RAPID DECAY PROPERTY AND 3-MANIFOLD GROUPS

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Abstract. We prove that the fundamental group of a connected, compact, irreducible 3-dimensional manifold without boundary M^3 has property of Rapid Decay if and only if M^3 does not have Sol-geometry.

1. Introduction

The property of Rapid Decay, or more briefly *property (RD)*, first appeared in a famous paper by Haagerup [9] (it is hence sometimes called "Haagerup inequality" [17]), where it was proved to hold for the finite rank free groups. This is in fact a property about the algebra $\mathbb{C}G$ of the group G: roughly speaking it says that there is a constant C such that for any f in $\mathbb{C}G$ the operator-norm of f ($\mathbb{C}G$ is equipped with the convolution product so that each function in $\mathbb{C}G$ can be considered as an operator) is bounded above by a constant times a certain weighted l^2-norm. It can be equivalently rephrased using Sobolev like spaces. Property (RD) has found applications for Novikov [4] and Baum-Connes [12] conjectures. Jolissaint [11] was the first to study this property as such. The aim of this short paper is to give a full characterization of the groups with property of Rapid Decay among the fundamental groups of compact 3-manifolds.

The 3-manifold groups do not need more advertising: the ideas and methods of 3-dimensional topology have widely spread through geometric group theory. Although a lot is known about their structure, specially since the proof of the Poincare conjecture [15], they present particularly rich phenomena and so form a natural class where to test properties or conjectures (see for instance [13]). The present work was more directly motivated by discussions with H. Oyono-Oyono who is interested in using the Rapid Decay for computations in K-theory.

A *closed* 3*-manifold* is a connected, compact 3-manifold without boundary. A 3-manifold is *irreducible* if every embedded 2-sphere bounds a 3-ball. If X is a complete metric space and $\mathrm{Isom}(X)$ its group of isometries, we say that a compact 3-manifold M^3 *has* X*-geometry* if the fundamental group of M^3, denoted by $\pi_1(M^3)$ (we omit the base-point), is a discrete subgroup of $\mathrm{Isom}(X)$ and $M^3 = X/\pi_1(M^3)$. The Lie group Sol is defined by the following split extension

$$0 \to \mathbb{R}^2 \to Sol \to \mathbb{R} \to 0$$

where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ by $(x, y) \mapsto (e^t x, e^{-t} y)$.

Theorem 1.1. *Let M^3 be a closed, irreducible 3-manifold. If M^3 does not have Sol-geometry, then the fundamental group of M^3 has property (RD).*

By [16][Theorem 5.3], a closed 3-manifold M^3 has Sol-geometry if and only if it is finitely covered by the suspension

$$\mathbb{T}^2 \times [0,1]/(x,1) \sim (\phi(x), 0)$$

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of an Anosov $\phi\colon \mathbb{T}^2 \to \mathbb{T}^2$ of the 2-dimensional torus $\mathbb{T}^2$. By [11][Proof of Corollary 3.1.9], the suspension of an Anosov of the torus does not have property (RD). Moreover, by [11][Proposition 2.1.5], if a finite-index subgroup of a group has property (RD), then the group itself has property (RD). We so get:

Corollary 1.2. *A closed, irreducible* 3*-manifold* M^3 *has property (RD) if and only if* M^3 *does not have Sol-geometry.*

2. Rapid Decay

We refer to [3] or [2] for more details about Rapid Decay property, more briefly termed *property (RD)*. We just give here a rough account of the material that we really need. A *length function* on a group Γ is a positive real function which is symmetric, subadditive with respect to the group operation and vanishes on the neutral element. For instance the word-length associated to a finite generating set is a length function. The notation $\mathbb{C}\Gamma$ denotes the set of complex-valued functions on Γ, with finite support. The *convolution* of two functions $f, g \in \mathbb{C}\Gamma$ is defined by $(f * g)(\gamma) = \sum_{\mu\in\Gamma} f(\mu)g(\mu^{-1}\gamma)$. We also need the l^2-norm $||f||_2 = \sqrt{\sum_{\gamma\in\Gamma} |f(\gamma)|^2}$, the operator norm $||f||_* = \sup\{||f * g||_2 \text{ s.t. } ||g||_2 = 1\}$ and a weighted l^2-norm $||f||_{L,s} = \sqrt{\sum_{\gamma\in\Gamma} |f(\gamma)|^2(1 + L(\gamma))^{2s}}$.

