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VARIATIONAL ASYMPTOTIC DERIVATION OF AN ELASTIC MODEL ARISING FROM THE PROBLEM OF 3D AUTOMATIC SEGMENTATION OF CARDIAC IMAGES

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Segmentation of 3D cardiac images using a deformable elastic model of the heart proved to be significantly improved by applying special boundary conditions on the elastic model [15]. The purpose of this paper is to derive those boundary conditions by means of a rigourous convergence result. We consider a simplified two-layer elastic shell model and show that when the thickness ε of the thin external fibrous layer tends to 0 it can be replaced by the above mentioned boundary conditions on the internal layer. A mixed variational formulation of the problem in curvilinear coordinates is introduced. This formulation is then scaled in order to be defined over an ε-independent domain. Finally, several *a priori* estimations on the solution are obtained which enable us to pass to the limit and prove our result.

Keywords: Segmentation; cardiac; elasticity; shell; mixed variational formulation; asymptotic analysis.

Mathematics Subject Classification 2000: 74B05, 74K25, 35B40, 35B45

1. Introduction

By means of Magnetic Resonance, one can get a clinical M.R. volume dataset. Such a volume dataset is denoted by a matrix V with X rows, Y columns and Z slices which represents a discrete grid of volume elements (or voxels)

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$v \in \{1,\dots,X\} \times \{1,\dots,Y\} \times \{1,\dots,Z\}$. For each voxel v, we denote by $I\colon \mathbb{N}^3 \to \mathbb{Z}$ the grey level function $v \mapsto I(v)$. Data are anisotropic with equal sampling in the x and y directions but a coarser density in the z direction. By image segmentation we refer to processes identifying all voxels which belong together according to a homogeneity criterion (most often a grey level criterion). Segmentation is required for the identification of the object (that is, the heart) in the M.R. volume data. Here, we deal with edge-based algorithms which try to detect the borderline of a structure (that is, the discontinuity surfaces of the "gradient" of the grey level function I). A force field is computed from the "gradient" of the function I by using a Gradient Vector Flow technique.

In order to address the problem of 3D automatic segmentation of cardiac M.R. multi-slices image sequences, a strategy based on an elastic simulation of the human heart has been proposed by Vincent *et al.* [16]. It can be summarized as follows: an *a priori* template (object) representing the heart is immersed into the image data and submitted to a force field which pulls the boundary of the object towards the image edges. This method has several advantages but one drawback concerns the regularity of the displacement field and the smoothness of the final object boundary. As an alternative to classical geometrical curvature-based boundary regularization techniques, Pham *et al.* [15] propose to add boundary constraints modeling crudely some biomechanical properties of the heart. They consider a simplified three-layer elastic model of the heart composed of a middle homogeneous isotropic layer and two surrounding thin layers of myocardial fibers with a directional structure. The aim of this model is to mimic the elastic properties of the heart resulting from the fiber structure of the muscle oriented in the longitudinal direction. It is an efficient tool for image segmentation but not a complete myocardium model. For a more realistic elastic model of the heart we refer to Caillerie *et al.* [6]. It is announced but not proved in [15] that the fibrous layers can be replaced by boundary conditions on the middle layer when the thickness of the external layers tends to 0. These conditions increase the stiffness of the boundary and smooth the displacement field at the interface of the elastic object by imposing preferential directions of deformation in the tangent space (see Fig. 1). We are not going here to get into the details of the numerical method used and refer the readers to Pham [14] and Pebay *et al.* [13]. However, it is worth noticing that the use of a 3D complex geometric template is necessary for the efficiency of the method. Therefore, we describe the thin layers with a shell-kind model using curvilinear coordinates. The purpose of this article is to obtain the above mentionned boundary conditions by means of a rigourous convergence result.

In order to simplify the mathematical analysis, we only consider two layers: an internal layer of fixed thickness ε_l and an external layer of thickness ε. These two layers have a common side, which is a surface $\hat{S}$ of $\mathbb{R}^3$. Therefore, the heart is represented by an elastic shell occupying a domain $\hat{\Omega}_\varepsilon = \hat{\Omega}^- \cup \hat{\Omega}^+_\varepsilon$, where $\hat{\Omega}^-$ is the internal layer and $\hat{\Omega}^+_\varepsilon$ is the external layer. The border of $\hat{\Omega}_\varepsilon$ is $\partial\hat{\Omega}_\varepsilon = \hat{\Gamma}^+_\varepsilon \cup \hat{\Gamma}^- \cup \hat{\Gamma}_{l,\varepsilon}$, where $\hat{\Gamma}^+_\varepsilon$ is the external border, $\hat{\Gamma}^-$ is the internal border and $\hat{\Gamma}_{l,\varepsilon}$ is the lateral border (see Fig. 2).

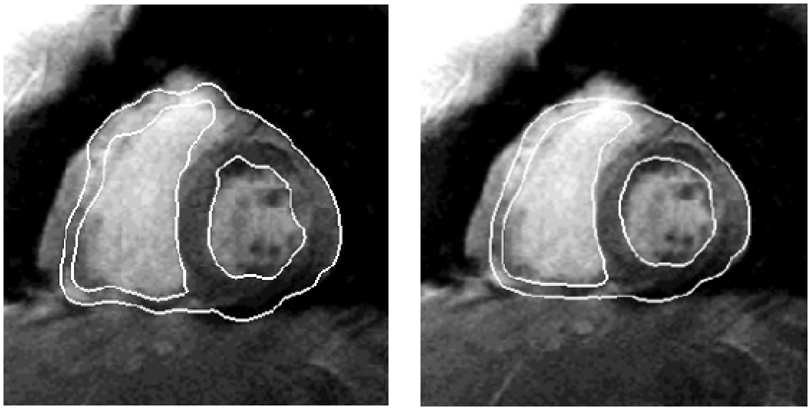

Fig. 1. Impact of the regularization on segmentation results for a mid-ventricular slice: without (left) and with (right) applied boundary conditions (from Pham [14]).

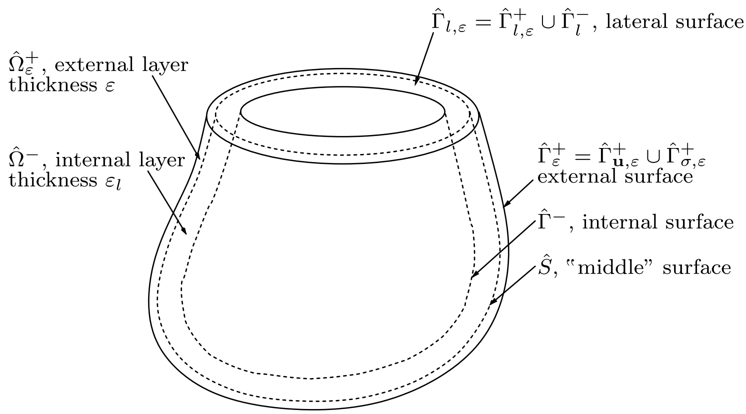

Fig. 2. The domain $\hat{\Omega}_\varepsilon = \hat{\Omega}^+_\varepsilon \cup \hat{\Omega}^-$.

We use the following classical conventions and notations throughout this work. Greek indices and exponents (except ε) belong to the set $\{1, 2\}$, whereas Latin indices belong to the set $\{1, 2, 3\}$. The summation convention with respect to repeated indices and exponents is systematically used. The Euclidean scalar product, the vector product and tensorial product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are denoted $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \otimes \mathbf{b}$, respectively; the Euclidean norm is denoted $|| \cdot ||$.

Let $(\mathbf{e}_i)$ be the canonical orthonormal basis of the Euclidean space $\mathbb{R}^3$. In cartesian coordinates the displacement field for any material point is represented by $\hat{\mathbf{u}} = \hat{u}_i \mathbf{e}_i$. The deformation is described by the Green–Lagrange strain tensor, which is linearized under the small deformation assumption:

$$\hat{e}_{ij}(\hat{\mathbf{u}}) = \frac{1}{2}\left(\frac{\partial \hat{u}_i}{\partial \hat{x}_j} + \frac{\partial \hat{u}_j}{\partial \hat{x}_i}\right).$$

If $\hat{\sigma}$ denotes the stress tensor, the constitutive law or stress-strain relation for the homogeneous isotropic internal layer takes the form:

$$\hat{\sigma}(\hat{\mathbf{u}}) = \lambda \, \mathrm{trace}(\hat{e}(\hat{\mathbf{u}}))I + 2\mu\hat{e}(\hat{\mathbf{u}}), \tag{1.1}$$

where λ and μ are the Lamé constants, and I is the identity tensor. Equivalently, we have,

$$\hat{e}(\hat{\mathbf{u}}) = \frac{1+\nu}{E}\hat{\sigma}(\hat{\mathbf{u}}) - \frac{\nu}{E}\, \mathrm{trace}(\hat{\sigma}(\hat{\mathbf{u}}))I, \tag{1.2}$$

where E is the Young modulus and ν the Poisson ratio. The following classical relations hold

$$\frac{\nu}{E} = \frac{\lambda}{4\mu(\lambda+\mu)}, \qquad \frac{1+\nu}{E} = \frac{1}{2\mu}. \tag{1.3}$$

If $\hat{\mathbf{d}}$ is the 3D orientation vector of fibers belonging to the tangent space and μ_e is the second Lamé coefficient for the external layer, the constitutive law for this layer reads as follows [15]:

$$\hat{\sigma}(\hat{\mathbf{u}}) = (\hat{\mathbf{d}} \cdot \hat{e}(\hat{\mathbf{u}})\hat{\mathbf{d}})\hat{\mathbf{d}} \otimes \hat{\mathbf{d}} + 2\mu_e\varepsilon\hat{e}(\hat{\mathbf{u}}). \tag{1.4}$$

We will show in Sec. 3.1 that the inverse relation is well defined for all $\varepsilon > 0$. In the context of bonded joint with soft material, similar constitutive law models have been proposed in [12] or in [3].

We assume that the elastic body is submitted to a volumic force field $\hat{\mathbf{f}}$ such that $\hat{\mathbf{f}} = 0$ in $\hat{\Omega}^+_\varepsilon$. The equilibrium state is expressed by:

$$\begin{cases} \mathrm{div}(\hat{\sigma}(\hat{\mathbf{u}})) + \hat{\mathbf{f}} = 0 & \text{in } \hat{\Omega}_\varepsilon, \\ \hat{\sigma}(\hat{\mathbf{u}}) = \lambda \, \mathrm{trace}(\hat{e}(\hat{\mathbf{u}}))I + 2\mu\hat{e}(\hat{\mathbf{u}}) & \text{in } \hat{\Omega}^-, \\ \hat{\sigma}(\hat{\mathbf{u}}) = (\hat{\mathbf{d}} \cdot \hat{e}(\hat{\mathbf{u}})\hat{\mathbf{d}})\hat{\mathbf{d}} \otimes \hat{\mathbf{d}} + 2\mu_e\varepsilon\hat{e}(\hat{\mathbf{u}}) & \text{in } \hat{\Omega}^+_\varepsilon, \\ \hat{\mathbf{u}} = 0 & \text{on } \hat{\Gamma}^- \cup \hat{\Gamma}_{l,\varepsilon} \cup \hat{\Gamma}^+_{\mathbf{u},\varepsilon}, \\ \hat{\sigma}\mathbf{n} = 0 & \text{on } \hat{\Gamma}^+_{\sigma,\varepsilon}, \\ \hat{\mathbf{u}}^- = \hat{\mathbf{u}}^+ \text{ and } \hat{\sigma}^-\mathbf{n} = \hat{\sigma}^+\mathbf{n} & \text{on } \hat{S}, \end{cases} \tag{1.5}$$

where $\hat{\Gamma}^+_\varepsilon = \hat{\Gamma}^+_{\mathbf{u},\varepsilon} \cup \hat{\Gamma}^+_{\sigma,\varepsilon}$ and $\mathrm{meas}(\hat{\Gamma}^+_{\mathbf{u},\varepsilon}) \neq 0$. $\hat{\mathbf{u}}^+$ (respectively $\hat{\mathbf{u}}^-$) is the restriction of $\hat{\mathbf{u}}$ to $\hat{\Omega}^+_\varepsilon$ (respectively $\hat{\Omega}^-$). The same notation applies to $\hat{\sigma}$. The vector $\mathbf{n}$ denotes the normal unit vector pointing outwards of $\hat{\Omega}_\varepsilon$ on $\hat{\Gamma}^+_{\sigma,\varepsilon}$ and outwards of $\hat{\Omega}^-$ on $\hat{S}$.

The goal of this work is to prove that when the thickness of the external layer, ε, tends to 0, the asymptotic model is given by:

$$\begin{cases} \mathrm{div}(\hat{\sigma}(\hat{\mathbf{u}})) + \hat{\mathbf{f}} = 0 & \text{in } \hat{\Omega}^-, \\ \hat{\sigma}(\hat{\mathbf{u}}) = \lambda \, \mathrm{trace}(\hat{e}(\hat{\mathbf{u}}))I + 2\mu\hat{e}(\hat{\mathbf{u}}) & \text{in } \hat{\Omega}^-, \\ \hat{\mathbf{u}} = 0 & \text{on } \hat{\Gamma}^- \cup \hat{\Gamma}^-_l, \\ \hat{\sigma}\mathbf{n} = -2\mu_e\hat{u}_n\mathbf{n} - \mu_e\hat{\mathbf{u}}_T & \text{on } \hat{S}, \end{cases} \tag{1.6}$$

where $\hat{\Gamma}^-_l$ is such that $\hat{\Gamma}_{l,\varepsilon} = \hat{\Gamma}^+_{l,\varepsilon} \cup \hat{\Gamma}^-_l$, $\hat{u}_n\mathbf{n}$ is the component of $\hat{\mathbf{u}}$ normal to the surface $\hat{S}$ and $\hat{\mathbf{u}}_T$ is the tangential component.

It is worth noticing here that $\hat{\Gamma}^+_{\mathbf{u},\varepsilon}$ and $\hat{\Gamma}^+_{\sigma,\varepsilon}$ disappear in the limit process. However, if, on the one hand, one can choose $\text{meas}(\hat{\Gamma}^+_{\sigma,\varepsilon}) = 0$, it is on the other hand necessary to have $\text{meas}(\hat{\Gamma}^+_{\mathbf{u},\varepsilon}) \neq 0$. The Dirichlet boundary condition on $\hat{\Gamma}^+_{\mathbf{u},\varepsilon}$ plays an important role in the proof of Theorem 6.7 at the end of the paper. It should also be noticed that the new boundary condition on $\hat{S}$ does not depend on the fibers direction $\hat{\mathbf{d}}$. If $\hat{\mathbf{d}}$ has a non zero component in the normal direction $\mathbf{n}$, the asymptotic model will be dramatically different.

An overview of the article is as follows. In the next section we collect most of the notation to be used in the remainder of the paper recalling basic notions on curvilinear coordinates. Using this notation in Sec. 3, we derive some estimations concerning the stress-strain relations in the internal layer, Ω^-, and in the external layer Ω^+_ε. In Sec. 4, we introduce the mixed variational formulation of the elasticity problem (1.5) and show its well-posedness. The problem is then reformulated in Sec. 5 over an ε-independent domain Ω. The main result of this paper is obtained in Sec. 6, in which we first prove several *a priori* estimations on the solution to the scaled problem before passing to the limit as ε tends to 0.

Let Ω be an open subset in $\mathbb{R}^3$. $L^2(\Omega)$, $||\cdot||_{0,\Omega}$ and $H^1(\Omega)$, $||\cdot||_{1,\Omega}$ denote the usual Sobolev spaces of real-valued functions. Boldface lowercase letters denote vector-valued functions and boldface uppercase letters denote matrix valued functions. The norms are denoted in the same way as for real-valued functions. For instance, if $\mathbf{v} \in (L^2(\Omega))^3$, we note $||\mathbf{v}||^2_{0,\Omega} = \sum_i ||v_i||^2_{0,\Omega}$.

