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Analyse numérique/*Numerical Analysis*

Coupling between scalar and vector potentials by the mortar element method

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Abstract The T–Ω formulation of the magnetic field is widely used in magnetodynamics. It allows the use of a scalar function in the computational domain and a vector quantity only in the conducting parts. Here we propose to approximate these two quantities on different meshes and to couple them by means of the mortar element method. ***To cite this article: Y. Maday et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 933–938.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Couplage entre potentiels scalaire et vecteur par la méthode des éléments joints

Résumé La formulation $T-\Omega$ pour le champ magnétique est largement utilisée en magnétodynamique. Elle permet d'utiliser une fonction scalaire dans tout le domaine de calcul et une quantité vectorielle seulement dans les conducteurs. On propose ici d'approcher ces deux quantités sur des maillages différents et de les coupler par la méthode des éléments avec joints. ***Pour citer cet article : Y. Maday et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 933–938.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

La simulation numérique de phénomènes electromagnétiques basse fréquence fait souvent appel au modèle des courants de Foucault. On s'interesse ici à la formulation en champs magnétique dans un domaine borné de $\mathbb{R}$ noté V composé de deux parties : un conducteur V_c et un isolant $V \setminus V_c$. Les équations sont alors (1) dans le conducteur et (2) dans l'isolant. Afin de ramener le calcul à des quantités scalaires, quand c'est possible, on a recours à l'introduction d'un potentiel scalaire $\widetilde{\Omega}$ défini dans V et un potentiel vecteur $\widetilde{T}$ limité à V_c tel que $H = \widetilde{T} - \nabla\widetilde{\Omega}$. Partant des équations variationelles définissant H (3) et (4), on obtient la formulation (7) pour déterminer $\widetilde{T}$ et $\widetilde{\Omega}$. L'espace $\mathrm{H}_0(\mathrm{curl}; V_c)$ est défini en (5) et $\mathrm{H}_0^1(V)$ est l'espace classique de Sobolev. Ce problème ne possèdant pas de solution unique, on choisit une condition (de type jauge) qui impose à $\Omega = \widetilde{\Omega} + \phi$, $\phi \in \mathrm{H}_0^1(V_c)$, d'être harmonique dans V_c. On aboutit alors à la formulation variationelle (10) où $b_c(W, v) := -\int_{V_c} W\nabla\mathcal{H}v$, pour $v \in \mathrm{H}_0^1(V)$ et $W \in \mathrm{H}_0(\mathrm{curl}; V_c)$ et $\mathcal{H}$ est l'opérateur de relèvément harmonique défini en (9). On montre d'abord que (11) a lieu pour une constante $0 < \gamma_1 < 1$, puis le lemme suivant sur la forme $a_g((W, w), (V, v)) := a_c(W, V) + b_c(W, v) + b_c(V, w) + a(w, v)$, pour $V, W \in \mathrm{H}_0(\mathrm{curl}; V_c)$ et $v, w \in \mathrm{H}_0^1(V)$:

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LEMME 0.1. – *La forme bilinéaire* $a_g(\cdot,\cdot)$ *est continue et elliptique sur* $\mathrm{H}_0(\mathrm{curl}; V_c) \times \mathrm{H}^1_0(V)$.

On en déduit que le problème (10) possède une solution et une seule.