Definition 2.1. [9, 11] A group Γ *has property (RD) with respect to a length function* L if there exist $C, s > 0$ such that, for each $f \in \mathbb{C}\Gamma$ one has

$$||f||_* \leq C||f||_{L,s}$$

Although compact, the above definition is not very tractable for our purpose. We thus give below another characterization due to Jolissaint. Îf L is a length function on a group Γ, we denote by $C_{r,L} = \{\gamma \in \Gamma \text{ s.t. } r - 1 < L(\gamma) \leq r\}$ the crown of radius r for the length function L and by $\chi_{r,L}$ the characteristic function of the crown $C_{r,L}$.

Proposition 2.2. [11]*[Proposition 1.2.6] A group* Γ *has property (RD) with respect to a length function L if and only if there exist* $c > 0$ *and* $r > 0$ *such that, if* $k, l, m \in \mathbb{N}$*, if* f *and* g *belong to* $\mathbb{C}\Gamma$ *with support in* $C_{k,L}$ *and* $C_{l,L}$ *respectively then*

$$||(f * g)\chi_{m,L}||_2 \leq c||f||_{r,L}||g||_2 \text{ whenever } |k - l| \leq m \leq k + l$$

and $||(f * g)\chi_{m,L}||_2 = 0$ *otherwise.*

3. Three-manifold groups

The ultimate goal of this section is to prove Theorem 1.1. All the 3-manifolds considered are connected, unless explicitly otherwise stated. They are assumed to have an infinite fundamental group since the result is otherwise obvious. Let us recall that an *incompressible torus* in a compact 3-manifold M^3 is an embedded torus $i\colon \mathbb{T}^2 \to M^3$ such that $i_\#\colon \pi_1(\mathbb{T}^2) \to \pi_1(M^3)$ is injective and $i(\mathbb{T}^2)$ is not parallel to a boundary component, i.e. cannot be isotoped into the boundary of M^3.

3.1. **Seifert fibred spaces and graph-manifolds.** A *Seifert fibred space* is a connected, compact, orientable, irreducible 3-manifold which is a union of circles C_α, called the *fibers* of M^3, such that each C_α admits a neighborhood $T(C_\alpha)$, homeomorphic by a fiber-preserving homeomorphism h_α to a *fibred solid torus* $\mathbb{T}^2_{p,q}$, i.e. the suspension $D^2 \times [0, 1]/(x, 1) \sim (r(x), 0)$ of a rotation r of the disc D^2 centered at the origin O and of angle $\frac{2\pi p}{q}$. The fibers of $\mathbb{T}^2_{p,q}$ are the orbits of the rotation r. The homeomorphism h_α is required to carry C_α to the r-orbit of O. A *graph-manifold* is a compact, orientable, irreducible 3-manifold M^3 which admits a finite union of incompressible tori $T_1, \cdots, T_r$ such that the closure of each connected component of $M^3 \setminus \bigcup_{i=1}^r T_i$ is a Seifert-fibred manifold.

Lemma 3.1. *Let M^3 be a Seifert fibred manifold. Then $\pi_1(M^3)$ has property (RD).*

Proof. By [16], $\pi_1(M^3)$ fits into a short exact sequence $1 \to \mathbb{Z} \to \pi_1(M^3) \to \pi_1(X) \to 1$, where $\pi_1(X)$ denotes the fundamental group of an orbifold X. It suffices to check that this sequence satisfies the conditions of "polynomial growth" of [11][Proposition 2.1.9] which gives the conclusion.