2. Preliminaries

2.1. *Curvilinear coordinates*

All needed notions of differential geometry may be found, e.g., in [8]. The presentation given in this section is very close to the one given in [9]. We consider a shell described by a surface $\hat{S}$, the thickness of which is $\varepsilon_l + \varepsilon$. We assume that the surface $\hat{S}$ is a bounded, two-dimensional submanifold of $\mathbb{R}^3$, which, for simplicity, admits an atlas consisting of one chart only. Let $\boldsymbol{\psi}$ be this chart. We are thus given once and for all a domain $w \subset \mathbb{R}^2$ and an injective mapping $\boldsymbol{\psi} \in \mathcal{C}^3(\bar{w}, \mathbb{R}^3)$, such that

$$\hat{S} = \boldsymbol{\psi}(\bar{w}).$$

We assume that w has a Lipschitz-continuous boundary, γ. Let $y = (y_\alpha)$ denote a generic point in the set $\bar{w}$ and let $\partial_\alpha = \partial/\partial y_\alpha$. Let $\boldsymbol{\psi}$ be such that the two vectors

$$\mathbf{a}_\alpha(y) = \partial_\alpha \boldsymbol{\psi}(y),$$

are linearly independent at all points $y \in \bar{w}$. They form the covariant basis of the tangent plane, $T(\hat{S})$, to the surface $\hat{S}$ at the point $\boldsymbol{\psi}(y)$. The two vectors $\mathbf{a}^\alpha(y)$ of the same tangent plane defined by the relations

$$\mathbf{a}^\alpha(y) \cdot \mathbf{a}_\beta(y) = \delta^\alpha_\beta,$$

constitute its contravariant basis. Let us also define

$$\mathbf{a}_3(y) = \mathbf{a}^3(y) = \frac{\mathbf{a}_1(y) \times \mathbf{a}_2(y)}{||\mathbf{a}_1(y) \times \mathbf{a}_2(y)||},$$

which is a chart-independent (modulo multiplication by -1) unit normal vector to the tangent plane. One then defines the metric tensor, $(a_{\alpha\beta})$ or $(a^{\alpha\beta})$ (in covariant or contravariant components), the curvature tensor, $(b_{\alpha\beta})$ or (b_α^β) (in covariant or mixed components), and the Christoffel symbols $\Gamma^\rho_{\alpha\beta}$, of the surface $\hat{S}$ by letting

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta, \quad a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta, \tag{2.1}$$

$$b_{\alpha\beta} = \mathbf{a}_3 \cdot \partial_\beta \mathbf{a}_\alpha, \quad b_\alpha^\beta = a^{\beta\alpha} b_{\sigma\alpha}, \tag{2.2}$$

$$\Gamma^\rho_{\alpha\beta} = \mathbf{a}^\rho \cdot \partial_\beta \mathbf{a}_\alpha. \tag{2.3}$$

Note the symmetries:

$$a_{\alpha\beta} = a_{\beta\alpha}, \quad a^{\alpha\beta} = a^{\beta\alpha}, \quad b_{\alpha\beta} = b_{\beta\alpha}, \quad \Gamma^\rho_{\alpha\beta} = \Gamma^\rho_{\beta\alpha}.$$

The area element along $\hat{S}$ is $\sqrt{a}dy$, where

$$a = \det(a_{\alpha\beta}). \tag{2.4}$$

The function a is continuous on the set $\bar{w}$ and there exits a constant a_0, such that

$$0 < a_0 \leq a(y), \quad \forall y \in \bar{w}. \tag{2.5}$$

For each $\varepsilon > 0$ we define the sets:

$$\begin{aligned}
\Omega_\varepsilon &= w \times]-\varepsilon_l, \varepsilon[,\\
\Omega_\varepsilon^+ &= w \times]0, \varepsilon[,\\
\Omega^- &= w \times]-\varepsilon_l, 0[,\\
\Gamma^+_{l,\varepsilon} &= \gamma \times [0, \varepsilon],\\
\Gamma^-_l &= \gamma \times [-\varepsilon_l, 0],\\
\Gamma^- &= w \times \{-\varepsilon_l\},\\
\Gamma^+_{\mathbf{u},\varepsilon} &= w_\mathbf{u} \times \{\varepsilon\},\\
\Gamma^+_{\sigma,\varepsilon} &= w_\sigma \times \{\varepsilon\},\\
S &= w \times \{0\},
\end{aligned}$$

with $w = w_\mathbf{u} \cup w_\sigma$ and $\text{meas}(w_\mathbf{u}) \neq 0$. Note that $\Gamma^+_{l,\varepsilon} \cup \Gamma^-_l \cup \Gamma^- \cup \Gamma^+_{\mathbf{u},\varepsilon} \cup \Gamma^+_{\sigma,\varepsilon} = \partial\Omega_\varepsilon$ constitutes a partition of the boundary of the set Ω_ε (see Fig. 3).

Let $x^\varepsilon = (x_i^\varepsilon)$ denote a generic point in $\bar{\Omega}_\varepsilon$, and let $\partial_i^\varepsilon = \partial/\partial x_i^\varepsilon$; hence $x_\alpha^\varepsilon = y_\alpha$ and $\partial_\alpha^\varepsilon = \partial_\alpha$. The initial configuration of the shell is the image of Ω_ε by the mapping $\mathbf{\Psi}$: $\bar{\Omega}_\varepsilon \to \mathbb{R}^3$ defined by

$$\mathbf{\Psi}(x^\varepsilon) = \boldsymbol{\psi}(y) + x_3^\varepsilon \mathbf{a}_3(y), \quad \forall x^\varepsilon = (y, x_3^\varepsilon) \in \bar{\Omega}_\varepsilon.$$

It can then be shown (cf. [8]) that there exists $\varepsilon_0 > 0$, such that the mapping $\mathbf{\Psi}$ is a $\mathcal{C}^2$-diffeomorphism, and the three vectors,

$$\mathbf{g}_i^\varepsilon(x^\varepsilon) = \partial_i^\varepsilon \mathbf{\Psi}(x^\varepsilon),$$

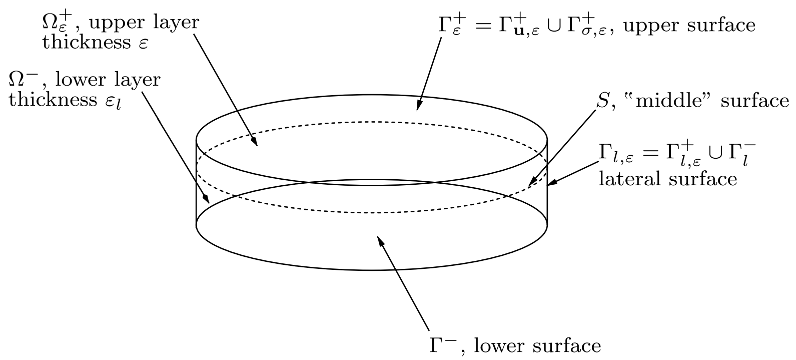

Fig. 3. The domain $\Omega_\varepsilon = \Omega_\varepsilon^+ \cup \Omega^-$.

are linearly independent at all points $x^\varepsilon \in \bar{\Omega}_\varepsilon$ for all $0 < \varepsilon < \varepsilon_0$. Therefore, we make a geometrical assumption on the thicknesses of the two layers of the shell heart model:

$$0 < \varepsilon \leq \varepsilon_l \leq \varepsilon_0.$$

The three vectors $\mathbf{g}_i^\varepsilon(x^\varepsilon)$ define the covariant basis at the point $\mathbf{\Psi}(x^\varepsilon)$. It is clear that $\mathbf{g}_3^\varepsilon = \mathbf{a}_3$ is the unit vector normal to $\hat{S}$. We choose it to be pointing outwards of Ω^- and for the remainder of this work, we use indifferently the notations $\mathbf{n}$ or $\mathbf{g}_3^\varepsilon$. The three vectors $\mathbf{g}^{i,\varepsilon}(x^\varepsilon)$ defined by

$$\mathbf{g}^{j,\varepsilon}(x^\varepsilon) \cdot \mathbf{g}_i^\varepsilon(x^\varepsilon) = \delta_i^j,$$

form the contravariant basis. One then defines the metric tensor (g_{ij}^ε) or $(g^{ij,\varepsilon})$ (in covariant or contravariant components) and the Christoffel symbols of the manifold $\mathbf{\Psi}(\bar{\Omega}_\varepsilon)$ by letting (we omit the explicit dependence on x^ε)

$$g_{ij}^\varepsilon = \mathbf{g}_i^\varepsilon \cdot \mathbf{g}_j^\varepsilon, \quad g^{ij,\varepsilon} = \mathbf{g}^{i,\varepsilon} \cdot \mathbf{g}^{j,\varepsilon},$$

$$\Gamma_{ij}^{p,\varepsilon} = \mathbf{g}^{p,\varepsilon} \cdot \partial_i^\varepsilon \mathbf{g}_j^\varepsilon.$$

Note the symmetries

$$g_{ij}^\varepsilon = g_{ji}^\varepsilon, \quad g^{ij,\varepsilon} = g^{ji,\varepsilon}, \quad \Gamma_{ij}^{p,\varepsilon} = \Gamma_{ji}^{p,\varepsilon},$$

and the relations

$$\Gamma_{\alpha 3}^{3,\varepsilon} = \Gamma_{33}^{p,\varepsilon} = 0 \quad \text{in } \bar{\Omega}_\varepsilon.$$

The volume element in the set $\mathbf{\Psi}(\Omega_\varepsilon)$ is $\sqrt{g^\varepsilon}dx^\varepsilon$, where

$$g^\varepsilon = \det(g_{ij}^\varepsilon). \tag{2.6}$$

2.2. *Vectors and tensors in curvilinear coordinates*

With all the notations defined in the preceding section, a vector field or a second order symmetric tensor field defined on the shell may be represented in the curvilinear system by its covariant or contravariant components:

$$v_i^\varepsilon \mathbf{g}^{i,\varepsilon} = v^{i,\varepsilon}\mathbf{g}_i^\varepsilon,$$
$$\tau_{ij}^\varepsilon \mathbf{g}^{i,\varepsilon} \otimes \mathbf{g}^{j,\varepsilon} = \tau^{ij,\varepsilon}\mathbf{g}_i^\varepsilon \otimes \mathbf{g}_j^\varepsilon.$$

One can relate covariant and contravariant components, thanks to the relations

$$\begin{aligned} v^{i,\varepsilon} &= g^{ik,\varepsilon} v_k^\varepsilon, & v_i^\varepsilon &= g_{ik}^\varepsilon v^{k,\varepsilon},\\ \tau^{ij,\varepsilon} &= \frac{1}{2}(g^{ik,\varepsilon}g^{jl,\varepsilon} + g^{jk,\varepsilon}g^{il,\varepsilon})\tau_{kl}^\varepsilon, & \tau_{ij}^\varepsilon &= \frac{1}{2}(g_{ik}^\varepsilon g_{jl}^\varepsilon + g_{jk}^\varepsilon g_{il}^\varepsilon)\tau^{kl,\varepsilon},\\ &= G^{ijkl,\varepsilon}\tau_{kl}^\varepsilon, & &= H_{ijkl}^\varepsilon \tau^{kl,\varepsilon}. \end{aligned}$$

Concerning the fourth-order tensors $(G^{ijkl,\varepsilon})$ and (H_{ijkl}^ε), the following relations hold for each $\varepsilon > 0$

$$G^{\alpha\beta k3,\varepsilon} = G^{333\alpha,\varepsilon} = 0,$$
$$H_{\alpha\beta k3}^\varepsilon = H_{333\alpha}^\varepsilon = 0,$$

and

$$G^{ijkl,\varepsilon} = G^{jikl,\varepsilon} = G^{klij,\varepsilon},$$
$$H_{ijkl}^\varepsilon = H_{jikl}^\varepsilon = H_{klij}^\varepsilon.$$

Both tensors are symmetric, positive definite, and uniform with respect to $x^\varepsilon \in \bar{\Omega}_\varepsilon$. The scalar product between two vectors, $u^{i,\varepsilon}\mathbf{g}_i^\varepsilon$ and $v_i^\varepsilon \mathbf{g}^{i,\varepsilon}$ can be written as

$$(u^{i,\varepsilon}\mathbf{g}_i^\varepsilon)\cdot(v_i^\varepsilon \mathbf{g}^{i,\varepsilon}) = u^{i,\varepsilon}v_i^\varepsilon.$$

The second-order inner product between two tensors can be written as

$$(\tau^{ij,\varepsilon}\mathbf{g}_i^\varepsilon \otimes \mathbf{g}_j^\varepsilon)\colon(\sigma_{ij}^\varepsilon \mathbf{g}^{i,\varepsilon} \otimes \mathbf{g}^{j,\varepsilon}) = \tau^{ij,\varepsilon}\sigma_{ij}^\varepsilon.$$

Using the fourth-order tensor $G^{ijkl,\varepsilon}$, this expression can be transformed to

$$\tau^{ij,\varepsilon}\sigma_{ij}^\varepsilon = G^{ijkl,\varepsilon}\tau_{kl}^\varepsilon \sigma_{ij}^\varepsilon.$$

Let us now introduce the vectorial notation which we will use. $\mathbf{S}$ denotes the set of all symmetric matrices of order 3. Any $(\tau_{ij}) \in \mathbf{S}$ can be represented by a vector $\boldsymbol{\tau} \in \mathbb{R}^6$:

$$\boldsymbol{\tau} = (\tau_{11}, \sqrt{2}\tau_{12}, \tau_{22}, \sqrt{2}\tau_{13}, \sqrt{2}\tau_{23}, \tau_{33})^T.$$

We also note

$$\boldsymbol{\tau}_T = (\tau_{11}, \sqrt{2}\tau_{12}, \tau_{22})^T, \quad \boldsymbol{\tau}_N = (\sqrt{2}\tau_{13}, \sqrt{2}\tau_{23}, \tau_{33})^T.$$

The fourth-order tensor $G^{ijkl,\varepsilon}$ can be represented by the 6×6 symmetric matrix $\mathbf{G}^\varepsilon$:

$$\mathbf{G}^\varepsilon = \left(\begin{array}{c|c} \mathbf{G}^\varepsilon_T & 0 \\ \hline 0 & \mathbf{G}^\varepsilon_N \end{array}\right),$$

with

$$\mathbf{G}^\varepsilon_T = \begin{pmatrix} g^{11,\varepsilon}g^{11,\varepsilon} & \sqrt{2}g^{11,\varepsilon}g^{12,\varepsilon} & g^{12,\varepsilon}g^{12,\varepsilon} \\ \sqrt{2}g^{11,\varepsilon}g^{12,\varepsilon} & g^{11,\varepsilon}g^{22,\varepsilon}+g^{12,\varepsilon}g^{12,\varepsilon} & \sqrt{2}g^{12,\varepsilon}g^{22,\varepsilon} \\ g^{12,\varepsilon}g^{12,\varepsilon} & \sqrt{2}g^{12,\varepsilon}g^{22,\varepsilon} & g^{22,\varepsilon}g^{22,\varepsilon} \end{pmatrix},$$

and

$$\mathbf{G}^\varepsilon_N = \begin{pmatrix} g^{11,\varepsilon}g^{33,\varepsilon} & g^{12,\varepsilon}g^{33,\varepsilon} & 0 \\ g^{12,\varepsilon}g^{33,\varepsilon} & g^{22,\varepsilon}g^{33,\varepsilon} & 0 \\ 0 & 0 & g^{33,\varepsilon}g^{33,\varepsilon} \end{pmatrix}.$$

Recalling that the (g^ε_{ij}) matrix is the inverse of the $(g^{ij,\varepsilon})$ matrix, we note that

$$(\mathbf{G}^\varepsilon)^{-1} = \mathbf{H}^\varepsilon = \left(\begin{array}{c|c} \mathbf{H}^\varepsilon_T & 0 \\ \hline 0 & \mathbf{H}^\varepsilon_N \end{array}\right),$$

with

$$\mathbf{H}^\varepsilon_T = \begin{pmatrix} g^\varepsilon_{11}g^\varepsilon_{11} & \sqrt{2}g^\varepsilon_{11}g^\varepsilon_{12} & g^\varepsilon_{12}g^\varepsilon_{12} \\ \sqrt{2}g^\varepsilon_{11}g^\varepsilon_{12} & g^\varepsilon_{11}g^\varepsilon_{22}+g^\varepsilon_{12}g^\varepsilon_{12} & \sqrt{2}g^\varepsilon_{12}g^\varepsilon_{22} \\ g^\varepsilon_{12}g^\varepsilon_{12} & \sqrt{2}g^\varepsilon_{12}g^\varepsilon_{22} & g^\varepsilon_{22}g^\varepsilon_{22} \end{pmatrix},$$

and

$$\mathbf{H}^\varepsilon_N = \begin{pmatrix} g^\varepsilon_{11}g^\varepsilon_{33} & g^\varepsilon_{12}g^\varepsilon_{33} & 0 \\ g^\varepsilon_{12}g^\varepsilon_{33} & g^\varepsilon_{22}g^\varepsilon_{33} & 0 \\ 0 & 0 & g^\varepsilon_{33}g^\varepsilon_{33} \end{pmatrix}.$$

In vectorial notation the second-order inner product between two symmetric tensors is written

$$\begin{aligned}(\tau^{ij,\varepsilon}\mathbf{g}^\varepsilon_i\otimes\mathbf{g}^\varepsilon_j)\colon(\sigma^\varepsilon_{ij}\mathbf{g}^{i,\varepsilon}\otimes\mathbf{g}^{j,\varepsilon}) &= G^{ijkl,\varepsilon}\tau^\varepsilon_{kl}\sigma^\varepsilon_{ij}, \\ &= \boldsymbol{\tau}^\varepsilon\cdot\mathbf{G}^\varepsilon\boldsymbol{\sigma}^\varepsilon, \\ &= \boldsymbol{\tau}^\varepsilon_T\cdot\mathbf{G}^\varepsilon_T\boldsymbol{\sigma}^\varepsilon_T+\boldsymbol{\tau}^\varepsilon_N\cdot\mathbf{G}^\varepsilon_N\boldsymbol{\sigma}^\varepsilon_N.\end{aligned}$$

The fact that $(G^{ijkl,\varepsilon})$ is symmetric, positive definite, and uniform with respect to $x^\varepsilon\in\bar{\Omega}_\varepsilon$ implies that there exists a constant $c^\varepsilon_G>0$ depending on Ω_ε only (thus on the small parameter ε), such that

$$\boldsymbol{\tau}\cdot\mathbf{G}^\varepsilon\boldsymbol{\tau}\geq c^\varepsilon_G\boldsymbol{\tau}\cdot\boldsymbol{\tau}=c^\varepsilon_G\tau_{ij}\tau_{ij}, \tag{2.7}$$

for all $x^\varepsilon\in\bar{\Omega}_\varepsilon$ and all $(\tau_{ij})\in\mathbf{S}$.

From the continuity of $x^\varepsilon\to\mathbf{G}^\varepsilon(x^\varepsilon)$ on $\bar{\Omega}^\varepsilon$ we also deduce that there exists a constant C^ε_G, such that

$$\boldsymbol{\tau}\cdot\mathbf{G}^\varepsilon\boldsymbol{\sigma}\leq C^\varepsilon_G||\boldsymbol{\tau}||||\boldsymbol{\sigma}|| \tag{2.8}$$

for all $x^\varepsilon\in\bar{\Omega}_\varepsilon$ and all $(\tau_{ij}),(\sigma_{ij})\in\mathbf{S}$.