Ce problème faisant intervenir une quantité scalaire sur V et une quantité vectorielle sur V_c, on peut penser à utiliser, dans le cadre d'une discrétisation en éléments finis, deux maillages différents : l'un sur V pour calculer Ω et indexé par H et l'autre sur V_c pour calculer T et indexé par h. On peut même envisager que les deux maillages soient contruit indépendamment (ce sera d'ailleurs naturellement le cas si le conducteur bouge dans V au cours du temps). La discrétisation de ce problème peut alors se faire en utilisant des éléments finis de type Nédélec pour T (espace noté $X_{0;h}(V_c)$) et des éléments finis standards pour Ω (espace noté $S_{0;H}(V)$). La forme b_c faisant intervenir un relèvement harmonique, doit être discrétisée. On doit tout d'abord définir un tel relèvement discret $\mathcal{H}_h$ défini en (12) mais aussi un opérateur de passage d'information Π_h du maillage de ∂V_c hérité du maillage de V à celui de ∂V_c hérité du maillage de V_c. Nous proposons de faire cette opération de manière optimale comme indiqué par (13), inspiré par la méthode des éléments avec joint, soit directement en suivant [2], soit à partir de [4] qui conduit à une définition de Π_h par matrice diagonale. On obtient, dès lors que le rapport h/H est assez petit, le fait que le problème (14) possède une solution unique et l'estimation d'erreur (15), valable dès que $T \in (\mathrm{H}^1(V_c))^3$ avec $\nabla \times T \in (\mathrm{H}^1(V_c))^3$ et $\Omega \in \mathrm{H}^\beta(V)$, pour un réel $\beta \leqslant 2$, où $b_h(W, v) := -\int_{V_c} W \nabla \mathcal{H}_h \Pi_h v$, pour tout $v \in S_{0;H}(V)$ et $W \in X_{0;h}(V_c)$. La forme matricielle du problème (14) est donnée en (16) où Q est la matrice associée à Π_h, S est associée au relèvement harmonique et B celle associée à la forme b_h. La méthode itérative de Gauss–Seidel (17) converge sans paramètre de relaxation.

1. Introduction

Low frequency electromagnetic devices are often modeled numerically on the basis of the eddy current formulation [1]. Two main families of formulations are widely used, the one based on magnetic and the one based on electric fields. Here, we restrict ourselves to the magnetic field approach. The entire space $\mathbb{R}^3$, is decomposed in the conducting region V_c and the external region $\mathbb{R}^3 \setminus V_c$. Denoting by H, B, J and E, respectively, the magnetic field, the magnetic flux density, the current density and the electric field, the quasi-stationary Maxwell equations restricted to the conducting region V_c read as follows:

$$\nabla \times H = J, \qquad \nabla \times E = -\partial_t B, \qquad \nabla \cdot B = 0. \tag{1}$$

The densities and the fields are linked by the constitutive properties, i.e., $J = \sigma E$, $B = \mu H$, where μ is the magnetic permeability and $\sigma \geqslant \sigma_0 > 0$ stands for the electric conductivity. Moreover, we assume that the material parameters are time independent and associated with a linear isotropic material. We suppose that no external source J_s is situated within the conducting regions. So, in the external insulating region $\mathbb{R}^3 \setminus V_c$, where σ is zero, we obtain the following field equations:

$$\nabla \times H = J_s, \qquad \nabla \cdot B = 0, \qquad B = \mu H. \tag{2}$$

The problem is well posed by imposing regularity conditions at infinity and suitable interface conditions on ∂V_c. In particular, $[H]_c \times n_c = 0$, $[B]_c \cdot n_c = 0$, where n_c is the outer normal to ∂V_c and $[v]_c$ stands for the jump of the quantity v at ∂V_c. Clearly, such a type of interface conditions has also to be verified at any surface where σ or μ is discontinuous. Additionally to the boundary conditions, we have to impose suitable initial values for the vector fields at a given time t_0. In particular, the initial condition on B has to give $\nabla \cdot B = 0$ and $[B] \cdot n = 0$ at any interface. We point out the fact that the vector fields J and $\partial_t B$ are automatically forced to be soleinodal by Eqs. (1). Of course, the condition $\nabla \cdot B = 0$ is satisfied at any time provided that it is verified by the initial condition. By introducing artificial boundary conditions, we can work on a bounded domain V. For simplicity, we assume that V_c is a simply connected polyhedral subdomain of V and $\overline{V}_c \subset V$. In a weak form, equations $\nabla \cdot B = 0$ and $B = \mu H$ in V can be written as follows:

$$\int_V \mu H \nabla v = 0, \quad \forall v \in \mathrm{H}^1_0(V), \tag{3}$$

where $H_0^1(V) = \{v \in L^2(V) \mid \nabla v \in (L^2(V))^3,\ v_{|\partial V} = 0\}$. From Eqs. (1) in V_c, we can write

$$\int_{V_c} \nabla \times H \nabla \times W + \int_{V_c} \sigma \partial_t(\mu H) W = 0, \quad \forall W \in H_0(\mathrm{curl}, V_c), \tag{4}$$

where

$$H_0(\mathrm{curl}, V_c) = \left\{ W \in \left(L^2(V_c)\right)^3 \mid \nabla \times W \in \left(L^2(V_c)\right)^3,\ W \times n_c = 0 \right\}. \tag{5}$$

For the current density J, the condition $\nabla \cdot J = 0$ suggests the introduction of a vector potential $\widetilde{T}$ such that $J = \nabla \times \widetilde{T}$. Then, in V_c, the difference between the vector potential $\widetilde{T}$ and the magnetic field H can be written as the gradient of a scalar function $\widetilde{\Omega}$, i.e., $H = \widetilde{T} - \nabla\widetilde{\Omega}$. A similar argument holds for the insulating region where we assume knowing a vector potential T_s such that $J_s = \nabla \times T_s$. Combining external and conducting regions we write H in V as $H = \widetilde{T} - \nabla\widetilde{\Omega}$, in V_c and $H = T_s - \nabla\widetilde{\Omega}$ in $V \setminus V_c$.

By eliminating the magnetic field H in (3) and (4), we obtain a coupled eddy current problem in terms of the electric potential $\widetilde{T}$ defined only in the conducting region V_c and the scalar potential $\widetilde{\Omega}$ defined everywhere in V. This system is completed with appropriate interface conditions over ∂V_c stating, e.g., that $\widetilde{\Omega}$ is continuous. This is nevertheless not enough to define $\widetilde{\Omega}$ and $\widetilde{T}$ uniquely. In fact $\nabla \cdot \widetilde{T}$ is not specified and thus there are many different gauge possibilities. One of them is to require that $\widetilde{T}$ has the same divergence as H in V_c but this eliminates $\widetilde{\Omega}$ on V_c. We prefer another condition, stated in Section 2.

2. Variational problem

In this section, we define a variational formulation based on the decomposition of H into $\widetilde{T}$ and $\nabla\widetilde{\Omega}$. We rightaway consider the system obtained after discretization in time of (3) and (4). Only implicit time discretization schemes can satisfy the stability condition. Having discretized the time derivative by means of a finite difference scheme of time step δt, then at each time step, we have to face a boundary value problem of the form

$$\begin{aligned} &\int_V \nabla\widetilde{\Omega}\nabla v - \int_{V_c} \widetilde{T}\nabla v = \int_{V\setminus V_c} f\nabla v, \quad \forall v \in H_0^1(V), \\ &\int_{V_c} \alpha\nabla \times \widetilde{T}\nabla \times W + \widetilde{T}W - \int_{V_c} \nabla\widetilde{\Omega}W = \int_{V_c} f_c W, \quad \forall W \in H_0(\mathrm{curl}, V_c), \end{aligned} \tag{6}$$

where the coefficient $\alpha > 0$ is constant. In the more general approach, it is uniformly positive definite and depends on the material parameters σ, μ as well as on the time step (e.g., $\alpha = \delta t/\mu\sigma$ in V_c). Note that the unknowns $\widetilde{T}$ and $\widetilde{\Omega}$ denote the approximation at the current time-step, f_c depends on the approximations of $\widetilde{T}$ and $\widetilde{\Omega}$ at the previous time-step, and f denotes the scaled source term depending on T_s. Additionally, $\widetilde{T}$ and $\widetilde{\Omega}$ have to satisfy the interface conditions at each time step: we choose here $\widetilde{T} \in H_0(\mathrm{curl}; V_c)$ and $\widetilde{\Omega} \in H_0^1(V)$. In our approach, the strong coupling between $\widetilde{T}$ and $\widetilde{\Omega}$ at the interface will be replaced by a weaker one. We consider the following discrete problem: find $(\widetilde{T}, \widetilde{\Omega}) \in H_0(\mathrm{curl}; V_c) \times H_0^1(V)$ such that