□

For proving Proposition 3.2 below, we will need a good understanding of the notion of Anosov diffeomorphism. An *Anosov* of $\mathbb{T}^2$ is a diffeomorphism $\phi\colon \mathbb{T}^2 \to \mathbb{T}^2$ such that there exist $\lambda > 1$ and a decomposition of the tangent bundle $T\mathbb{T}^2 = T^u\mathbb{T}^2 \oplus T^s\mathbb{T}^2$ such that at each point $x \in \mathbb{T}^2$ $T_x\mathbb{T}^2 = T_x^u\mathbb{T}^2 \oplus T_x^s\mathbb{T}^2$ satisfies $||D_x\phi(v)|| = \lambda||v||$ (resp. $||D_x\phi(v)|| = \frac{1}{\lambda}||v||$) for all $v \in T_x^u\mathbb{T}^2$ (resp. for all $v \in T_x^s\mathbb{T}^2$). Equivalently up to isotopy ϕ lifts to a linear automorphism of $\mathbb{Z}^2$ with two real eigenvalues $\lambda > 1$ and $\frac{1}{\lambda}$. The associated eigenspaces give the unstable and stable directions at each point.

Proposition 3.2. *The fundamental group of a graph-manifold has property (RD).*

Proof. By Lemma 3.1 we can assume that the considered manifold M^3 is not Seifert. By definition of a graph-manifold there are incompressible tori $T_1, \cdots, T_r$ such that the closure of each connected component S_j of $M \setminus \bigcup_{i=1}^r T_i$ is Seifert. Now a Seifert fibred manifold with boundary fibers over the circle and the monodromy is periodic [10]. By [11][Corollary 2.1.10] each one has property (RD). This is also true for the $\mathbb{Z}\oplus\mathbb{Z}$-subgroups along which the $\pi_1(S_j)$ are amalgamated.

Lemma 3.3. *There exists $V \geq 0$ such that, if g is an element of $\pi_1(M^3)$ contained in a conjugate of a subgroup associated to a Seifert-component S_j or a torus T_i, then any geodesic representing this element crosses at most V lifts of tori T_i.*

Proof. The main difficulty of this lemma holds in the case where some of the gluing-maps between the boundary tori are Anosov diffeomorphisms.

□

From Lemma 3.3, some computations, which are to detail, allow us to prove the inequality given in Proposition 2.2. This same proposition gives the conclusion.

□

3.2. **Atoroidal 3-manifolds.** A compact 3-manifold is *atoroidal* if it does not contain any incompressible torus.

Proposition 3.4. *Let M^3 be a closed, irreducible, orientable, atoroidal 3-manifold. If M^3 does not have Sol-geometry, then $\pi_1(M^3)$ has property (RD).*

Proof. By [15] M^3 has one of the eight following geometries: $\mathbb{E}^3, \mathbb{H}^3, \mathbb{S}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \widetilde{SL_2\mathbb{R}}, \mathrm{Nil}, \mathrm{Sol}$. The six geometries $\mathbb{E}^3, \mathbb{S}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \widetilde{SL_2\mathbb{R}}, \mathrm{Nil}$ correspond to Seifert fibred manifolds. By Lemma 3.1, in order to prove Proposition 3.4 it suffices to prove it for manifolds with $\mathbb{H}^3$-geometry, i.e. hyperbolic manifolds. But this is a result of Jolissaint [11][Theorem 3.2.1] that the fundamental group of such a manifold has property (RD). □

3.3. **Relative hyperbolicity and 3-manifolds.** Relative hyperbolicity was first formulated by Gromov in [8], then reformulated by Farb [6], followed by Bowditch [1] and Osin [14]. The introduction of this notion is motivated by the following result of Drutu-Sapir:

Theorem 3.5. [5] *Let G be a group which is strongly hyperbolic relative to a finite family of subgroups $\mathcal{H}$. If all the subgroups in $\mathcal{H}$ have property (RD) then G has property (RD).*

By the JSJ-decomposition [10], given any closed, irreducible, orientable 3-manifold M^3 there exist a family of maximal graph-submanifolds $\mathcal{G}_1, \cdots, \mathcal{G}_r$ in M^3 such that the closure of each connected component of $M^3 \setminus \bigcup_{i=1}^r$ is a compact 3-manifold with hyperbolic, finite volume interior, the boundary of which is a union of tori.