It is clear that if we consider the restrictions of functions $G^{ijkl,\varepsilon}$ to Ω^-, inequality (2.7) still holds with an ε-independent constant $C_G > 0$. To emphasize the fact that the restriction to Ω^- of geometrical quantities such as $\mathbf{g}_i^\varepsilon$, $G^{ijkl,\varepsilon}$, ... are ε-independent, we omit the exponents ε in what follows. For example, $\mathbf{g}_i$ denotes the restriction of $\mathbf{g}_i^\varepsilon$ to Ω^-. We then have

$$\boldsymbol{\tau} \cdot \mathbf{G}\boldsymbol{\tau} \geq c_G \boldsymbol{\tau} \cdot \boldsymbol{\tau} = c_G \tau_{ij} \tau_{ij}, \tag{2.9}$$

for all $x^\varepsilon \in \bar{\Omega}^-$ and all $(\tau_{ij}) \in \mathbf{S}$.

To conclude this section, let us recall that given the covariant components $(u_i^\varepsilon) = \mathbf{u}^\varepsilon$ of an arbitrary displacement field $u_i^\varepsilon \mathbf{g}^{i,\varepsilon}$ of the points of the shell, the covariant components of the linearized strain tensor read

$$\begin{aligned} e_{\alpha\beta}^\varepsilon(\mathbf{u}^\varepsilon) &= \frac{1}{2}(\partial_\alpha^\varepsilon u_\beta^\varepsilon + \partial_\beta^\varepsilon u_\alpha^\varepsilon) - \Gamma_{\alpha\beta}^{p,\varepsilon} u_p^\varepsilon, \\ e_{\alpha 3}^\varepsilon(\mathbf{u}^\varepsilon) &= \frac{1}{2}(\partial_\alpha^\varepsilon u_3^\varepsilon + \partial_3^\varepsilon u_\alpha^\varepsilon) - \Gamma_{\alpha 3}^{\rho,\varepsilon} u_\rho^\varepsilon, \\ e_{33}^\varepsilon(\mathbf{u}^\varepsilon) &= \partial_3^\varepsilon u_3^\varepsilon. \end{aligned} \tag{2.10}$$

Using our vectorial notation, the associated vector of $\mathbb{R}^6$ is denoted by

$$\mathbf{e}^\varepsilon(\mathbf{u}^\varepsilon) = (\mathbf{e}_T^\varepsilon(\mathbf{u}^\varepsilon),\ \mathbf{e}_N^\varepsilon(\mathbf{u}^\varepsilon)).$$

3. Strain-Stress Relation

In this section the strain-stress relations in the internal and external layers are expressed using the vectorial notation. We introduce a new basis of $\mathbb{R}^3$ in order to derive estimations (3.3), (3.4) and (3.7), (3.8) which are needed in the remaining part of the paper.

3.1. *Strain-stress relation in the external layer*

Assume that the linearized strain tensor is described by its *contravariant* components, $e^{kl,\varepsilon}(\mathbf{u}^\varepsilon)$ and that the stress tensor is described by its *covariant* components $\sigma_{ij}^\varepsilon(\mathbf{u}^\varepsilon)$. Assume that the orientation vector of fibers is tangent to the surface $\hat{S}$ and that it is defined by its covariant components, d_α. These components are assumed to be x_3-independent, that is to say, $d_\alpha = d_\alpha(x_1, x_2)$. We also assume that $d_\alpha \in \mathcal{C}^0(\bar{w}, \mathbb{R})$ and that for all $(x_1, x_2) \in \bar{w}$, $\mathbf{d} \neq 0$.

Omiting the explicit dependence on $\mathbf{u}^\varepsilon$, the constitutive law (1.4) for the external fibrous layer then reads

$$\begin{aligned} \sigma_{ij}^\varepsilon \mathbf{g}^{i,\varepsilon} \otimes \mathbf{g}^{j,\varepsilon} &= (e^{kl,\varepsilon} d_k d_l) d_i d_j \mathbf{g}^{i,\varepsilon} \otimes \mathbf{g}^{j,\varepsilon} \\ &\quad + 2\mu_e \varepsilon \frac{1}{2}(g_{ik}^\varepsilon g_{jl}^\varepsilon + g_{jk}^\varepsilon g_{il}^\varepsilon) e^{kl,\varepsilon} \mathbf{g}^{i,\varepsilon} \otimes \mathbf{g}^{j,\varepsilon}. \end{aligned} \tag{3.1}$$

This relation can be written as

$$\sigma_{ij}^\varepsilon = B_{ijkl}^\varepsilon e^{kl,\varepsilon}, \tag{3.2}$$

where

$$B^{\varepsilon}_{ijkl} = d_i d_j d_k d_l + 2\mu_e \varepsilon \frac{1}{2}(g^{\varepsilon}_{ik} g^{\varepsilon}_{jl} + g^{\varepsilon}_{jk} g^{\varepsilon}_{il}).$$

Note the symmetries

$$B^{\varepsilon}_{ijkl} = B^{\varepsilon}_{jikl} = B^{\varepsilon}_{klij}.$$

Since $d_3 = 0$ the following relation holds

$$B^{\varepsilon}_{\alpha\beta k3} = B^{\varepsilon}_{\alpha 333} = 0.$$

The fourth-order symmetric tensor (B^{ε}_{ijkl}) defined by its covariant components is known as the stiffness tensor. In order to establish the mixed variational formulation of the problem, we need to use the inverse relation and the associated compliance tensor $(C^{ijkl,\varepsilon})$ defined by its contravariant components. Let us show that $(C^{ijkl,\varepsilon})$, the inverse of (B^{ε}_{ijkl}), exists for all $\varepsilon > 0$.

The contravariant components $C^{ijkl,\varepsilon}\colon \bar{\Omega}^{+}_{\varepsilon} \to \mathbb{R}$ of the compliance tensor $(C^{ijkl,\varepsilon})$ are obtained by inverting the matrix of covariant components of the stiffness tensor, $B^{\varepsilon}_{ijkl}\colon \bar{\Omega}^{+}_{\varepsilon} \to \mathbb{R}$. In vectorial notation, relation (3.2) reads

$$\boldsymbol{\sigma}^{\varepsilon} = \mathbf{B}^{\varepsilon}\mathbf{e}^{\varepsilon},$$

where

$$\boldsymbol{\sigma}^{\varepsilon} = (\sigma^{\varepsilon}_{11}, \sqrt{2}\sigma^{\varepsilon}_{12}, \sigma^{\varepsilon}_{22}, \sqrt{2}\sigma^{\varepsilon}_{13}, \sqrt{2}\sigma^{\varepsilon}_{23}, \sigma^{\varepsilon}_{33})^T,$$
$$\mathbf{e}^{\varepsilon} = (e^{11,\varepsilon}, \sqrt{2}e^{12,\varepsilon}, e^{22,\varepsilon}, \sqrt{2}e^{13,\varepsilon}, \sqrt{2}e^{23,\varepsilon}, e^{33,\varepsilon})^T.$$

$\mathbf{B}^{\varepsilon}$ is the 6×6 matrix defined by

$$\mathbf{B}^{\varepsilon} = \mathbf{D} + 2\mu_e \varepsilon \mathbf{H}^{\varepsilon},$$

with

$$\mathbf{B}^{\varepsilon} = \left(\begin{array}{c|c} \mathbf{B}^{\varepsilon}_T & 0 \\ \hline 0 & \mathbf{B}^{\varepsilon}_N \end{array}\right), \quad \mathbf{H}^{\varepsilon} = \left(\begin{array}{c|c} \mathbf{H}^{\varepsilon}_T & 0 \\ \hline 0 & \mathbf{H}^{\varepsilon}_N \end{array}\right), \quad \mathbf{D} = \left(\begin{array}{c|c} \mathbf{D}_T & 0 \\ \hline 0 & 0 \end{array}\right),$$

and

$$\mathbf{D}_T = \begin{pmatrix} (d_1)^4 & \sqrt{2}(d_1)^3 d_2 & (d_1)^2(d_2)^2 \\ \sqrt{2}(d_1)^3 d_2 & 2(d_1)^2(d_2)^2 & \sqrt{2}d_1(d_2)^3 \\ (d_1)^2(d_2)^2 & \sqrt{2}d_1(d_2)^3 & (d_2)^4 \end{pmatrix}.$$

$\mathbf{H}^{\varepsilon}$ is symmetric, positive definite and uniform with respect to $x^{\varepsilon} \in \bar{\Omega}_{\varepsilon}$ similar to the fourth-order tensor $(H^{ijkl,\varepsilon})$ is. $\mathbf{D}$ is symmetric and non-negative as its rank is one and its only non-zero eigenvalue is trace$(\mathbf{D}_T) > 0$. Consequently, for all $\varepsilon > 0$, $\mathbf{B}^{\varepsilon}$ is symmetric, positive definite and therefore invertible. Moreover, as $(\mathbf{H}^{\varepsilon}_N)^{-1} = \mathbf{G}^{\varepsilon}_N$, we have

$$(\mathbf{B}^{\varepsilon})^{-1} = \mathbf{C}^{\varepsilon} = \left(\begin{array}{c|c} (\mathbf{B}^{\varepsilon}_T)^{-1} & 0 \\ \hline 0 & \dfrac{1}{2\mu_e \varepsilon}\mathbf{G}^{\varepsilon}_N \end{array}\right).$$

In order to obtain a simple expression for $(\mathbf{B}_T^\varepsilon)^{-1}$ one has to notice that $\mathbf{H}_T^\varepsilon$ is symmetric, positive definite and $\mathbf{D}_T$ is symmetric, positive definite and uniform with respect to $x^\varepsilon \in \bar{\Omega}_\varepsilon$. Therefore, it follows from a classical result (see Appendix A) on the simultaneous reduction of two quadratic forms that there exists a 3×3 invertible matrix $\mathbf{P}_T^\varepsilon$, such that

$$(\mathbf{P}_T^\varepsilon)^T \mathbf{H}_T^\varepsilon \mathbf{P}_T^\varepsilon = \mathbf{I},$$
$$(\mathbf{P}_T^\varepsilon)^T \mathbf{D}_T \mathbf{P}_T^\varepsilon = \mathbf{S}^\varepsilon,$$

with

$$\mathbf{S}^\varepsilon = \begin{pmatrix} s_\varepsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that for all $\varepsilon > 0$ and for all $x^\varepsilon \in \bar{\Omega}_\varepsilon$, $s_\varepsilon > 0$.

We obtain

$$(\mathbf{P}_T^\varepsilon)^T \mathbf{B}_T^\varepsilon \mathbf{P}_T^\varepsilon = \begin{pmatrix} s_\varepsilon + 2\mu_e\varepsilon & 0 & 0 \\ 0 & 2\mu_e\varepsilon & 0 \\ 0 & 0 & 2\mu_e\varepsilon \end{pmatrix},$$

and therefore

$$(\mathbf{Q}_T^\varepsilon)^T (\mathbf{B}_T^\varepsilon)^{-1} \mathbf{Q}_T^\varepsilon = \begin{pmatrix} \dfrac{1}{s_\varepsilon + 2\mu_e\varepsilon} & 0 & 0 \\ 0 & \dfrac{1}{2\mu_e\varepsilon} & 0 \\ 0 & 0 & \dfrac{1}{2\mu_e\varepsilon} \end{pmatrix},$$

where

$$\mathbf{Q}_T^\varepsilon = ((\mathbf{P}_T^\varepsilon)^T)^{-1}.$$

The columns of the matrix

$$\mathbf{Q}^\varepsilon = \left(\begin{array}{c|c} \mathbf{Q}_T^\varepsilon & 0 \\ \hline 0 & \mathbf{I}_3 \end{array}\right)$$

define a new basis of $\mathbb{R}^6$. Any $(\tau_{ij}) \in \mathbf{S}$ represented by a vector

$$\boldsymbol{\tau} = (\tau_{11}, \sqrt{2}\tau_{12}, \tau_{22}, \sqrt{2}\tau_{13}, \sqrt{2}\tau_{23}, \tau_{33})^T,$$

in the canonical basis of $\mathbb{R}^6$ is represented by a vector $\tilde{\boldsymbol{\tau}}$ in this new basis. We have $\boldsymbol{\tau}_T = \mathbf{Q}_T^\varepsilon \tilde{\boldsymbol{\tau}}_T$ and $\boldsymbol{\tau}_N = \tilde{\boldsymbol{\tau}}_N$.

Since $\mathbf{G}_N^\varepsilon$ is symmetric, positive definite and uniform with respect to $x^\varepsilon \in \bar{\Omega}_\varepsilon$, there exist two constants $C_G^\varepsilon > 0$ and $c_G^\varepsilon > 0$ depending on Ω_ε, such that

$$\begin{aligned}
\boldsymbol{\tau} \cdot \mathbf{C}^\varepsilon \boldsymbol{\tau} &= \boldsymbol{\tau} \cdot (\mathbf{B}^\varepsilon)^{-1} \boldsymbol{\tau} = \tilde{\boldsymbol{\tau}} \cdot (\mathbf{Q}^\varepsilon)^T (\mathbf{B}^\varepsilon)^{-1} \mathbf{Q}^\varepsilon \tilde{\boldsymbol{\tau}}, \\
&= \tilde{\boldsymbol{\tau}}_T \cdot (\mathbf{Q}_T^\varepsilon)^T (\mathbf{B}_T^\varepsilon)^{-1} \mathbf{Q}_T^\varepsilon \tilde{\boldsymbol{\tau}}_T + \frac{1}{2\mu_e \varepsilon} \boldsymbol{\tau}_N \cdot \mathbf{G}_N^\varepsilon \boldsymbol{\tau}_N, \\
&= \frac{1}{s_\varepsilon + 2\mu_e \varepsilon} \tilde{\tau}_{11}^2 + \frac{1}{\mu_e \varepsilon} \tilde{\tau}_{12}^2 + \frac{1}{2\mu_e \varepsilon} \tilde{\tau}_{22}^2 + \frac{1}{2\mu_e \varepsilon} \boldsymbol{\tau}_N \cdot \mathbf{G}_N^\varepsilon \boldsymbol{\tau}_N, \\
&\geq \frac{1}{s_\varepsilon + 2\mu_e \varepsilon} \tilde{\tau}_{11}^2 + \frac{1}{\mu_e \varepsilon} \tilde{\tau}_{12}^2 + \frac{1}{2\mu_e \varepsilon} \tilde{\tau}_{22}^2 + \frac{c_G^\varepsilon}{2\mu_e \varepsilon} (2\tau_{13}^2 + 2\tau_{23}^2 + \tau_{33}^2), \qquad (3.3)
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\sigma} \cdot \mathbf{C}^\varepsilon \boldsymbol{\tau} &= \boldsymbol{\sigma} \cdot (\mathbf{B}^\varepsilon)^{-1} \boldsymbol{\tau} = \tilde{\boldsymbol{\sigma}} \cdot (\mathbf{Q}^\varepsilon)^T (\mathbf{B}^\varepsilon)^{-1} \mathbf{Q}^\varepsilon \tilde{\boldsymbol{\tau}}, \\
&= \tilde{\boldsymbol{\sigma}}_T \cdot (\mathbf{Q}_T^\varepsilon)^T (\mathbf{B}_T^\varepsilon)^{-1} \mathbf{Q}_T^\varepsilon \tilde{\boldsymbol{\sigma}}_T + \frac{1}{2\mu_e \varepsilon} \boldsymbol{\sigma}_N \cdot \mathbf{G}_N^\varepsilon \boldsymbol{\tau}_N, \\
&= \frac{1}{s_\varepsilon + 2\mu_e \varepsilon} \tilde{\sigma}_{11} \tilde{\tau}_{11} + \frac{1}{\mu_e \varepsilon} \tilde{\sigma}_{12} \tilde{\tau}_{12} + \frac{1}{2\mu_e \varepsilon} \tilde{\sigma}_{22} \tilde{\tau}_{22} + \frac{1}{2\mu_e \varepsilon} \boldsymbol{\sigma}_N \cdot \mathbf{G}_N^\varepsilon \boldsymbol{\tau}_N, \\
&\leq \frac{1}{s_\varepsilon + 2\mu_e \varepsilon} \tilde{\sigma}_{11} \tilde{\tau}_{11} + \frac{1}{\mu_e \varepsilon} \tilde{\sigma}_{12} \tilde{\tau}_{12} + \frac{1}{2\mu_e \varepsilon} \tilde{\sigma}_{22} \tilde{\tau}_{22} \\
&\quad + \frac{C_G^\varepsilon}{2\mu_e \varepsilon} (2\sigma_{13}^2 + 2\sigma_{23}^2 + \sigma_{33}^2)^{1/2} (2\tau_{13}^2 + 2\tau_{23}^2 + \tau_{33}^2)^{1/2}, \qquad (3.4)
\end{aligned}$$

for all $(\tau_{ij}), (\sigma_{ij}) \in \mathbf{S}$ and all $x^\varepsilon \in \bar{\Omega}_\varepsilon^+$.