$$\begin{aligned} &a\left(\widetilde{\Omega}, v\right) + \hat{b}_c\left(\widetilde{T}, v\right) = \int_{V\setminus V_c} f\nabla v, \quad \forall v \in H_0^1(V), \\ &a_c\left(\widetilde{T}, W\right) + \hat{b}_c\left(W, \widetilde{\Omega}\right) = \int_{V_c} f_c W, \quad \forall W \in H_0(\mathrm{curl}; V_c). \end{aligned} \tag{7}$$

Here, the bilinear forms are defined by

$$\begin{aligned} a_c\big(\widetilde{T}, W\big) &:= \int_{V_c} \big(\alpha \nabla \times \widetilde{T} \nabla \times W + \widetilde{T} W\big), \quad \forall T, W \in \mathrm{H}_0(\mathrm{curl}; V_c),\\ \hat{b}_c(W, v) &:= -\int_{V_c} W \nabla v, \qquad \forall W \in \mathrm{H}_0(\mathrm{curl}; V_c),\ \forall v \in \mathrm{H}^1_0(V),\\ a\big(\widetilde{\Omega}, v\big) &:= \int_V \nabla \widetilde{\Omega} \nabla v, \qquad \forall \Omega,\ v \in \mathrm{H}^1_0(V). \end{aligned} \tag{8}$$

The bilinear forms are continuous with respect to the associated norms.

Now, it is easy to see that if $(\widetilde{T}, \widetilde{\Omega})$ is a solution of (7), then $(\widetilde{T} + \nabla\phi, \widetilde{\Omega} + \phi)$, $\phi \in \mathrm{H}^1_0(V_c)$, is a solution as well. To achieve uniqueness, we choose ϕ such that $\Omega = \widetilde{\Omega} + \phi$ is harmonic on V_c; to this purpose, we introduce the harmonic extension operator $\mathcal{H} : \mathrm{H}^1(V_c) \to \mathrm{H}^1(V_c)$ verifying

$$\mathcal{H}v_{|\partial V_c} := v_{|\partial V_c}, \quad \int_{V_c} \nabla \mathcal{H} v \nabla w = 0, \quad \forall w \in \mathrm{H}^1_0(V_c), \tag{9}$$

and state the modified variational problem: find $(T, \Omega) \in \mathrm{H}_0(\mathrm{curl}; V_c) \times \mathrm{H}^1_0(V)$ such that

$$\begin{aligned} a(\Omega, v) + b_c(T, v) &= \int_{V \setminus V_c} f \nabla v, \quad \forall v \in \mathrm{H}^1_0(V),\\ a_c(T, W) + b_c(W, \Omega) &= \int_{V_c} f_c W, \quad \forall W \in \mathrm{H}_0(\mathrm{curl}; V_c), \end{aligned} \tag{10}$$

where $b_c(W, v) := -\int_{V_c} W \nabla \mathcal{H} v$, for $v \in \mathrm{H}^1_0(V)$ and $W \in \mathrm{H}_0(\mathrm{curl}; V_c)$. In the following, $\|\cdot\|_{0;D}$ stands for the L^2-norm on the open bounded set D.

LEMMA 2.1. – *There exists a constant* $0 < \gamma_1 < 1$ *depending only on* V_c *and* V *such that*

$$\|\nabla \mathcal{H} v\|_{0;V_c} \leqslant \gamma_1 \|\nabla v\|_{0;V}, \quad \forall v \in \mathrm{H}^1_0(V). \tag{11}$$

For homogeneous Dirichlet boundary conditions, Lemma 2.1 is sufficient to establish the ellipticity of $a_g((W, w), (V, v)) := a_c(W, V) + b_c(W, v) + b_c(V, w) + a(w, v)$, where $V, W \in \mathrm{H}_0(\mathrm{curl}; V_c)$ and $v, w \in \mathrm{H}^1_0(V)$, which is the bilinear form associated with the variational problem (10).