Proposition 3.6. *Let M^3 be a closed, irreducible, orientable 3-manifold. Then the fundamental group of M^3 is strongly hyperbolic relative to a family formed by*

(1) the subgroups G_i corresponding to certain conjugates of the fundamental groups of the maximal graph-submanifolds $\mathcal{G}_1, \cdots, \mathcal{G}_r$ in M^3,
(2) the $\mathbb{Z} \oplus \mathbb{Z}$-subgroups corresponding to the incompressible tori in $M^3 \setminus \bigcup_{i=1}^r \mathcal{G}_i$.

Proof. The fundamental group of M^3 is the fundamental group of a graph of groups Γ satisfying the following properties:

- the vertex groups are the G_i's, together with the fundamental groups H_j of finite volume hyperbolic 3-manifolds with cusps,
- the edge groups are $\mathbb{Z} \oplus \mathbb{Z}$-subgroups,
- two vertex groups G_i and G_j, $i \neq j$, are not adjacent.

The graph Γ becomes a graph of strongly relatively hyperbolic groups when considering each edge-group, and each vertex-group G_i strongly hyperbolic relative to itself, whereas each vertex-group H_j is considered as a group strongly hyperbolic relative to the $\mathbb{Z} \oplus \mathbb{Z}$-subgroups of the cusps [6]. Since the cusps subgroups are malnormal, if T is the universal covering of Γ, there is a uniform bound M on the length of the corridors in T: if $\mathcal{C}$ is a corridor in the tree of spaces $\mathcal{T}$ over the tree T and $\pi \colon \mathcal{T} \to T$ is the projection, then $\pi(\mathcal{C}$ has diameter smaller than M in T. It follows from the combination theorem of [7] that the fundamental group of Γ is strongly hyperbolic relative to a family of subgroups as given by Proposition 3.6.

□

Proof of Theorem 1.1. Let M^3 be a closed, irreducible, orientable 3-manifold which does not have Sol-geometry. If M^3 is atoroidal then, by Proposition 3.4 $\pi_1(M^3)$ has property (RD).

By Proposition 3.2 the fundamental group of a graph-manifold has property (RD). This is also true for $\mathbb{Z} \oplus \mathbb{Z}$. By Proposition 3.6, if M^3 is not atoroidal then $\pi_1(M^3)$ is strongly hyperbolic relative to a finite family of subgroups which either are fundamental groups of graph-manifolds or are isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Since each one has property (RD), by Theorem 3.5 the fundamental group of M^3 has property (RD).

Assume now that M^3 is not orientable. Then the orientable double cover of M^3, denoted by M^3_0, is a closed, orientable, irreducible 3-manifold. Therefore, by which precedes, the fundamental group of M^3_0 has property (RD). On the other hand, it is of index 2 in $\pi_1(M^3)$. Thus, by [11][Proposition 2.1.5], $\pi_1(M^3_0)$ has property (RD) if and only if $\pi_1(M^3)$ does. We so get that $\pi_1(M^3)$ has property (RD). □

Remark 3.7. By Kneser theorem, any closed 3-manifold M^3 admits a unique factorization in prime summands. By Van Kampen, the fundamental group of M^3 is the free product of the fundamental groups of the prime summands. By Theorem 1.1 the fundamental group of a prime summand has property (RD) unless the manifold obtained from this prime summand by capping-off the spheres in its boundary has Sol-geometry. Thus, by [11], $\pi_1(M^3)$ has property (RD) if and only if no prime summand of M^3 has Sol-geometry after capping-off the spheres in its boundary.

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