3.2. *Strain-stress relation in the internal layer*

In the curvilinear coordinate system, the stress-strain relation (1.2) for the homogeneous isotropic internal layer can be written as

$$e^{ij} = A^{ijkl} \sigma_{kl}, \qquad (3.5)$$

where the fourth-order symmetric tensor A is represented by its contravariant components $A^{ijkl} \colon \bar{\Omega}^- \to \mathbb{R}$

$$A^{ijkl} = \frac{1+\nu}{2E} \left(g^{ik} g^{jl} + g^{jk} g^{il} \right) - \frac{\nu}{E} g^{ij} g^{kl}. \qquad (3.6)$$

Note the symmetries

$$A^{ijkl} = A^{jikl} = A^{klij},$$

and the relations

$$A^{\alpha\beta k3} = A^{\alpha 333} = 0.$$

It is classical that A is positive definite and uniform with respect to $x^\varepsilon \in \bar{\Omega}^-$. With our vectorial notation and the change of basis defined by the matrix $\mathbf{Q}$, the following relations hold: There exist two constants $C_A > 0$ and $c_A > 0$ depending on Ω^-, such that

$$\begin{aligned}\boldsymbol{\tau}\cdot\mathbf{A}\boldsymbol{\tau} &= \tilde{\boldsymbol{\tau}}\cdot(\mathbf{Q})^T\mathbf{A}\mathbf{Q}\tilde{\boldsymbol{\tau}},\\ &= \tilde{\boldsymbol{\tau}}_T\cdot(\mathbf{Q}_T)^T\mathbf{A}_T\mathbf{Q}_T\tilde{\boldsymbol{\tau}}_T + \boldsymbol{\tau}_N\cdot\mathbf{A}_N\boldsymbol{\tau}_N,\\ &\geq c_A(2\tilde{\tau}_{13}^2 + 2\tilde{\tau}_{23}^2 + \tilde{\tau}_{33}^2 + 2\tau_{13}^2 + 2\tau_{23}^2 + \tau_{33}^2),\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\boldsymbol{\sigma}\cdot\mathbf{A}\boldsymbol{\tau} &= \tilde{\boldsymbol{\sigma}}\cdot\mathbf{Q}^T\mathbf{A}\mathbf{Q}\tilde{\boldsymbol{\tau}},\\ &= \tilde{\boldsymbol{\sigma}}_T\cdot(\mathbf{Q}_T)^T\mathbf{A}_T\mathbf{Q}_T\tilde{\boldsymbol{\sigma}}_T + \boldsymbol{\sigma}_N\cdot\mathbf{A}_N\boldsymbol{\tau}_N,\\ &\leq C_A(\tilde{\sigma}_{11}^2 + 2\tilde{\sigma}_{12}^2 + \tilde{\sigma}_{22}^2 + 2\sigma_{13}^2 + 2\sigma_{23}^2 + \sigma_{33}^2)^{1/2}\\ &\quad\times(\tilde{\tau}_{11}^2 + 2\tilde{\tau}_{12}^2 + \tilde{\tau}_{22}^2 + 2\tau_{13}^2 + 2\tau_{23}^2 + \tau_{33}^2)^{1/2}.\end{aligned} \tag{3.8}$$

for all $(\tau_{ij}), (\sigma_{ij}) \in \mathbf{S}$ and all $x^\varepsilon \in \bar{\Omega}^-$.

4. Mixed Variational Formulation in Curvilinear Coordinates

This section aims to give the mixed variational formulation of the elasticity problem (1.5) using the notation introduced in the preceding sections. Well-posedness is then proved thanks to Brezzi's theorem.

The unknowns of the mixed variational formulation of the problem expressed in curvilinear coordinates are:

- the vector field

$$\mathbf{u}^\varepsilon = (u_i^\varepsilon)\colon \bar{\Omega}_\varepsilon \to \mathbb{R}^3,$$

  where the three functions $u_i^\varepsilon\colon \bar{\Omega}_\varepsilon \to \mathbb{R}$ are the covariant components of the displacement field of the points of the shell;
- the symmetric tensor field

$$\sigma^\varepsilon = (\sigma_{ij}^\varepsilon)\colon \bar{\Omega}_\varepsilon \to \mathbb{R}^9,$$

  where the nine functions $\sigma_{ij}^\varepsilon\colon \bar{\Omega}_\varepsilon \to \mathbb{R}$ are the covariant components of the stress tensor.

In what follows $\mathbf{v}^+$ (respectively $\mathbf{v}^-$) denotes the restriction of $\mathbf{v}$ to Ω^+ (respectively Ω^-). Let us introduce some functional spaces, namely

$$\begin{aligned}\mathbf{V}^\varepsilon = \{&\mathbf{v},\ \mathbf{v}^- \in (H^1(\Omega^-))^3,\ \mathbf{v}^+ \in H^1(\Omega_\varepsilon^+)^3,\\ &\mathbf{v} = 0 \text{ on } \Gamma^- \cup \Gamma_l \cup \Gamma_{\mathbf{u}}^+,\ \mathbf{v}^- = \mathbf{v}^+ \text{ on } S\}.\end{aligned}$$

$\mathbf{V}^\varepsilon$ is the Hilbert space of admissible displacement fields compatible with the transition condition on S. It is equipped with the norm

$$||\mathbf{v}||_{1,\Omega_\varepsilon} = [||\mathbf{v}||_{1,\Omega_\varepsilon^+}^2 + ||\mathbf{v}||_{1,\Omega^-}^2]^{1/2}.$$

Also, $\Sigma^\varepsilon = \{\tau = (\tau_{ij}) \in (L^2(\Omega_\varepsilon))^9,\ \tau_{ij} = \tau_{ji}\}$ is the Hilbert space of stress tensors. It is equipped with the norm

$$||\tau||_{0,\Omega_\varepsilon} = \left[\sum_{i,j} ||\tau_{ij}||^2_{0,\Omega_\varepsilon}\right]^{1/2}.$$

We assume that the applied volumic force field is defined by its contravariant components, $f^i \mathbf{g}_i^\varepsilon$, and make the following assumption

$$\begin{aligned} &\mathbf{f} = (f^i) = 0, \quad \text{in } \Omega_\varepsilon^+, \\ &\mathbf{f} \in (L^2(\Omega^-))^3. \end{aligned}$$

From the equations of the strong formulation of the elasticity problem (1.5), one classically deduces the mixed variational formulation expressed in terms of the curvilinear coordinates x_i^ε of the reference configuration $\mathbf{\Psi}(\bar{\Omega}_\varepsilon)$. The unknowns $\mathbf{u}^\varepsilon$ and σ^ε satisfy:

$$\begin{cases} \mathbf{u}^\varepsilon \in \mathbf{V}^\varepsilon, \quad \sigma^\varepsilon \in \Sigma^\varepsilon, \\ \displaystyle\int_{\Omega^-} A^{ijkl} \sigma_{kl}^\varepsilon \tau_{ij} \sqrt{g} dx^\varepsilon + \int_{\Omega_\varepsilon^+} C^{ijkl,\varepsilon} \sigma_{kl}^\varepsilon \tau_{ij} \sqrt{g^\varepsilon} dx^\varepsilon \\ \quad \displaystyle = \int_{\Omega^-} G^{ijkl} e_{kl}(\mathbf{u}^\varepsilon) \tau_{ij} \sqrt{g} dx^\varepsilon + \int_{\Omega_\varepsilon^+} G^{ijkl,\varepsilon} e_{kl}^\varepsilon(\mathbf{u}^\varepsilon) \tau_{ij} \sqrt{g^\varepsilon} dx^\varepsilon, \quad \forall \tau \in \Sigma^\varepsilon, \\ \displaystyle\int_{\Omega^-} G^{ijkl} e_{kl}(\mathbf{v}) \sigma_{ij}^\varepsilon \sqrt{g} dx^\varepsilon + \int_{\Omega_\varepsilon^+} G^{ijkl,\varepsilon} e_{kl}^\varepsilon(\mathbf{v}) \sigma_{ij}^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon \\ \quad \displaystyle = \int_{\Omega^-} f^i v_i \sqrt{g} dx^\varepsilon, \quad \forall \mathbf{v} \in \mathbf{V}^\varepsilon. \end{cases}$$

Using vectorial notation, we define:

$$\begin{aligned} A^\varepsilon(\sigma, \tau) &= \int_{\Omega^-} [\tilde{\boldsymbol{\sigma}}_T \cdot (\mathbf{Q}_T)^T \mathbf{A}_T \mathbf{Q}_T \tilde{\boldsymbol{\tau}}_T + \boldsymbol{\sigma}_N \cdot \mathbf{A}_N \boldsymbol{\tau}_N] \sqrt{g} dx^\varepsilon \\ &\quad + \int_{\Omega_\varepsilon^+} [\tilde{\boldsymbol{\sigma}}_T \cdot (\mathbf{Q}_T^\varepsilon)^T \mathbf{C}_T^\varepsilon \mathbf{Q}_T^\varepsilon \tilde{\boldsymbol{\tau}}_T + \boldsymbol{\sigma}_N \cdot \mathbf{C}_N^\varepsilon \boldsymbol{\tau}_N] \sqrt{g^\varepsilon} dx^\varepsilon, \\ &= \int_{\Omega^-} [\tilde{\boldsymbol{\sigma}}_T \cdot (\mathbf{Q}_T)^T \mathbf{A}_T \mathbf{Q}_T \tilde{\boldsymbol{\tau}}_T + \boldsymbol{\sigma}_N \cdot \mathbf{A}_N \boldsymbol{\tau}_N] \sqrt{g} dx^\varepsilon \\ &\quad + \int_{\Omega_\varepsilon^+} \left[\frac{1}{s_\varepsilon + 2\mu_e \varepsilon} \tilde{\sigma}_{11} \tilde{\tau}_{11} + \frac{1}{\mu_e \varepsilon} \tilde{\sigma}_{12} \tilde{\tau}_{12} + \frac{1}{2\mu_e \varepsilon} \tilde{\sigma}_{22} \tilde{\tau}_{22} \right. \\ &\quad \left. + \frac{1}{2\mu_e \varepsilon} \boldsymbol{\sigma}_N \cdot \mathbf{G}_N^\varepsilon \boldsymbol{\tau}_N \right] \sqrt{g^\varepsilon} dx^\varepsilon, \\ B^\varepsilon(\mathbf{v}, \tau) &= \int_{\Omega^-} [\tilde{\mathbf{e}}_T(\mathbf{v}) \cdot (\mathbf{Q}_T)^T \mathbf{G}_T \mathbf{Q}_T \tilde{\boldsymbol{\tau}}_T + \mathbf{e}_N(\mathbf{v}) \cdot \mathbf{G}_N \boldsymbol{\tau}_N] \sqrt{g} dx^\varepsilon \end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{\Omega_\varepsilon^+} [\tilde{\mathbf{e}}_T^\varepsilon(\mathbf{v}) \cdot (\mathbf{Q}_T^\varepsilon)^T \mathbf{G}_T^\varepsilon \mathbf{Q}_T^\varepsilon \tilde{\boldsymbol{\tau}}_T + \mathbf{e}_N^\varepsilon(\mathbf{v}) \cdot \mathbf{G}_N^\varepsilon \boldsymbol{\tau}_N]\sqrt{g^\varepsilon} dx^\varepsilon, \\
&= \int_{\Omega^-} [\tilde{\mathbf{e}}_T(\mathbf{v}) \cdot \tilde{\boldsymbol{\tau}}_T + \mathbf{e}_N(\mathbf{v}) \cdot \mathbf{G}_N \boldsymbol{\tau}_N]\sqrt{g} dx^\varepsilon \\
&\quad + \int_{\Omega_\varepsilon^+} [\tilde{\mathbf{e}}_T^\varepsilon(\mathbf{v}) \cdot \tilde{\boldsymbol{\tau}}_T + \mathbf{e}_N^\varepsilon(\mathbf{v}) \cdot \mathbf{G}_N^\varepsilon \boldsymbol{\tau}_N]\sqrt{g^\varepsilon} dx^\varepsilon, \\
L(\mathbf{v}) &= \int_{\Omega^-} f^i v_i \sqrt{g} dx^\varepsilon.
\end{aligned}
\tag{4.1}
$$

With this notation the variational mixed formulation reads

$$\mathbf{u}^\varepsilon \in \mathbf{V}^\varepsilon, \quad \sigma^\varepsilon \in \Sigma^\varepsilon, \tag{4.2}$$

$$A^\varepsilon(\sigma^\varepsilon, \tau) = B^\varepsilon(\mathbf{u}^\varepsilon, \tau), \quad \forall \tau \in \Sigma^\varepsilon, \tag{4.3}$$

$$B^\varepsilon(\mathbf{v}, \sigma^\varepsilon) = L(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}^\varepsilon. \tag{4.4}$$

and the following result holds.

Theorem 4.1. *There exists a unique solution* $(\mathbf{u}^\varepsilon, \sigma^\varepsilon)$ *to problem* (4.2)–(4.4). *Moreover, there exist two positive constants,* C_σ^ε *and* $C_{\mathbf{u}}^\varepsilon$ *depending on* ε *only such that*

$$
\begin{aligned}
||\sigma^\varepsilon||_{0,\Omega^\varepsilon} &\leq C_\sigma^\varepsilon ||\mathbf{f}||_{0,\Omega^-}, \\
||\mathbf{u}^\varepsilon||_{1,\Omega^\varepsilon} &\leq C_{\mathbf{u}}^\varepsilon ||\mathbf{f}||_{0,\Omega^-}.
\end{aligned}
$$

Proof. It is a direct consequence of Brezzi's theorem [4] (also see Babuška and Aziz [1]). Let us first note that since $\boldsymbol{\sigma} = \mathbf{Q}^\varepsilon \tilde{\boldsymbol{\sigma}}$, since $x^\varepsilon \to \mathbf{Q}^\varepsilon(x^\varepsilon)$ is continuous on $\bar{\Omega}^\varepsilon$ and since $\mathbf{Q}^\varepsilon$ is invertible for all $x^\varepsilon \in \bar{\Omega}^\varepsilon$, there exist two constants $c^\varepsilon, C^\varepsilon > 0$, such that

$$c^\varepsilon ||\tilde{\sigma}||_{0,\Omega^\varepsilon} \leq ||\sigma||_{0,\Omega^\varepsilon} \leq C^\varepsilon ||\tilde{\sigma}||_{0,\Omega^\varepsilon}, \quad \forall \sigma \in \Sigma^\varepsilon. \tag{4.5}$$

Since $x^\varepsilon \to g^\varepsilon(x^\varepsilon)$ is continuous on $\bar{\Omega}^\varepsilon$ and strictly positive, there exist two constants $g_0^\varepsilon, g_1^\varepsilon > 0$, such that

$$g_0^\varepsilon \leq g^\varepsilon \leq g_1^\varepsilon, \quad \forall x^\varepsilon \in \bar{\Omega}^\varepsilon. \tag{4.6}$$

From (4.5), (4.6), (3.4) and (3.8), we deduce that the bilinear form $A^\varepsilon(\sigma, \tau)$ is continuous on $\Sigma^\varepsilon \times \Sigma^\varepsilon$. There exists a positive constant M_A^ε, such that

$$|A^\varepsilon(\sigma, \tau)| \leq M_A^\varepsilon ||\sigma||_{0,\Omega^\varepsilon} ||\tau||_{0,\Omega^\varepsilon}, \quad \forall \sigma, \ \tau \in \Sigma^\varepsilon.$$

Since $x^\varepsilon \to \Gamma_{ij}^{k,\varepsilon}$ is continuous on $\bar{\Omega}^\varepsilon$, it follows from (2.10) that there exists a constant $C^\varepsilon > 0$, such that

$$||e^\varepsilon(\mathbf{v})||_{0,\Omega_\varepsilon} \leq C^\varepsilon ||\mathbf{v}||_{1,\Omega_\varepsilon}, \quad \forall \mathbf{v} \in (H^1(\Omega_\varepsilon))^3. \tag{4.7}$$

We deduce from (4.5) to (4.7) and (2.8) that the bilinear form $B^\varepsilon(\mathbf{v}, \tau)$ is continuous on $\mathbf{V}^\varepsilon \times \Sigma^\varepsilon$.