LEMMA 2.2. – *The bilinear form* $a_g(\cdot,\cdot)$ *is continuous and elliptic on* $\mathrm{H}_0(\mathrm{curl}; V_c) \times \mathrm{H}^1_0(V)$.

Proof. – We start by considering $a_g((W, w), (W, w))$, for $(W, w) \in \mathrm{H}_0(\mathrm{curl}; V_c) \times \mathrm{H}^1_0(V)$, in more detail. With the Hilbert space $\mathrm{H}_0(\mathrm{curl}; V_c)$, we associate the energy norm $|||\cdot|||_{V_c}$ defined by

$$\big|\|W\|\big|^2_{V_c} := \|W\|^2_{0;V_c} + \alpha \|\nabla \times W\|^2_{0;V_c}, \quad \forall W \in \mathrm{H}_0(\mathrm{curl}; V_c).$$

This norm is equivalent to the standard Hilbert space norm. The L^2-orthogonality of $\mathcal{H}w$ yields

$$\begin{aligned} a_g((W, w), (W, w)) &= \big|\|W\|\big|^2_{V_c} + \|\nabla w\|^2_{0;V} - 2\int_{V_c} W \nabla \mathcal{H} w\\ &\geqslant \big|\|W\|\big|^2_{V_c} + \|\nabla w\|^2_{0;V} - 2\|W\|_{0;V_c} \|\nabla \mathcal{H} w\|_{0;V_c}. \end{aligned}$$

Now, we can use (11) to bound $\|\nabla \mathcal{H} w\|_{0;V_c}$ in terms of $\|\nabla w\|_{0;V}$ and we get

$$\begin{aligned} a_g\big((W, w), (W, w)\big) &\geqslant \big|\|W\|\big|^2_{V_c} + \|\nabla w\|^2_{0;V} - 2\gamma_1 \big|\|W\|\big|_{V_c} \|\nabla w\|_{0;V}\\ &\geqslant (1 - \gamma_1)\big(\big|\|W\|\big|^2_{V_c} + \|\nabla w\|^2_{0;V}\big). \end{aligned}$$

The last inequality shows that the ellipticity constant only depends on the constant in (11) and that to obtain a "good" ellipticity constant, it is sufficient that γ_1 is not too close to one. □

The unique solvability of the variational problem (10) is a consequence of Lemma 2.2 and of the continuity of the bilinear forms. The first equation in (10) and the definition of the harmonic extension yields $b_c(T,v)=0$, $\forall v\in \mathrm{H}_0^1(V_c)$. Moreover, taking $W=\nabla v$, for a function $v\in \mathrm{H}_0^1(V_c)$, in the second equation of (10), we get that T is divergence free as soon as f_c is divergence free. Hence, T is implicitly gauged, and Ω restricted to V_c is harmonic.

Remark 1. – The constant γ_1 does not depend on the shape regularity of $V\setminus V_c$ and V_c. In particular for time dependent problems where V_c may be moving, this might be important. More precisely, γ_1 does not depend on the distance between ∂V_c and ∂V. In the limit case that this distance tends to zero, γ_1 is still bounded away from one. In this situation we work with a ficticious domain V_g such that $\overline{V}\subset V_g$ and $V_g\setminus V$ is shape regular. Due to the homogeneous Dirichlet boundary conditions of $v\in \mathrm{H}_0^1(V)$ we extend v by zero to an element in $\mathrm{H}_0^1(V_g)$. Moreover, the ellipticity constant does not degenerate if the distance between ∂V_c and ∂V tends to zero.

