\|^2 ds$$

is well-defined and is a Lyapunov function for the origin (see Exercise 3.13 for a proof). This gives another proof of the asymptotical stability of the origin for the differential system $x' = Ax$.

However, the real advantage of this function L is that it will still be a Lyapunov function for nonlinear differential systems in the neighbourhood of an equilibrium point. To do this, we approximate the nonlinear differential system by its linear part. This is the main idea of the following theorem.

Fix a C^1 vector field $X : U \to \mathbb{R}^d$ and an equilibrium point p_0 of the differential system $x' = X(x)$. We denote by $dX(p_0)$ the differential of X at the point p_0.

Theorem 3.13 (Routh criterion)**.** *Suppose that all the eigenvalues of the differential $dX(p_0)$ have negative real parts. Then the equilibrium point p_0 is asymptotically stable. Moreover, if the differential $dX(p_0)$ has one eigenvalue with positive real part, then the equilibrium point p_0 is not stable.*

For a proof of the first part of this theorem, see Exercise 3.13.