We deduce from (4.5), (4.6), (3.3) and (3.7) that there exists a constant $m_A^\varepsilon > 0$, such that

$$A^\varepsilon(\sigma,\sigma) \geq m_A^\varepsilon ||\sigma||^2_{0,\Omega^\varepsilon}, \quad \forall \sigma \in \Sigma^\varepsilon.$$

Eventually, the inf-sup condition

$$\inf_{\substack{\mathbf{v}\in\mathbf{V}^\varepsilon \\ ||\mathbf{v}||_{1,\Omega_\varepsilon}=1}} \sup_{\substack{\tau\in\Sigma^\varepsilon \\ ||\tau||_{0,\Omega_\varepsilon}=1}} B(\mathbf{v},\tau) > 0$$

follows essentially from Korn's inequality in curvilinear coordinates (see, for example, [8]). There exists a constant $C^\varepsilon = C^\varepsilon(\Omega_\varepsilon, \mathbf{\Psi}, \Gamma_l \cup \Gamma^- \cup \Gamma^+_{\mathbf{u}})$, such that

$$||\mathbf{v}||_{1,\Omega_\varepsilon} \leq C^\varepsilon ||e^\varepsilon(\mathbf{v})||_{0,\Omega_\varepsilon}, \quad \forall \mathbf{v} \in \mathbf{V}^\varepsilon.$$

This condition can be written as: there exists a constant $\beta^\varepsilon > 0$, such that

$$\sup_{\tau\in\Sigma^\varepsilon} \frac{B^\varepsilon(\mathbf{v},\tau)}{||\tau||_{0,\Omega^\varepsilon}} \geq \beta^\varepsilon ||\mathbf{v}||_{1,\Omega^\varepsilon}, \quad \forall \mathbf{v} \in V^\varepsilon,$$

and one then has the classical bounds:

$$||\sigma^\varepsilon||_{0,\Omega^\varepsilon} \leq \frac{1}{\beta^\varepsilon}\left(1 + \frac{M_A^\varepsilon}{m_A^\varepsilon}\right)||\mathbf{f}||_{0,\Omega^-},$$
$$||\mathbf{u}^\varepsilon||_{1,\Omega^\varepsilon} \leq \frac{M_A^\varepsilon}{(\beta^\varepsilon)^2}\left(1 + \frac{M_A^\varepsilon}{m_A^\varepsilon}\right)||\mathbf{f}||_{0,\Omega^-}. \qquad \square$$

5. Formulation over a Domain Independent of ε

Let us define the sets

$$\begin{aligned} \Omega &= w \times]-\varepsilon_l, 1[, \\ \Omega^+ &= w \times]0,1[, \\ \Omega^- &= w \times]-\varepsilon_l, 0[, \\ \Gamma_l^+ &= \gamma \times [0,1], \\ \Gamma_l^- &= \gamma \times [-\varepsilon_l, 0], \\ \Gamma^- &= w \times \{-\varepsilon_l\}, \\ \Gamma^+_{\mathbf{u}} &= w_{\mathbf{u}} \times \{1\}, \\ \Gamma^+_\sigma &= w_\sigma \times \{1\}. \end{aligned}$$

Let $x = (x_i)$ denote a generic point in the set $\bar{\Omega}$, and let $\partial_i = \partial/\partial x_i$. With $x^\varepsilon \in \bar{\Omega}_\varepsilon$, we associate the point $x = (x_i) \in \bar{\Omega}$, defined by

$$\begin{aligned} x_\alpha &= x^\varepsilon_\alpha \ (=y_\alpha), \\ x_3 &= x_3^\varepsilon \quad \text{if } x^\varepsilon \in \Omega^-, \\ x_3 &= (x_3^\varepsilon/\varepsilon) \quad \text{if } x^\varepsilon \in \Omega^+_\varepsilon. \end{aligned}$$

We thus have

$$\begin{aligned} \partial_\alpha^\varepsilon &= \partial_\alpha, \\ \partial_3^\varepsilon &= \partial_3 && \text{if } x^\varepsilon \in \Omega^-, \\ \partial_3^\varepsilon &= (\partial_3/\varepsilon) && \text{if } x^\varepsilon \in \Omega_\varepsilon^+. \end{aligned}$$

The functions

$$g_{ij},\ g^{ij},\ g,\ \Gamma_{ij}^p\colon\ \bar{\Omega}^- \to \mathbb{R},$$

are not affected by the scaling. On the other hand, with these same functions defined on $\bar{\Omega}_\varepsilon^+$,

$$g_{ij}^\varepsilon,\ g^{ij,\varepsilon},\ g^\varepsilon,\ \Gamma_{ij}^{p,\varepsilon}\colon\ \bar{\Omega}_\varepsilon^+ \to \mathbb{R},$$

we associate the functions

$$g_{ij}(\varepsilon),\ g^{ij}(\varepsilon),\ g(\varepsilon),\ \Gamma_{ij}^p(\varepsilon)\colon\ \bar{\Omega}^+ \to \mathbb{R},$$

defined for all $x^\varepsilon \in \Omega_\varepsilon^+$ by

$$\begin{aligned} g_{ij}(\varepsilon)(x) &= g_{ij}^\varepsilon(x^\varepsilon), \quad g^{ij}(\varepsilon)(x) = g^{ij,\varepsilon}(x^\varepsilon), \\ g(\varepsilon)(x) &= g^\varepsilon(x^\varepsilon), \quad \Gamma_{ij}^p(\varepsilon)(x) = \Gamma_{ij}^{p,\varepsilon}(x^\varepsilon). \end{aligned} \tag{5.1}$$

With the unknowns $\mathbf{u}^\varepsilon\colon \bar{\Omega}_\varepsilon \to \mathbb{R}^3$ and $\sigma^\varepsilon\colon \bar{\Omega}_\varepsilon \to \mathbb{R}^9$ of problem (4.2)–(4.4), we associate the scaled unknowns $\mathbf{u}(\varepsilon)\colon \bar{\Omega} \to \mathbb{R}^3$ and $\sigma(\varepsilon)\colon \bar{\Omega} \to \mathbb{R}^9$, defined by

$$\begin{aligned} \mathbf{u}(\varepsilon)(x) &= \mathbf{u}^\varepsilon(x^\varepsilon) \quad \forall x^\varepsilon \in \bar{\Omega}_\varepsilon, \\ \sigma(\varepsilon)(x) &= \sigma^\varepsilon(x^\varepsilon) \quad \forall x^\varepsilon \in \bar{\Omega}_\varepsilon. \end{aligned}$$

With any vector field $\mathbf{v} = (v_i) \in H^1(\Omega^+)^3$, we associate the symmetric tensor $(e_{ij}(\varepsilon)(\mathbf{v})) \in (L^2(\Omega^+)^9)$, defined by

$$\begin{aligned} e_{\alpha\beta}(\varepsilon)(\mathbf{v}) &= \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma_{\alpha\beta}^p(\varepsilon) v_p, \\ e_{\alpha 3}(\varepsilon)(\mathbf{v}) &= \frac{1}{2}\left(\partial_\alpha v_3 + \frac{1}{\varepsilon}\partial_3 v_\alpha\right) - \Gamma_{\alpha 3}^\rho(\varepsilon) v_\rho, \\ e_{33}(\varepsilon)(\mathbf{v}) &= \frac{1}{\varepsilon}\partial_3 v_3. \end{aligned}$$

Let us now introduce the functional spaces $\mathbf{V}$ and Σ:

$$\begin{aligned} \mathbf{V} = \{\mathbf{v},\ \mathbf{v}^- &\in (H^1(\Omega^-))^3,\ \mathbf{v}^+ \in (H^1(\Omega^+))^3, \\ &\mathbf{v} = 0 \text{ on } \Gamma^- \cup \Gamma_l \cup \Gamma_{\mathbf{u}}^+,\ \mathbf{v}^- = \mathbf{v}^+ \text{ on } S\}. \end{aligned}$$

$\mathbf{V}$ is the Hilbert space of admissible displacement fields compatible with the transition condition on S. Also

$$\Sigma = \{\tau = (\tau_{ij}) \in L^2(\Omega)^9,\ \tau_{ij} = \tau_{ji}\}$$

is the Hilbert space of stress tensors.

Eventually, the following notations are used in the scaled variational mixed formulation.

$$\begin{aligned}
A(\varepsilon)(\sigma,\tau) &= A^-(\sigma,\tau) + A^+(\varepsilon)(\sigma,\tau),\\
A^-(\sigma,\tau) &= \int_{\Omega^-} [\tilde{\boldsymbol{\sigma}}_T \cdot (\mathbf{Q}_T)^T \mathbf{A}_T \mathbf{Q}_T \tilde{\boldsymbol{\tau}}_T + \boldsymbol{\sigma}_N \cdot \mathbf{A}_N \boldsymbol{\tau}_N]\sqrt{g}dx,\\
A^+(\varepsilon)(\sigma,\tau) &= \int_{\Omega^+} \left[\frac{\varepsilon}{s(\varepsilon)+2\mu_e\varepsilon}\tilde{\sigma}_{11}\tilde{\tau}_{11} + \frac{1}{\mu_e}\tilde{\sigma}_{12}\tilde{\tau}_{12} + \frac{1}{2\mu_e}\tilde{\sigma}_{22}\tilde{\tau}_{22}\right.\\
&\qquad \left. + \frac{1}{2\mu_e}\boldsymbol{\sigma}_N \cdot \mathbf{G}_N(\varepsilon)\boldsymbol{\tau}_N\right]\sqrt{g(\varepsilon)}dx,\\
B(\varepsilon)(\mathbf{v},\tau) &= B^-(\mathbf{v},\tau) + B^+(\varepsilon)(\mathbf{v},\tau),\\
B^-(\mathbf{v},\tau) &= \int_{\Omega^-} [\tilde{\mathbf{e}}_T(\mathbf{v}) \cdot \tilde{\boldsymbol{\tau}}_T + \mathbf{e}_N(\mathbf{v}) \cdot \mathbf{G}_N \boldsymbol{\tau}_N]\sqrt{g}\,dx,\\
B^+(\varepsilon)(\mathbf{v},\tau) &= \int_{\Omega^+} [\varepsilon\tilde{\mathbf{e}}_T(\varepsilon)(\mathbf{v}) \cdot \tilde{\boldsymbol{\tau}}_T + \varepsilon\mathbf{e}_N(\varepsilon)(\mathbf{v}) \cdot \mathbf{G}_N(\varepsilon)\boldsymbol{\tau}_N]\sqrt{g(\varepsilon)}\,dx,\\
L(\mathbf{v}) &= \int_{\Omega^-} f^i v_i \sqrt{g}\,dx.
\end{aligned}$$

The scaled unknowns $\mathbf{u}(\varepsilon)$ and $\sigma(\varepsilon)$ solve the scaled variational mixed formulation, (5.2)–(5.4), now posed over the set Ω, and thus over a domain which is independent of ε,

$$\mathbf{u}(\varepsilon) \in \mathbf{V}, \quad \sigma(\varepsilon) \in \Sigma, \tag{5.2}$$

$$A(\varepsilon)(\sigma(\varepsilon),\tau) = B(\varepsilon)(\mathbf{u}(\varepsilon),\tau) \quad \forall\tau \in \Sigma, \tag{5.3}$$

$$B(\varepsilon)(\mathbf{v},\sigma(\varepsilon)) = L(\mathbf{v}) \quad \forall\mathbf{v} \in \mathbf{V}. \tag{5.4}$$

In the following lemmas, we gather properties needed in the sequel concerning the behavior of different functions as $\varepsilon \to 0$. $||\cdot||_{0,\infty,\bar{\Omega}^+}$ denotes the usual norm of the space $\mathcal{C}^0(\bar{\Omega}^+)$. The constant ε_0 is defined in Sec. 2.1.

Lemma 5.1. *The functions* $g_{ij}(\varepsilon)$, $g^{ij}(\varepsilon)$, $g(\varepsilon)$, $\Gamma^p_{ij}(\varepsilon)$ *are defined as in* (5.1) *and the functions* a_{ij}, a^{ij}, a, $\Gamma^\rho_{\alpha\beta}$, $b_{\alpha\beta}$, b^ρ_α *are defined as in* (2.1)–(2.3). *All the functions* $a_{ij},\ldots,b^\rho_\alpha \in \mathcal{C}^0(\bar{w})$ *are identified with functions in* $\mathcal{C}^0(\bar{\Omega}^+)$. *Then there exist constants* $C > 0$ (*all denoted by the same symbol*) *such that*

$$||g_{\alpha\beta}(\varepsilon) - a_{\alpha\beta}||_{0,\infty,\bar{\Omega}^+} \le C\varepsilon, \tag{5.5}$$

$$||g^{\alpha\beta}(\varepsilon) - a^{\alpha\beta}||_{0,\infty,\bar{\Omega}^+} \le C\varepsilon, \tag{5.6}$$

$$g_{i3}(\varepsilon) = g^{i3}(\varepsilon) = \delta_{i3}, \tag{5.7}$$

$$||g(\varepsilon) - a||_{0,\infty,\bar{\Omega}^+} \le C\varepsilon, \tag{5.8}$$

$$||\Gamma^\rho_{\alpha\beta}(\varepsilon) - \Gamma^\rho_{\alpha\beta}||_{0,\infty,\bar{\Omega}^+} \le C\varepsilon, \tag{5.9}$$

$$||\Gamma^3_{\alpha\beta}(\varepsilon) - b_{\alpha\beta}||_{0,\infty,\bar{\Omega}^+} \le C\varepsilon, \tag{5.10}$$

$$||\Gamma^\rho_{\alpha3}(\varepsilon) + b^\rho_\alpha||_{0,\infty,\bar{\Omega}^+} \le C\varepsilon, \tag{5.11}$$

$$\Gamma^3_{\alpha3}(\varepsilon) = \Gamma^p_{33}(\varepsilon) = 0. \tag{5.12}$$

Proof. The proof can be found in [9, Lemma 3.1] and completed in [10, Lemma 3.1]. The main argument is the fact that $\mathbf{g}_\alpha(\varepsilon) = \mathbf{a}_\alpha + \varepsilon x_3 \partial_\alpha \mathbf{a}_3$ and $\mathbf{g}_3(\varepsilon) = \mathbf{a}_3$. □

Lemma 5.2. *There exist constants g_0, g_1, such that*

$$0 < g_0 \le g(\varepsilon) \le g_1, \quad \forall \varepsilon \in]0, \varepsilon_0], \ \forall x \in \bar{\Omega}^+, \tag{5.13}$$

$$0 < g_0 \le g \le g_1, \quad \forall x \in \bar{\Omega}^-. \tag{5.14}$$

Proof. (5.14) follows from the continuity of the strictly positive function g on $\bar{\Omega}^-$. (5.13) follows from (2.5) and (5.8). □

Let us define the 6×6 matrix $\mathbf{G}(0)$ by

$$\mathbf{G}(0) = \left(\begin{array}{c|c} \mathbf{G}_T(0) & 0 \\ \hline 0 & \mathbf{G}_N(0) \end{array} \right),$$

where

$$\mathbf{G}_T(0) = \begin{pmatrix} a^{11}a^{11} & \sqrt{2}a^{11}a^{12} & a^{12}a^{12} \\ \sqrt{2}a^{11}a^{12} & a^{11}a^{22} + a^{12}a^{12} & \sqrt{2}a^{12}a^{22} \\ a^{12}a^{12} & \sqrt{2}a^{12}a^{22} & a^{22}a^{22} \end{pmatrix},$$

and

$$\mathbf{G}_N(0) = \begin{pmatrix} a^{11}a^{33} & a^{12}a^{33} & 0 \\ a^{12}a^{33} & a^{22}a^{33} & 0 \\ 0 & 0 & a^{33}a^{33} \end{pmatrix}.$$

From Lemma 5.1 we easily deduce that there exists a constant $C > 0$, such that

$$||(\mathbf{G}(\varepsilon))_{ij} - (\mathbf{G}(0))_{ij}||_{0,\infty,\bar{\Omega}^+} \le C\varepsilon, \tag{5.15}$$

where the 6×6 matrix $\mathbf{G}(\varepsilon)$ is defined in a obvious way.

Lemma 5.3. *There exist two constants $c_G > 0$ and $C_G > 0$ independent of ε, such that*

$$\boldsymbol{\tau} \cdot \mathbf{G_N}(\varepsilon)\boldsymbol{\tau} \ge c_G||\boldsymbol{\tau}||^2, \quad \forall \varepsilon \in [0, \varepsilon_0], \ \forall x \in \bar{\Omega}^+, \ \forall \boldsymbol{\tau} \in \mathbb{R}^3. \tag{5.16}$$

$$\boldsymbol{\tau} \cdot \mathbf{G_N}\boldsymbol{\tau} \ge c_G||\boldsymbol{\tau}||^2, \quad \forall x \in \bar{\Omega}^-, \ \forall \boldsymbol{\tau} \in \mathbb{R}^3. \tag{5.17}$$

$$\boldsymbol{\sigma} \cdot \mathbf{G_N}(\varepsilon)\boldsymbol{\tau} \le C_G||\boldsymbol{\sigma}||||\boldsymbol{\tau}||, \quad \forall \varepsilon \in [0, \varepsilon_0], \ \forall x \in \bar{\Omega}^+, \ \forall \boldsymbol{\tau} \in \mathbb{R}^3. \tag{5.18}$$

$$\boldsymbol{\sigma} \cdot \mathbf{G_N}\boldsymbol{\tau} \le C_G||\boldsymbol{\sigma}||||\boldsymbol{\tau}||, \quad \forall x \in \bar{\Omega}^-, \ \forall \boldsymbol{\tau} \in \mathbb{R}^3. \tag{5.19}$$

Proof. We only detail the proof of (5.16). From (2.7) we deduce that for each $\varepsilon > 0$, there exists $c_G(\varepsilon) > 0$, such that

$$\boldsymbol{\tau} \cdot \mathbf{G_N}(\varepsilon)\boldsymbol{\tau} \ge c_G(\varepsilon)||\boldsymbol{\tau}||^2,$$

for all $x \in \bar{\Omega}^+$ and all $\boldsymbol{\tau} \in \mathbb{R}^3$.

$\mathbf{G}_N(0)$ is clearly symmetric, positive definite and uniform with respect to $x \in \bar{\Omega}^+$. Therefore, there exists a constant $c_{G0} > 0$, such that

$$\boldsymbol{\tau} \cdot \mathbf{G}_{\mathbf{N}}(0)\boldsymbol{\tau} \geq c_{G0}||\boldsymbol{\tau}||^2,$$

for all $x \in \bar{\Omega}^+$ and all $\boldsymbol{\tau} \in \mathbb{R}^3$.

The continuity of the mapping

$$(x, \varepsilon, \boldsymbol{\tau}) \in \bar{\Omega}^+ \times [0, \varepsilon_0] \times \mathcal{B} \to \boldsymbol{\tau} \cdot \mathbf{G}(\varepsilon)(x)\boldsymbol{\tau},$$

where $\mathcal{B} = \{\boldsymbol{\tau} \in \mathbb{R}^3, ||\boldsymbol{\tau}|| = 1\}$, and the compacity of the domain lead to the existence of a constant c_G, such that relation (5.16) holds for $0 \leq \varepsilon \leq \varepsilon_0$. □

Lemma 5.4. *There exists a constant $C > 0$, such that*

$$||s(\varepsilon) - s(0)||_{0,\infty,\bar{\Omega}^+} \leq C\varepsilon, \quad \forall \varepsilon > 0, \tag{5.20}$$

where $s(0) = \mathrm{trace}(\mathbf{D}_T\mathbf{G}_T(0))$.