3. Discretization

For the discretization of the vector field T, we propose to use the lowest order Nédélec finite elements, also known as edge elements [4]. We denote $X_{0;h}(V_c)$ the associated finite element space. For a given quasi-uniform simplicial triangulation $\mathcal{T}_h$ of V_c, the finite element functions are curl-conforming and the degrees of freedom are vector circulations over the edges e of $\mathcal{T}_h$. We note that $T\in X_{0;h}(V_c)$ has a vanishing tangential component on ∂V_c. Thus the degrees of freedom of $X_{0;h}(V_c)$ are associated with the interior edges of $\mathcal{T}_h$. The domain V is associated with a second quasi-uniform simplicial triangulation $\mathcal{T}_H$. The discretization of Ω on V is based on standard conforming elements of the lowest order and we denote by $S_{0;H}(V)$ the associated finite element space. The finite element functions are now piecewise linear and H^1-conforming and the degrees of freedom are associated with the interior vertices of the triangulation $\mathcal{T}_H$. We denote $S_h(V_c)$ the space of standard conforming elements of first order associated with $\mathcal{T}_h$ on V_c. We remark that no boundary conditions are imposed on $S_h(V_c)$. Moreover, we assume that ∂V_c can be written as union of faces in $\mathcal{T}_h$. The trace space of $S_h(V_c)$ on ∂V_c is called $W_h(\partial V_c)$.

To formulate the discrete version of the variational problem (10), we have to specify a discrete operator replacing the harmonic extension (9). A natural choice is to involve the discrete harmonic extension $\mathcal{H}_h$ defined as a map $\mathcal{H}_h : W_h(\partial V_c)\to S_h(V_c)$ verifying

$$\mathcal{H}_h v_{|\partial V_c} := v_{|\partial V_c}, \quad \int_{V_c}\nabla\mathcal{H}_h v\nabla w=0, \quad \forall w\in S_h(V_c)\cap \mathrm{H}_0^1(V_c). \tag{12}$$

However in general, the restriction of $v\in S_{0;H}(V)$ on ∂V_c is not an element in $W_h(\partial V_c)$. Thus, we cannot apply directly the discrete harmonic extension to the restriction of $v\in S_{0;H}(V)$ on ∂V_c. To overcome this difficulty, we introduce a projection operator on the boundary. This operator is well known in the mortar finite element context [2] and can be defined in terms of a Lagrange multiplier space $M_h(\partial V_c)$:

$$\Pi_h : \mathrm{L}^2(\partial V_c)\longrightarrow W_h(\partial V_c), \quad \int_{\partial V_c}\Pi_h v\nu=\int_{\partial V_c} v\nu, \quad \forall \nu\in M_h(\partial V_c). \tag{13}$$

To obtain a well defined operator Π_h, the Lagrange multiplier space $M_h(\partial V_c)$ has to be well chosen. There are many possibilities but, for simplicity reasons, we restrict ourselves to two choices. In the first case, $M_h(\partial V_c) := W_h(\partial V_c)$, as in [2]. In contrast to mortar finite element methods with many subdomains, no modification of $W_h(\partial V_c)$ is required since ∂V_c is a closed surface without crosspoints. Then, the operator Π_h is a L^2-projection and, numerically, a mass matrix system has to be solved. In a second case, we replace the standard hat functions by piecewise linear dual basis functions [5] in the definition of $M_h(\partial V_c)$. Then, Π_h is a quasi L^2-projection having the same qualitative stability properties as before. The advantage is that the mass matrix system is diagonal. These choices guarantee that the operator Π_h is H^s-stable for $0\leqslant s\leqslant 1$. Furthermore, it satisfies the approximation property in the $\mathrm{H}^{1/2}$-norm and in the $\mathrm{H}^{-1/2}$-norm. In terms of the discrete harmonic extension and the operator Π_h, we formulate the new discrete variational problem: find $(T_h,\Omega_H)\in X_{0;h}(V_c)\times S_{0;H}(V)$ such that