There exist two constants s_0 and s_1, such that

$$0 < s_0 \leq s(\varepsilon) \leq s_1, \quad \forall x \in \bar{\Omega}^+, \ \forall \varepsilon > 0. \tag{5.21}$$

Proof. The scaled matrices $\mathbf{H}_T(\varepsilon), \mathbf{P}_T(\varepsilon), \mathbf{S}(\varepsilon)$ are defined in an obvious way on $\bar{\Omega}^+$ for all $\varepsilon > 0$. Since

$$\begin{aligned} s(\varepsilon) &= \mathrm{trace}(\mathbf{S}(\varepsilon)) = \mathrm{trace}(\mathbf{P}_T(\varepsilon)^T\mathbf{D}_T\mathbf{P}_T(\varepsilon)) \\ &= \mathrm{trace}(\mathbf{D}_T\mathbf{P}_T(\varepsilon)\mathbf{P}_T(\varepsilon)^T) = \mathrm{trace}(\mathbf{D}_T\mathbf{G}_T(\varepsilon)), \end{aligned}$$

we deduce from (5.15) that

$$||s(\varepsilon) - \mathrm{trace}(\mathbf{D}_T\mathbf{G}_T(0))||_{0,\infty,\bar{\Omega}^+} \leq C\varepsilon.$$

In order to infer (5.21), it remains to show that

$$s(0) = \mathrm{trace}(\mathbf{D}_T\mathbf{G}_T(0)) > 0, \quad \forall x \in \bar{\Omega}^+.$$

As for $\mathbf{G}(0)$, $\mathbf{H}(0)$ is defined in an obvious way using the functions a_{ij}. $\mathbf{H}(0)$ is symmetric, positive definite and uniform with respect to $x \in \bar{\Omega}^+$. We proceed as in Sec. 3.1. There exists an invertible matrix $\mathbf{P}_0$, such that

$$\begin{aligned} \mathbf{P}_0^T\mathbf{H}_T(0)\mathbf{P}_0 &= \mathbf{I}, \\ \mathbf{P}_0^T\mathbf{D}_T\mathbf{P}_0 &= \mathrm{diag}(s_0, 0, 0), \end{aligned}$$

with $s_0 > 0$, $\forall x \in \bar{\Omega}^+$. Since $\mathbf{G}_T(0)^{-1} = \mathbf{H}_T(0) = (\mathbf{P}_0\mathbf{P}_0^T)^{-1}$, it is clear that

$$s(0) = \mathrm{trace}(\mathbf{D}_T\mathbf{G}_T(0)) = \mathrm{trace}(\mathbf{D}_T\mathbf{P}_0\mathbf{P}_0^T) = \mathrm{trace}(\mathbf{P}_0^T\mathbf{D}_T\mathbf{P}_0) = s_0.$$

□

6. Asymptotic Analysis

In this section, we establish our main result. The goal is to pass to the limit as $\varepsilon \to 0$ in the scaled variational mixed formulation (5.2)–(5.4), in order to derive the asymptotic formulation and obtain the announced boundary conditions on the surface S. This is achieved in two steps. In Sec. 6.1, we obtain several *a priori* estimations on the sequences, $(\mathbf{u}(\varepsilon))_{\varepsilon>0}$ and $(\sigma(\varepsilon))_{\varepsilon>0}$, presented in Lemma 6.1 through Lemma 6.4. All these estimations are then used in Sec. 6.2 in which we let $\varepsilon \to 0$ to obtain the limit formulation, which is presented in Theorem 6.6. Eventually, we show in Theorem 6.7 how the solution of the asymptotic problem can be explicitely computed in Ω^+ and deduce boundary conditions on S.

6.1. *A priori estimations on* $(\mathbf{u}(\varepsilon))_{\varepsilon>0}$ *and* $(\boldsymbol{\sigma}(\varepsilon))_{\varepsilon>0}$

Lemma 6.1. *Let* $(\mathbf{u}(\varepsilon),\ \sigma(\varepsilon))$ *be the solution to problem* (5.2)–(5.4). *There exist constants* $C_1, C_2 > 0$, *such that for all* $\varepsilon \in]0, \varepsilon_0]$,

$$\left[\sum_{\alpha,\beta} ||\tilde{e}_{\alpha\beta}(\mathbf{u}(\varepsilon))||^2_{0,\Omega^-}\right]^{1/2} \le C_1 \left[\sum_{\alpha,\beta} ||\tilde{\sigma}_{\alpha\beta}(\varepsilon)||^2_{0,\Omega^-}\right]^{1/2} \tag{6.1}$$

and

$$\begin{aligned} &[2||e_{13}(\mathbf{u}(\varepsilon))||^2_{0,\Omega^-} + 2||e_{23}(\mathbf{u}(\varepsilon))||^2_{0,\Omega^-} + ||e_{33}(\mathbf{u}(\varepsilon))||^2_{0,\Omega^-}]^{1/2} \\ &\quad \le C_2[2||\sigma_{13}(\varepsilon)||^2_{0,\Omega^-} + 2||\sigma_{23}(\varepsilon)||^2_{0,\Omega^-} + ||\sigma_{33}(\varepsilon)||^2_{0,\Omega^-}]^{1/2}. \end{aligned} \tag{6.2}$$

Proof. In (5.3), let us choose $\tau_{ij} = 0$ in Ω^+ and $\tilde{\tau}_{\alpha\beta} = \tilde{e}_{\alpha\beta}(\mathbf{u}(\varepsilon))$ in Ω^-.

$$\int_{\Omega^-} \tilde{\boldsymbol{\sigma}}_T(\varepsilon) \cdot (\mathbf{Q}_T)^T \mathbf{A}_T \mathbf{Q}_T \tilde{\mathbf{e}}_T(\mathbf{u}(\varepsilon))\sqrt{g}dx = \int_{\Omega^-} \tilde{\mathbf{e}}_T(\mathbf{u}(\varepsilon)) \cdot \tilde{\mathbf{e}}_T(\mathbf{u}(\varepsilon))\sqrt{g}dx.$$

Using (5.14), (3.8) and Cauchy–Schwarz's inequality we obtain

$$\begin{aligned} &\int_{\Omega^-} \tilde{\boldsymbol{\sigma}}_T(\varepsilon) \cdot (\mathbf{Q}_T)^T \mathbf{A}_T \mathbf{Q}_T \tilde{\mathbf{e}}_T(\mathbf{u}(\varepsilon))\sqrt{g}dx \\ &\quad \le \sqrt{g_1} C_A \left[\sum_{\alpha,\beta} ||\tilde{\sigma}_{\alpha\beta}(\varepsilon)||^2_{0,\Omega^-}\right]^{1/2} \left[\sum_{\alpha,\beta} ||\tilde{e}_{\alpha\beta}(\mathbf{u}(\varepsilon))||^2_{0,\Omega^-}\right]^{1/2}. \end{aligned}$$

With (5.14) and (3.8) we have

$$\int_{\Omega^-} \tilde{\mathbf{e}}_T(\varepsilon)(\mathbf{u}(\varepsilon)) \cdot \tilde{\mathbf{e}}_T(\varepsilon)(\mathbf{u}(\varepsilon))\sqrt{g}dx \ge \sqrt{g_0} \sum_{\alpha,\beta} ||\tilde{e}_{\alpha\beta}(\mathbf{u}(\varepsilon))||^2_{0,\Omega^-}$$

and we conclude that the first inequality is verified.

The second inequality is proved in the same way choosing $\tau_{i3} = e_{i3}(\mathbf{u}(\varepsilon))$ in Ω^-, and using (5.17) and (5.19). ☐

Lemma 6.2. *Let* $(\mathbf{u}(\varepsilon),\ \sigma(\varepsilon))$ *be the solution to problem* (5.2)–(5.4). *There exist positive constants* $C_3,\ C_4,\ C_5,\ C_6$ *and* C_7, *such that for all* $\varepsilon \in]0, \varepsilon_0]$,

$$||\tilde{\sigma}_{\alpha\beta}(\varepsilon)||_{0,\Omega^-} \leq C_3, \tag{6.3}$$

$$||\sigma_{i3}(\varepsilon)||_{0,\Omega^-} \leq C_4, \tag{6.4}$$

$$||\tilde{\sigma}_{11}(\varepsilon)||_{0,\Omega^+} \leq C_5\sqrt{\frac{2\mu_e\varepsilon + s_1}{\varepsilon}}, \tag{6.5}$$

$$||\tilde{\sigma}_{\alpha 2}(\varepsilon)||_{0,\Omega^+} \leq C_6, \tag{6.6}$$

$$||\sigma_{i3}(\varepsilon)||_{0,\Omega^+} \leq C_7. \tag{6.7}$$

Proof. Let us choose $\tilde{\tau}_{\alpha\beta} = \tilde{\sigma}_{\alpha\beta}(\varepsilon)$, $\tilde{\tau}_{i3} = \tilde{\sigma}_{i3}(\varepsilon)$ in (5.3) and $\mathbf{v} = \mathbf{u}(\varepsilon)$ in (5.4). We obtain

$$A(\varepsilon)(\sigma(\varepsilon),\ \sigma(\varepsilon)) = L(\mathbf{u}(\varepsilon)).$$

From (5.13), (5.14), (3.7) and (5.21), we deduce

$$\begin{aligned}A(\varepsilon)(\sigma(\varepsilon),\ \sigma(\varepsilon)) \geq \sqrt{g_0}c_A\Bigg(&\sum_{\alpha\beta}||\tilde{\sigma}_{\alpha\beta}(\varepsilon)||^2_{0,\Omega^-}\\
&+2||\sigma_{13}(\varepsilon)||^2_{0,\Omega^-}+2||\sigma_{23}(\varepsilon)||^2_{0,\Omega^-}+||\sigma_{33}(\varepsilon)||^2_{0,\Omega^-}\Bigg)\\
&+\frac{\varepsilon}{2\mu_e\varepsilon+s_1}\sqrt{g_0}||\tilde{\sigma}_{11}(\varepsilon)||^2_{0,\Omega^+}\\
&+\frac{1}{2\mu_e}\sqrt{g_0}\big(2||\tilde{\sigma}_{12}(\varepsilon)||^2_{0,\Omega^+}+||\tilde{\sigma}_{22}(\varepsilon)||^2_{0,\Omega^+}\big)\\
&+\frac{\hat{c}_G}{2\mu_e}\sqrt{g_0}\big(2||\sigma_{13}(\varepsilon)||^2_{0,\Omega^+}+2||\sigma_{23}(\varepsilon)||^2_{0,\Omega^+}+||\sigma_{33}(\varepsilon)||^2_{0,\Omega^+}\big).\end{aligned}$$

Cauchy–Schwarz's inequality gives

$$L(\mathbf{u}(\varepsilon)) \leq \sqrt{g_1}\left[\sum_i ||f^i||^2_{0,\Omega^-}\right]^{1/2}\left[\sum_i ||u_i(\varepsilon)||^2_{0,\Omega^-}\right]^{1/2}.$$

From the three-dimensional Korn inequality in curvilinear coordinates [8], we deduce that there exists a constant $C = C(\Omega^-, \mathbf{\Psi}, \Gamma_l^- \cup \Gamma^-) > 0$, such that

$$\left[\sum_i ||u_i(\varepsilon)||^2_{0,\Omega^-}\right]^{1/2} \leq C\left[\sum_{i,j}||e_{ij}(\mathbf{u}(\varepsilon))||^2_{0,\Omega^-}\right]^{1/2}.$$

There exists an ε-independent constant $C_Q > 0$ (which is a norm of matrix Q on Ω^-), such that

$$\begin{aligned}\sum_{i,j}||e_{ij}(\mathbf{u}(\varepsilon))||^2_{0,\Omega^-} \leq C_Q\sum_{\alpha,\beta}&||\tilde{e}_{\alpha\beta}(\mathbf{u}(\varepsilon))||^2_{0,\Omega^-}+2||e_{13}(\mathbf{u}(\varepsilon))||^2_{0,\Omega^-}\\
&+2||e_{23}(\mathbf{u}(\varepsilon))||^2_{0,\Omega^-}+||e_{33}(\mathbf{u}(\varepsilon))||^2_{0,\Omega^-},\end{aligned}$$

and using Lemma 6.1, we obtain

$$\sum_i ||u_i(\varepsilon)||^2_{0,\Omega^-} \le C^2C_1^2C_Q \sum_{\alpha,\beta} ||\tilde{\sigma}_{\alpha\beta}(\varepsilon)||^2_{0,\Omega^-} + C^2C_2^2\big[2||\sigma_{13}(\varepsilon)||^2_{0,\Omega^-} + 2||\sigma_{23}(\varepsilon)||^2_{0,\Omega^-} + ||\sigma_{33}(\varepsilon)||^2_{0,\Omega^-}\big].$$

This eventually leads to

$$\left[\sum_i ||u_i(\varepsilon)||^2_{0,\Omega^-}\right]^{1/2} \le \sqrt{\max(C^2C_1^2C_Q, C^2C_2^2)}\left[\sum_{\alpha,\beta} ||\tilde{\sigma}_{\alpha\beta}(\varepsilon)||^2_{0,\Omega^-} + 2||\sigma_{13}(\varepsilon)||^2_{0,\Omega^-} + 2||\sigma_{23}(\varepsilon)||^2_{0,\Omega^-} + ||\sigma_{33}(\varepsilon)||^2_{0,\Omega^-}\right]^{1/2},$$

which completes the proof. □

It is worth noticing here the particular form of estimate (6.5) in the preceding lemma. This estimate is sufficient since in the limit process we will only use the fact that $\sqrt{\varepsilon}||\tilde{\sigma}_{11}(\varepsilon)||_{0,\Omega^+}$ is bounded as $\varepsilon \to 0$ (see the proof of Theorem 6.6 at the end of the paper).

Lemma 6.3. *Let* $(\mathbf{u}(\varepsilon),\ \sigma(\varepsilon))$ *be the solution to problem* (5.2)–(5.4). *There exist three constants* C_8, C_9 *and* $C_{10} > 0$, *such that*

$$\left[\sum_{\alpha,\beta} ||\tilde{e}_{\alpha\beta}(\varepsilon)(\mathbf{u}(\varepsilon))||^2_{0,\Omega^+}\right]^{1/2} \le C_8\frac{1}{\varepsilon}\left[\left(\frac{\varepsilon}{2\mu_e\varepsilon + s_0}\right)^2 ||\tilde{\sigma}_{11}(\varepsilon)||^2_{0,\Omega^+} + \left(\frac{1}{2\mu_e}\right)^2 ||\tilde{\sigma}_{12}(\varepsilon)||^2_{0,\Omega^+} + \left(\frac{1}{2\mu_e}\right)^2 ||\tilde{\sigma}_{22}(\varepsilon)||^2_{0,\Omega^+}\right]^{1/2}, \tag{6.8}$$

$$\left[\sum_{\alpha} ||e_{\alpha 3}(\varepsilon)(\mathbf{u}(\varepsilon))||^2_{0,\Omega^+}\right]^{1/2} \le C_9\frac{1}{\varepsilon}\left[\sum_{\alpha} ||\sigma_{\alpha 3}(\varepsilon)||^2_{0,\Omega^+}\right]^{1/2}, \tag{6.9}$$

$$||e_{33}(\varepsilon)(\mathbf{u}(\varepsilon))||_{0,\Omega^+} \le C_{10}\frac{1}{\varepsilon}||\sigma_{33}(\varepsilon)||_{0,\Omega^+}. \tag{6.10}$$

Proof. In (5.3), let us choose successively:
$\tau_{ij} = 0$ in Ω^-, $\tilde{\tau}_{\alpha\beta} = \tilde{e}_{\alpha\beta}(\varepsilon)(\mathbf{u}(\varepsilon))$ and $\tau_{i3} = 0$ in Ω^+,
$\tau_{ij} = 0$ in Ω^-, $\tilde{\tau}_{\alpha\beta} = 0$, $\tau_{33} = 0$ and $\tau_{\alpha 3} = e_{\alpha 3}(\varepsilon)(\mathbf{u}(\varepsilon))$ in Ω^+,
$\tau_{ij} = 0$ in Ω^-, $\tilde{\tau}_{\alpha\beta} = 0$, $\tau_{\alpha 3} = 0$ and $\tau_{33} = e_{33}(\varepsilon)(\mathbf{u}(\varepsilon))$ in Ω^+. □

Lemma 6.4. *Let* $(\mathbf{u}(\varepsilon),\ \sigma(\varepsilon))$ *be the solution to problem* (5.2)–(5.4). *There exist constants* C_{11} *and* $C_{12} > 0$, *such that*

$$||\partial_3 u_3(\varepsilon)||_{0,\Omega^+} \leq C_{11}, \tag{6.11}$$

$$||\partial_3 u_\alpha(\varepsilon)||_{0,\Omega^+} \leq C_{12}. \tag{6.12}$$

Proof. Since $e_{33}(\varepsilon)(\mathbf{u}(\varepsilon)) = \frac{1}{\varepsilon}\partial_3 u_3(\varepsilon)$, we directly deduce from estimates (6.10) and (6.7) that

$$||\partial_3 u_3(\varepsilon)||_{0,\Omega^+} \leq C_{11}.$$

The following relation holds

$$||\partial_3 u_\alpha(\varepsilon)||^2_{0,\Omega^+} = \varepsilon||\partial_3 u^\varepsilon_\alpha||^2_{0,\Omega^+_\varepsilon}.$$