$$
\begin{aligned}
a(\Omega_H, v) + b_h(T_h, v) &= \int_{V\setminus V_c} f\nabla v, \quad \forall v \in S_{0;H}(V),\\
a_c(T_h, W) + b_h(W, \Omega_H) &= \int_{V_c} f_c W, \quad \forall W \in X_{0;h}(V_c),
\end{aligned}
\tag{14}
$$

where $b_h(W, v) := -\int_{V_c} W\nabla\mathcal{H}_h\Pi_h v$, for $v \in S_{0;H}(V)$ and $W \in X_{0;h}(V_c)$. Lemma 2.1 has an analogous discrete version that yields to prove an optimal error estimate (*see* [3] for further details).

LEMMA 3.1. – *For h/H small enough, problem* (14) *has a unique solution and there exists a constant C independent of the meshsize and a real $\beta \leqslant 2$ such that, for $T \in (\mathrm{H}^1(V_c))^3$ with $\nabla\times T \in (\mathrm{H}^1(V_c))^3$ and $\Omega \in \mathrm{H}^\beta(V)$, we have*

$$
\big|\|T - T_h\|\big|_{V_c} + \|\Omega - \Omega_H\|_{1;V} \leqslant C\big(h\big(\|T\|_{1;V_c} + \|\nabla\times T\|_{1;V_c}\big) + \mathrm{H}^{\beta-1}\|\Omega\|_{\beta;V}\big). \tag{15}
$$

4. Algorithmics

We conclude our analysis by presenting a numerical algorithm to solve the discrete problem (14). Let us denote by A_c and A the standard stiffness matrices associated with the bilinear forms $a_c(\cdot,\cdot)$ and $a(\cdot,\cdot)$ respectively. Let Q and S be the matrices associated, the first, with the mortar projection Π_h from $S_{0;H}(V)$ to $W_h(\partial V_c)$ and, the second, with the harmonic extension $\mathcal{H}_h$ from $W_h(\partial V_c)$ to $S_h(V_c)$. The algebraic form of the discrete problem (14) reads:[1] find two vectors T_h and Ω_H solution of the linear system

$$
A_cT_h + BSQ\Omega_H = F_c, \qquad A\Omega_H + Q^tS^tB^tT_h = F, \tag{16}
$$

where B is a rectangular stiffness matrix associated with the bilinear form $b_h(T_h, \Omega_H)$ and ^{t} denotes the transpose operator. As a linear iterative solver for (16), we suggest the use of a block Gauss–Seidel method, i.e., starting from Ω_H^n, first we compute T_h^{n+1} and then Ω_H^{n+1} by means of

$$
A_cT_h^{n+1} + BSQ\Omega_H^n = F_c, \qquad A\Omega_H^{n+1} + Q^tS^tB^tT_h^{n+1} = F. \tag{17}
$$

This algorithm converges thanks to the following lemma, whose proof relies on Lemma 2.1 and on the continuity and coerciveness of the bilinear forms $a(\cdot,\cdot)$ and $a_c(\cdot,\cdot)$.

LEMMA 4.1. – *Let us denote by e^n the error $\Omega_H - \Omega_H^n$ at the iteration n on the vector Ω_H. The mapping $e^n \to e^{n+1}$ is a strict contraction over $S_H(V)$: there exists a constant $0 < \theta < 1$ such that*

$$
a\big(e^{n+1}, e^{n+1}\big) < \theta a\big(e^n, e^n\big).
$$

The convergence of Ω_H^n to Ω_H yields the one of T_h^n to T_h. In (16), the application of B, Q, Q^t and B^t is standard. We do not assemble S and S^t but solve two homogeneous Dirichlet boundary value problems. Thus, at each step n, we have to solve two Dirichlet boundary value problems on $S_h(V_c)$, one curl problem on V_c and one Dirichlet problem on $S_{0;H}(V)$.

[1] In what follows, we use the same letters T_h and Ω_H for the discrete functions and their vectors of unknowns in the appropriated respective basis.

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