It is possible to extend $\mathbf{u}^\varepsilon$ by 0 to the ε-independent domain $\Omega^+_{\varepsilon_0}$ and apply Korn's inequality in curvilinear coordinates [8]. We deduce

$$||\partial_3 u^\varepsilon_\alpha||^2_{0,\Omega^+_\varepsilon} \leq C\sum_{i,j}||e^\varepsilon_{ij}(\mathbf{u}^\varepsilon)||^2_{0,\Omega^+_\varepsilon},$$

with

$$||e^\varepsilon_{ij}(\mathbf{u}^\varepsilon)||^2_{0,\Omega^+_\varepsilon} = \varepsilon||e_{ij}(\varepsilon)(\mathbf{u}(\varepsilon))||^2_{0,\Omega^+}.$$

We therefore have

$$\begin{aligned}||\partial_3 u_\alpha(\varepsilon)||^2_{0,\Omega^+} \leq\ & \hat{C}\varepsilon^2\sum_{\alpha,\beta}||\tilde{e}_{\alpha\beta}(\varepsilon)(\mathbf{u}(\varepsilon))||^2_{0,\Omega^+}\\ &+ 2C\varepsilon^2\sum_{\alpha}||e_{\alpha 3}(\varepsilon)(\mathbf{u}(\varepsilon))||^2_{0,\Omega^+}\\ &+ C\varepsilon^2||e_{33}(\varepsilon)(\mathbf{u}(\varepsilon))||^2_{0,\Omega^+},\end{aligned}$$

and we conclude using Lemmas 6.3 and 6.2 in order to bound the righthand side of the previous inequality. □

6.2. *Asymptotic analysis as* $\varepsilon \to 0$

Let us introduce the functional spaces V_3, $\mathbf{V}^*$ and Σ^*:

$$V_3(\Omega^+) = \left\{v \in L^2(\Omega^+),\ \frac{\partial v}{\partial x_3} \in L^2(\Omega^+),\ v = 0 \text{ on } \Gamma^+_l \cup \Gamma^+_{\mathbf{u}}\right\},$$

$$\begin{aligned}\mathbf{V}^* = \{&\mathbf{v},\ \mathbf{v}^- \in (H^1(\Omega^-))^3,\ \mathbf{v}^+ \in (V_3(\Omega^+))^3,\\ &\mathbf{v} = 0 \text{ on } \Gamma^- \cup \Gamma_l \cup \Gamma^+_{\mathbf{u}},\ \mathbf{v}^- = \mathbf{v}^+ \text{ on } S\}.\end{aligned}$$

$V_3(\Omega^+)$ and $\mathbf{V}^*$ are Hilbert spaces with the norms

$$||v||_{V_3(\Omega^+)} = \left\|\frac{\partial v}{\partial x_3}\right\|_{0,\Omega^+},$$

$$||\mathbf{v}||_{\mathbf{V}^*} = \left[\sum_i ||v_i||^2_{1,\Omega^-} + \left\|\frac{\partial v_i}{\partial x_3}\right\|^2_{0,\Omega^+}\right]^{1/2}.$$

It is possible to define the trace $v_{|\partial\Omega^-} \in H^{1/2}(\partial\Omega^-) \subset L^2(\partial\Omega^-)$ of $v \in H^1(\Omega^-)$ on the boundary $\partial\Omega^-$ of Ω^-. The trace on $\partial\Omega^+$ of an element $v \in V_3(\Omega^+)$ can also be defined and particularly $v_{|S} \in L^2_{\text{loc}}(S)$ (see Theorem B.2 of the Appendix B). Σ^* is the Hilbert space defined by

$$\Sigma^* = \{\tau = (\tau_{ij}),\ \tau_{ij} = \tau_{ji},\ \tau_{ij} \in L^2(\Omega) \text{ for } (i,j) \neq (1,1),\ \tau_{11} \in L^2(\Omega^-)\}.$$

The following notations are used in the limit scaled variational mixed formulation:

$$\begin{aligned}
A^*(\sigma,\tau) &= A^-(\sigma,\tau) + A^{*+}(\sigma,\tau),\\
A^{*+}(\sigma,\tau) &= \int_{\Omega^+} \left[\frac{1}{\mu_e}\tilde{\sigma}_{12}\tilde{\tau}_{12} + \frac{1}{2\mu_e}\tilde{\sigma}_{22}\tilde{\tau}_{22} + \frac{1}{2\mu_e}\boldsymbol{\sigma}_N \cdot \mathbf{G}_N(0)\boldsymbol{\tau}_N\right]\sqrt{a}dx,\\
B^*(\mathbf{v},\tau) &= B^-(\mathbf{v},\tau) + B^{*+}(\mathbf{v},\tau),\\
B^{*+}(\mathbf{v},\tau) &= \int_{\Omega^+} [(\partial_3\mathbf{v})_N \cdot \mathbf{G}_N(0)\boldsymbol{\tau}_N]\sqrt{a}dx,
\end{aligned}$$

where the vector $(\partial_3\mathbf{v})_N = \left(\frac{1}{\sqrt{2}}\partial_3 v_1, \frac{1}{\sqrt{2}}\partial_3 v_2, \partial_3 v_3\right)^T$. In the remaining part of this paper the arrows $\to$ and $\rightharpoonup$ denote strong and weak convergence as $\varepsilon \to 0$, respectively.

Lemma 6.5. *Let* $(\mathbf{u}(\varepsilon),\ \sigma(\varepsilon))$ *be the solution to the scaled variational mixed formulation* (5.2)–(5.4). *Then, there exists a subsequence, still denoted by* $(\mathbf{u}(\varepsilon),\ \sigma(\varepsilon))$ *for convenience, and there exists* $(\mathbf{u}^*, \sigma^*) \in \mathbf{V}^* \times \Sigma^*$, *such that*

$$\tilde{\sigma}_{11}(\varepsilon) \rightharpoonup \tilde{\sigma}^*_{11} \quad \text{in } L^2(\Omega^-), \tag{6.13}$$

$$\tilde{\sigma}_{\alpha 2}(\varepsilon) \rightharpoonup \tilde{\sigma}^*_{\alpha 2} \quad \text{in } L^2(\Omega), \tag{6.14}$$

$$\sigma_{i3}(\varepsilon) \rightharpoonup \sigma^*_{i3} \quad \text{in } L^2(\Omega), \tag{6.15}$$

$$\mathbf{u}(\varepsilon) \rightharpoonup \mathbf{u}^* \quad \text{in } \mathbf{V}^*. \tag{6.16}$$

Proof. Points (6.13)–(6.15) are direct consequences of Lemma 6.2.

Let us prove (6.16). From (6.1) and (6.3), we deduce that $\tilde{e}_{\alpha\beta}(\mathbf{u}(\varepsilon))$ is bounded in $L^2(\Omega^-)$. From (6.2) and (6.4), we deduce that $e_{i3}(\mathbf{u}(\varepsilon))$ is bounded in $L^2(\Omega^-)$. Therefore, $e_{ij}(\mathbf{u}(\varepsilon))$ is bounded in $L^2(\Omega^-)$, and Korn's inequality (see [8]) applied on Ω^- yields to the boundedness of $u_i(\varepsilon)$ in $H^1(\Omega^-)$. From Lemma 6.4, we deduce that $u_i(\varepsilon)$ is bounded in $V_3(\Omega^+)$. Consequently, there exists a subsequence $u_i(\varepsilon) \rightharpoonup u_i^*$ in $H^1(\Omega^-) \cup V_3(\Omega^+)$.

Since $u_i(\varepsilon) = 0$ on $\Gamma^- \cup \Gamma_l \cup \Gamma^+_{\mathbf{u}}$, $u_i^* = 0$ on $\Gamma^- \cup \Gamma_l \cup \Gamma^+_{\mathbf{u}}$. Since $u_i^+(\varepsilon)(x_1,x_2,0) = u_i^-(\varepsilon)(x_1,x_2,0)$ in $L^2_{\text{loc}}(S)$ and $u_i^-(\varepsilon)(x_1,x_2,0) \in H^{1/2}(S)$, we have that $u_i^+(\varepsilon)(x_1,x_2,0) = u_i^-(\varepsilon)(x_1,x_2,0)$ in $H^{1/2}(S)$ and therefore in $L^2(S)$. Thus, we obtain that $u_i^{*+} = u_i^{*-}$ a.e. on S and $\mathbf{u}^* \in \mathbf{V}^*$. □

Theorem 6.6. $(\mathbf{u}^*, \sigma^*)$ *solves the scaled mixed variational problem*:

$$\mathbf{u}^* \in \mathbf{V}^*, \quad \sigma^* \in \Sigma^*, \tag{6.17}$$

$$A^*(\sigma^*,\tau) = B^*(\mathbf{u}^*,\tau), \quad \forall \tau \in \Sigma, \tag{6.18}$$

$$B^*(\mathbf{v},\sigma^*) = L(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}. \tag{6.19}$$

Proof. The result is obtained by passing to the limit as $\varepsilon \to 0$ in (5.2)–(5.4).

(i) The terms $A^-(\sigma(\varepsilon), \tau)$, $B^-(\mathbf{u}(\varepsilon), \tau)$ and $B^-(\mathbf{v}, \sigma(\varepsilon))$:

Using Lemma 6.5, it is clear that

$$A^-(\sigma(\varepsilon), \tau) \to A^-(\sigma^*, \tau),$$
$$B^-(\mathbf{u}(\varepsilon), \tau) \to B^-(\mathbf{u}^*, \tau),$$
$$B^-(\mathbf{v}, \sigma(\varepsilon)) \to B^-(\mathbf{v}, \sigma^*).$$

(ii) The term $A^+(\varepsilon)(\sigma(\varepsilon), \tau)$:

From (5.8) (cf. Lemma 5.1), we know that $\sqrt{g(\varepsilon)} \to \sqrt{a}$ in $\mathcal{C}^0(\bar{\Omega}^+)$. From Lemma 6.2, we deduce that $\sqrt{\varepsilon}\tilde{\sigma}_{11}(\varepsilon)$ is bounded in $L^2(\Omega^+)$ for $0 < \varepsilon \leq \varepsilon_0$ and since $\frac{\sqrt{\varepsilon}}{s(\varepsilon)+2\mu_e\varepsilon} \to 0$ in $\mathcal{C}^0(\bar{\Omega}^+)$,

$$\int_{\Omega^+} \frac{\varepsilon}{s(\varepsilon) + 2\mu_e \varepsilon} \tilde{\sigma}_{11}(\varepsilon)\tilde{\tau}_{11}\sqrt{g(\varepsilon)}\, dx \to 0.$$

Then, using (6.6), (6.7) (cf. Lemma 6.2) and (5.15), we conclude that

$$A^+(\varepsilon)(\sigma(\varepsilon), \tau) \to A^{*+}(\sigma^*, \tau).$$

(iii) The term $B^+(\varepsilon)(\mathbf{v}, \sigma(\varepsilon))$:

(5.9) and (5.10) (cf. Lemma 5.1) lead to

$$\begin{aligned} e_{\alpha\beta}(\varepsilon)(\mathbf{v}) &= \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^p_{\alpha\beta}(\varepsilon) v_p \to \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^\sigma_{\alpha\beta} v_\sigma - b_{\alpha\beta} v_3 \\ &= e_{\alpha\beta}(0)(\mathbf{v}), \end{aligned}$$

in $L^2(\Omega^+)$ for all $\mathbf{v} \in (H^1(\Omega^+))^3$. Since $\sqrt{\varepsilon}\tilde{\sigma}_{11}(\varepsilon)$, $\tilde{\sigma}_{12}(\varepsilon)$ and $\tilde{\sigma}_{22}(\varepsilon)$ are bounded in $L^2(\Omega^+)$,

$$\int_{\Omega^+} \varepsilon \tilde{\mathbf{e}}_T(\varepsilon)(\mathbf{v}) \cdot \tilde{\boldsymbol{\sigma}}_T(\varepsilon)\sqrt{g(\varepsilon)} dx \to 0.$$

We recall that $e_{\alpha 3}(\varepsilon)(\mathbf{v}) = \frac{1}{2}(\partial_\alpha v_3 + \frac{1}{\varepsilon}\partial_3 v_\alpha) - \Gamma^\sigma_{\alpha 3}(\varepsilon) v_\sigma$. Using (5.11) (cf. Lemma 5.1), we deduce that

$$\varepsilon e_{\alpha 3}(\varepsilon)(\mathbf{v}) \to \frac{1}{2}\partial_3 v_\alpha,$$

in $L^2(\Omega^+)$ for all $\mathbf{v} \in (H^1(\Omega^+))^3$. We also have

$$\varepsilon e_{33}(\varepsilon)(\mathbf{v}) \to \partial_3 v_3,$$

in $L^2(\Omega^+)$ for all $\mathbf{v} \in (H^1(\Omega^+))^3$. Therefore, we conclude that

$$\int_{\Omega^+} \varepsilon \mathbf{e}_N(\varepsilon)(\mathbf{v}) \cdot \mathbf{G}_N(\varepsilon)\boldsymbol{\sigma}_N(\varepsilon)\sqrt{g(\varepsilon)} dx \to \int_{\Omega^+} (\partial_3 \mathbf{v})_N \cdot \mathbf{G}_N(0)\boldsymbol{\sigma}^*_N \sqrt{a} dx,$$

and

$$B^+(\varepsilon)(\mathbf{v}, \sigma(\varepsilon)) \to B^{*+}(\mathbf{v}, \sigma^*).$$

(iv) The term $B^+(\varepsilon)(\mathbf{u}(\varepsilon),\tau)$:

Let us show that $\varepsilon\partial_\alpha u_i(\varepsilon) \rightharpoonup 0$ in $L^2(\Omega^+)$. From (6.5), (6.6) (cf. Lemma 6.2) and (6.8) (cf. Lemma 6.3), we deduce that $\varepsilon\tilde{e}_{\alpha\beta}(\varepsilon)(\mathbf{u}(\varepsilon))$ is bounded in $L^2(\Omega^+)$. Therefore, $\varepsilon e_{\alpha\beta}(\varepsilon)(\mathbf{u}(\varepsilon))$ is also bounded in $L^2(\Omega^+)$. Since $\Gamma^p_{\alpha\beta}(\varepsilon)$ is bounded in $\mathcal{C}^0(\bar{\Omega}^+)$ and $u_p(\varepsilon)$ is bounded in $L^2(\Omega^+)$, we deduce from

$$\varepsilon e_{\alpha\beta}(\varepsilon)(\mathbf{u}(\varepsilon)) = \varepsilon\left(\frac{1}{2}(\partial_\alpha u_\beta(\varepsilon) + \partial_\beta u_\alpha(\varepsilon)) - \Gamma^p_{\alpha\beta}(\varepsilon)u_p(\varepsilon)\right),$$

that $\varepsilon\partial_1 u_1(\varepsilon)$, $\varepsilon\partial_2 u_2(\varepsilon)$ and $\varepsilon(\partial_1 u_2(\varepsilon) + \partial_2 u_1(\varepsilon))$ are bounded in $L^2(\Omega^+)$. In the same way,

$$\varepsilon e_{\alpha 3}(\varepsilon)(\mathbf{u}(\varepsilon)) = \frac{1}{2}(\varepsilon\partial_\alpha u_3(\varepsilon) + \partial_3 u_\alpha(\varepsilon)) - \varepsilon\Gamma^\sigma_{\alpha 3}(\varepsilon)u_\sigma(\varepsilon)$$

is bounded in $L^2(\Omega^+)$ and since $\partial_3 u_i(\varepsilon)$ (cf. Lemma 6.4) and $\varepsilon\Gamma^\sigma_{\alpha 3}(\varepsilon)u_\sigma(\varepsilon)$ are bounded in $L^2(\Omega^+)$, this implies that $\varepsilon\partial_\alpha u_3(\varepsilon)$ is bounded in $L^2(\Omega^+)$. We then apply the *classical* Korn inequality to $e(\mathbf{u})$ on Ω^+ to obtain the boundedness of $\varepsilon\partial_1 u_2(\varepsilon)$ and $\varepsilon\partial_2 u_1(\varepsilon)$. To sum up, $\varepsilon\partial_j u_i(\varepsilon)$ is bounded in $L^2(\Omega^+)$.

Hence $\varepsilon u_i(\varepsilon)$ is bounded in $H^1(\Omega^+)$ and there exists a subsequence, still denoted by $\varepsilon u_i(\varepsilon)$, which converges weakly to some v_i in $H^1(\Omega^+)$. The trace of v_i on $\Gamma^+_{\mathbf{u}}$ is 0 since the trace of $u_i(\varepsilon)$ on $\Gamma^+_{\mathbf{u}}$ is 0. Moreover, $\varepsilon\partial_3 u_\alpha \to 0$ in $L^2(\Omega^+)$ and therefore $\partial_3 v_i = 0$ a.e in Ω^+. We conclude that $v_i = 0$ and that $\varepsilon\partial_\alpha u_i(\varepsilon) \rightharpoonup 0$ in $L^2(\Omega^+)$.

As a consequence $\varepsilon e_{\alpha\beta}(\varepsilon)(\mathbf{u}(\varepsilon)) \rightharpoonup 0$ and therefore $\varepsilon\tilde{e}_{\alpha\beta}(\varepsilon)(\mathbf{u}(\varepsilon)) \rightharpoonup 0$, $\varepsilon\big(\frac{1}{2}(\partial_\alpha u_3(\varepsilon) - \Gamma^\rho_{\alpha 3}(\varepsilon)u_\rho(\varepsilon)\big) \rightharpoonup 0$ in $L^2(\Omega^+)$. Eventually,

$$B^+(\varepsilon)(\mathbf{u}(\varepsilon),\tau) \to B^{*+}(\mathbf{u}^*,\tau). \qquad \square$$

Theorem 6.7. *In the domain Ω^+, the displacement field $\mathbf{u}^*$ is given by*

$$u^*_\alpha(x_1,x_2,x_3) = \frac{1}{\mu_e}\sigma^{*-}_{\alpha 3}(x_1,x_2,0)(x_3-1), \quad a.e \text{ in } \Omega^+, \tag{6.20}$$

$$u^*_3(x_1,x_2,x_3) = \frac{1}{2\mu_e}\sigma^{*-}_{33}(x_1,x_2,0)(x_3-1) \quad a.e \text{ in } \Omega^+. \tag{6.21}$$

Proof. In (6.18), let us choose $\tau = 0$ in Ω^-, $\tau_{\alpha\beta} = 0$ in Ω^+ and $\boldsymbol{\tau}_N = \frac{1}{\sqrt{a}}\mathbf{G}_N(0)^{-1}\big[\frac{1}{2\mu_e}\boldsymbol{\sigma}^*_N - (\partial_3\mathbf{u}^*)_N\big]$ in Ω^+. This leads to

$$\int_{\Omega^+}\left\|\frac{1}{2\mu_e}\boldsymbol{\sigma}^*_N - (\partial_3\mathbf{u}^*)_N\right\|^2 dx = 0,$$

that is to say,

$$\frac{1}{\mu_e}\sigma^*_{\alpha 3} - \partial_3 u^*_\alpha = 0, \quad \text{in } L^2(\Omega^+), \tag{6.22}$$

$$\frac{1}{2\mu_e}\sigma^*_{33} - \partial_3 u^*_3 = 0, \quad \text{in } L^2(\Omega^+). \tag{6.23}$$

In (6.19), let us choose $\mathbf{v} = 0$ in Ω^- and $\mathbf{v} \in (\mathcal{D}(\Omega^+))^3$ in Ω^+. Then

$$\int_{\Omega^+} [(\partial_3 \mathbf{v})_N \cdot \mathbf{G}_N(0)\boldsymbol{\sigma}^*_N]\sqrt{a}dx = 0 = -\int_{\Omega^+} [(\mathbf{v})_N \cdot \mathbf{G}_N(0)\partial_3\boldsymbol{\sigma}^*_N]\sqrt{a}dx,$$

where the vector $(\mathbf{v})_N = \left(\frac{1}{\sqrt{2}}v_1, \frac{1}{\sqrt{2}}v_2, v_3\right)^T$.

It follows that

$$\partial_3 \boldsymbol{\sigma}^*_N = 0, \tag{6.24}$$

in $(\mathcal{D}'(\Omega^+))^3$ and therefore in $(L^2(\Omega^+))^3$.

From (6.22)–(6.24), we deduce that $\partial_3\partial_3 u^*_i = 0$ in $L^2(\Omega^+)$. Since the trace of u^*_i on $\Gamma^+_{\mathbf{u}}$ is 0, we obtain

$$u^*_i(x_1, x_2, x_3) = c_i(x_3 - 1), \quad \text{a.e in } \Omega^+,$$

where

$$c_i = -u^{*+}_i(x_1, x_2, 0) = \partial_3 u^*_i(x_1, x_2, x_3). \tag{6.25}$$

From (6.22), (6.24) and (6.25), we deduce that the trace of σ^+_{i3} on $\partial\Omega^+$ belongs to $L^2(\partial\Omega^+)$. Also

$$c_\alpha = \frac{1}{\mu_e}\sigma^{*+}_{\alpha 3}(x_1, x_2, 0) = -u^{*+}_\alpha(x_1, x_2, 0) \quad \text{in } L^2(S),$$
$$c_3 = \frac{1}{2\mu_e}\sigma^{*+}_{33}(x_1, x_2, 0) = -u^{*+}_3(x_1, x_2, 0) \quad \text{in } L^2(S).$$

It remains to be shown that $\sigma^{*+}_{i3} = \sigma^{*-}_{i3}$ on S.

We first show that $\boldsymbol{\sigma}^{*+}_N = 0$ on Γ^+_σ.

In (6.19) let us choose $\mathbf{v} \in \mathbf{K}$, such that $\mathbf{v} \in (H^1(\Omega^+))^3$, $\mathbf{v} = 0$ on $S \cup \Gamma^+_{\mathbf{u}} \cup \Gamma^+_l$ and $\mathbf{v} = 0$ in Ω^-. We obtain using Green's formula

$$\begin{aligned} B^{*+}(\mathbf{v}, \sigma^*) &= 0, \\ &= -\int_{\Omega^+} (\mathbf{v})_N \cdot \mathbf{G}_N(0)(\partial_3\boldsymbol{\sigma}^*_N)\sqrt{a}dx + \int_{\Gamma^+_\sigma} (\mathbf{v})_N \cdot \mathbf{G}_N(0)\boldsymbol{\sigma}^*_N\sqrt{a}dx. \end{aligned}$$

Using (6.24) results in

$$\int_{\Gamma^+_\sigma} (\mathbf{v})_N \cdot \mathbf{G}_N(0)\boldsymbol{\sigma}^*_N\sqrt{a}dx = 0, \quad \forall \mathbf{v} \in (L^2(\Gamma^+_\sigma))^3,$$

which implies

$$\boldsymbol{\sigma}^{*+}_N = 0, \quad \text{in } (L^2(\Gamma^+_\sigma))^3. \tag{6.26}$$

Let us now transform Eq. (6.19),

$$B^-(\mathbf{v}, \sigma^*) + B^{*+}(\mathbf{v}, \sigma^*) = L(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{K},$$

using Green's formula (and (6.26)) and going back to cartesian coordinates. This gives

$$B^{*+}(\mathbf{v},\sigma^*) = \int_{\Omega^+} [(\partial_3 \mathbf{v})_N \cdot \mathbf{G}_N(0)\boldsymbol{\sigma}_N^*]\sqrt{a}dx,$$
$$= -\int_{\Omega^+} (\mathbf{v})_N \cdot \mathbf{G}_N(0)(\partial_3 \boldsymbol{\sigma}_N^*)\sqrt{a}dx - \int_S (\mathbf{v})_N^+ \cdot \mathbf{G}_N(0)\boldsymbol{\sigma}_N^{*+}\sqrt{a}dx_1\,dx_2.$$

Using (6.24) results in

$$B^{*+}(\mathbf{v},\sigma^*) = -\int_S (\mathbf{v})_N^+ \cdot \mathbf{G}_N(0)\boldsymbol{\sigma}_N^{*+}\sqrt{a}dx_1 dx_2,$$

and going back to cartesian coordinates

$$B^{*+}(\mathbf{v},\sigma^*) = -\int_{\hat{S}} \hat{\mathbf{v}}^+ \cdot \hat{\sigma}^{*+}\mathbf{n}\,ds.$$

Moreover, we have

$$B^-(\mathbf{v},\tau) = \int_{\Omega^-} [\tilde{\mathbf{e}}_T(\mathbf{v}) \cdot \tilde{\boldsymbol{\sigma}}_T^* + \mathbf{e}_N(\mathbf{v}) \cdot \mathbf{G}_N \boldsymbol{\sigma}_N^*]\sqrt{g}dx,$$
$$= \int_{\hat{\Omega}^-} \hat{e}(\hat{\mathbf{v}})\colon \hat{\sigma}^*\, d\hat{x}.$$

Since $\mathrm{div}(\hat{\sigma}(\varepsilon)) = \hat{\mathbf{f}} \in (L^2(\hat{\Omega}^-))^3$, $\mathrm{div}(\hat{\sigma}^*)$ belongs to $(L^2(\hat{\Omega}^-))^3$ and $\hat{\sigma}^*$ belongs to $H(\mathrm{div}, \hat{\Omega}^-)$ (see Appendix B). Therefore, we can define $\hat{\sigma}\mathbf{n}_{|\hat{S}} \in H^{-1/2}(\hat{S})$ and we have Green's formula

$$B^-(\mathbf{v},\tau) = -\int_{\hat{\Omega}^-} \mathrm{div}(\hat{\sigma}^*) \cdot \hat{\mathbf{v}}\,d\hat{x} + \langle \hat{\sigma}^{*-}\mathbf{n}, \hat{\mathbf{v}}^- \rangle_{(H^{-1/2}(\hat{S}))^3,(H^{1/2}(\hat{S}))^3}.$$

Eventually, since

$$L(\mathbf{v}) = \int_{\Omega^-} f^i v_i \sqrt{g}dx = \int_{\hat{\Omega}^-} \hat{\mathbf{f}} \cdot \hat{\mathbf{v}}\,d\hat{x},$$

we obtain

$$-\int_{\hat{S}} \hat{\mathbf{v}} \cdot \hat{\sigma}^{*+}\mathbf{n}\,ds \;+ \langle \hat{\sigma}^{*-}\mathbf{n}, \hat{\mathbf{v}} \rangle_{(H^{-1/2}(\hat{S}))^3,(H^{1/2}(\hat{S}))^3} = 0, \quad \forall \hat{\mathbf{v}} \in (L^2(\hat{S}))^3.$$

Therefore, $\hat{\sigma}^{*-}\mathbf{n} = \hat{\sigma}^{*+}\mathbf{n}$ in $(H^{-1/2}(\hat{S}))^3$ but since $\hat{\sigma}^{*+}\mathbf{n} \in (L^2(\hat{S}))^3$ the equality holds in $(L^2(\hat{S}))^3$. In curvilinear coordinates this reads $\boldsymbol{\sigma}_N^{*+}(x_1,x_2,0) = \boldsymbol{\sigma}_N^{*-}(x_1,x_2,0)$ in $(L^2(S))^3$ and the proof is complete. □

To conclude, let us show that the limit displacement and stress tensor fields satisfy in $\hat{\Omega}^-$ the equation of the elasticity problem (1.6) announced in the introduction of the paper. The result is expressed in the cartesian coordinate system.

Theorem 6.8. $\hat{\mathbf{u}}^*$ *and* $\hat{\boldsymbol{\sigma}}^*$ *satisfy*:

$$\begin{cases} \operatorname{div}(\hat{\sigma}^*) + \hat{\mathbf{f}} = 0 & \text{a.e in } \hat{\Omega}^-, \\ \hat{\sigma}^* = \lambda \text{ trace}(\hat{e}(\hat{\mathbf{u}}^*))I + 2\mu\hat{e}(\hat{\mathbf{u}}^*) & \text{a.e in } \hat{\Omega}^-, \\ \hat{\mathbf{u}}^* = 0 & \text{a.e on } \hat{\Gamma}^- \cup \hat{\Gamma}_l^-, \\ \hat{\sigma}^*\mathbf{n} = -2\mu_e \hat{u}_n^* \mathbf{n} - \mu_e \hat{\mathbf{u}}_T^* & \text{a.e on } \hat{S}. \end{cases} \tag{6.27}$$

Proof. Since $\mathbf{u}^* \in \mathbf{V}^*$ it is clear that $\hat{\mathbf{u}}^* = 0$ a.e on $\hat{\Gamma}^- \cup \hat{\Gamma}_l^-$. Choosing $x_3 = 0$ in (6.20) and (6.21) of Theorem 6.7, we deduce that

$$\mu_e u_\alpha^* \mathbf{g}^\alpha + 2\mu_e u_3^* \mathbf{g}^3 = -\sigma_{\alpha 3}^* \mathbf{g}^\alpha - \sigma_{33}^* \mathbf{g}^3 \quad \text{a.e on } S,$$

which is exactly the boundary condition expected on $\hat{S}$ expressed in curvilinear coordinates. Let us now obtain the stress-strain compartment equation. Choosing $\tau^+ = 0$ in (6.18) of Theorem 6.6 leads to

$$A^-(\sigma^*, \tau) = B^-(\mathbf{u}^*, \tau).$$

Going back to cartesian coordinates, this equation reads

$$\int_{\hat{\Omega}^-} \hat{A}_{ijkl} \hat{\sigma}_{kl}^* \hat{\tau}_{ij} \, d\hat{x} = \int_{\hat{\Omega}^-} \hat{e}_{ij}(\hat{\mathbf{u}}^*) \hat{\tau}_{ij} \, d\hat{x},$$

where

$$\hat{A}_{ijkl} = \frac{1+\nu}{2E}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) - \frac{\nu}{E}\delta_{ij}\delta_{kl}.$$

Since this holds for all $\tau_{ij} = \tau_{ji} \in L^2(\Omega^-)$, we obtain that $\hat{A}_{ijkl}\hat{\sigma}_{kl}^* = \hat{e}_{ij}(\hat{\mathbf{u}}^*)$ a.e in $\hat{\Omega}^-$. This relation can also be written

$$\hat{e}(\hat{\mathbf{u}}^*) = \frac{1+\nu}{E}\hat{\sigma}^* - \frac{\nu}{E} \text{ trace}(\hat{\sigma}^*)I,$$

which is equivalent to

$$\hat{\sigma}^* = \lambda \text{ trace}(\hat{e}(\hat{\mathbf{u}}^*))I + 2\mu\hat{e}(\hat{\mathbf{u}}^*).$$

Eventually, in order to obtain the equilibrium equation, one may choose in (6.19) of Theorem 6.6 $\mathbf{v}$ such that $\mathbf{v}^+ = 0$, $\mathbf{v} \in (H^1(\Omega^-))^3$ and $\mathbf{v} = 0$ on $\Gamma^- \cup \Gamma_l^- \cup S$. □

It should be noted that this last problem is wellposed. One can easily deduce this by formulating a mixed variational formulation (in cartesian coordinates) and check that assumptions of Theorem 1.2, p. 47 of the book by Brezzi and Fortin [5] are satisfied.

Appendix A

In this first appendix, we recall a result concerning the simultaneous reduction of two quadratic forms.

Let $\mathbf{A}$ be a symmetric, positive definite $n \times n$ matrix and $\mathbf{B}$ a symmetric $n \times n$ matrix. Using the matrix $\mathbf{A}$, one can define the scalar product $(.,.)_\mathbf{A}$ on $\mathbb{R}^n$ by

$$(x, y)_\mathbf{A} = X^T \mathbf{A} Y, \quad \forall x, y \in \mathbb{R}^n,$$

where X and Y are the $n \times 1$ matrices of x and y in the canonical basis. We define the quadratic form $q_{\mathbf{B}}$ by

$$q_{\mathbf{B}}(x) = X^T \mathbf{B} X, \quad \forall x \in \mathbb{R}^n.$$

There exists a unique linear operator $f \colon \mathbb{R}^n \to \mathbb{R}^n$, which is symmetric for the scalar product $(.,.)_{\mathbf{A}}$, such that

$$q_{\mathbf{B}}(x) = (x, f(x))_{\mathbf{A}}, \quad \forall x \in \mathbb{R}^n.$$

Let $\mathbf{C}$ be the matrix of f in the canonical basis. We have

$$X^T \mathbf{B} X = X^T \mathbf{A} \mathbf{C} X, \quad \forall X \in \mathbb{R}^n$$

and therefore $\mathbf{AC} = \mathbf{B}$. Since $\mathbf{C}$ is the matrix of a symmetric linear operator, it is diagonalizable in a basis which is orthonormal with regard to the scalar product $(.,.)_{\mathbf{A}}$. Hence, there exist an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$, such that

$$\mathbf{P}^T \mathbf{A} \mathbf{P} = \mathbf{I}_n, \tag{A.1}$$

$$\mathbf{P}^{-1} \mathbf{C} \mathbf{P} = \mathbf{D}. \tag{A.2}$$

From (A.1) we deduce that $\mathbf{A}^{-1} = \mathbf{P}\mathbf{P}^T$ and replacing $\mathbf{C}$ by $\mathbf{A}^{-1}\mathbf{B}$ in (A.2), we deduce that $\mathbf{P}^{-1}\mathbf{P}\mathbf{P}^T\mathbf{B}\mathbf{P} = \mathbf{D}$. To sum up, we have that

$$\mathbf{P}^T \mathbf{A} \mathbf{P} = \mathbf{I}_n \quad \text{and} \quad \mathbf{P}^T \mathbf{B} \mathbf{P} = \mathbf{D}.$$

Appendix B

In this appendix, we recall two traces theorems. Let Ω be a Lipschitz continuous open subset of $\mathbb{R}^3$. Let us define the Hilbert space $H(\mathrm{div}, \Omega)$ by

$$H(\mathrm{div}, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^3;\ \mathrm{div}(\mathbf{v}) \in L^2(\Omega)\}.$$

Theorem B.1. *The mapping* $\gamma_n \colon \mathbf{v} \to \mathbf{v} \cdot \mathbf{n}_{|\partial\Omega}$ *is a linear continuous operator from* $H(\mathrm{div}, \Omega)$ *into* $H^{-1/2}(\partial\Omega)$.

For a proof the reader is referred to Theorem 2.5, p. 27 of the book by Girault and Raviart [11].

For $1 \leq i \leq 3$, let $a_i \colon \Omega \to \mathbb{R}$ be C^1 functions such that $\sum_{i=1}^3 \partial_i a_i$ is bounded. Let us define the Hilbert space H by

$$H = \left\{ \phi \in L^2(\Omega);\ \sum_{i=1}^{3} a_i \partial_i \phi \in L^2(\Omega) \right\}.$$

The following result holds.

Theorem B.2. *Assume the functions* a_i *satisfy the previous hypothesis. Then for* $S \subset \partial\Omega$ *a part of the boundary of positive measure, the mapping* $\gamma_S \colon \phi \to \phi_{|S}$ *is a linear continuous operator from* H *into* $L^2_{\mathrm{loc}}\left(S, \sum_{i=1}^3 a_i n_I \, d\sigma\right)$, *where* $\mathbf{n}$ *is the outward normal.*

For a proof the reader is referred to Bardos [2], p. 205.

References

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