How singularities change our point of view

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Introduction I

The presence of singularities affects the geometry of hypersurfaces and of their complement.

This happens for projective complex hypersurfaces (studied via the methods of classical algebraic geometry) as well for their traces on the real affine space.

These last geometric objects are obviously more important in Computer Aided Design and Geometric Modeling.

Introduction II

In these applications the data are corrupted with noise. So, often, the singularities do not appear frankly, but produce ill-conditioned behaviors.

Therefore a difficulty of the question is to force them to show up.

In other words find the “nearest” singular situation and analyze its deformations.

Introduction II

Singularity theory, includes the computation and analysis of their infinitesimal (i.e. small) deformations into simpler singularities.

Whitney, Thom, Mather, Arnol'd and their coworkers and followers have classified representations, then studied useful stratifications of the classifying spaces (e.g. Jets spaces).

This leads to notions of genericity and stability which can help modelize the "real word".

We refer to R. Thom in his pioneering book on Structural stability and morphogenesis and to his developments with Zeeman of the Catastrophe Theory.

Introduction III

Deforming further a singularity, one get eventually a smooth (i.e. singularity-free) object. This can be called **"smoothing"**.

The analysis of the deformation provides a better understanding of the singularity and often allows calculations of invariants attached to the singularity.

Conversely, in some cases it is interesting to perform the reverse transformation, that we can call **"singularizing"** a geometric situation.

This reverse transformation can seem strange to the applied mathematician who generally chooses models described by regular functions.

Introduction IV

This amounts to consider that a complicated smooth object, say with "ill-conditioned curvatures" is easier to describe and better understood, if it is viewed as a (small) deformation of a simpler singular object.

However, this is of common use in every day life, for instance when I describe my desk as a rectangular table with rounded corners :

the natural **"nearer" simple and structured** object is a rectangle, which is a singular curve.

Our aim is to describe this process in some scientific situations then relate it to better mastered mathematical computational tools.

Examples I

In this first section, we provide examples in applied sciences, beside the ones presented by Thom and Zeeman in their theory, where it is convenient to consider singularities in order to better understand a situation or overtake a difficulty.

- Shape from shadow,
- Image processing,
- Molder of an almost conic surface,
- Mechanical parts.

Shape from shadow

- In Computer vision a method of automatic recognition and separation of objects, is to lightup an object from positioned sources. Then consider the curves traced on the object separating shadow and light. Geometrically this is interpreted via the concept of critical loci of some mapping.

- Under reasonable assumptions there are a finite number of types of singularities which may occur on these curves for generically positioned sources. This gives rise to classifications, used to better view, represent and analyze these objects.

See the work by Donati, Merle and Stolfi.

See also the work of Y. L. Kergosien. Here after is one of his interesting picture of a stable projection of an iliac bone.

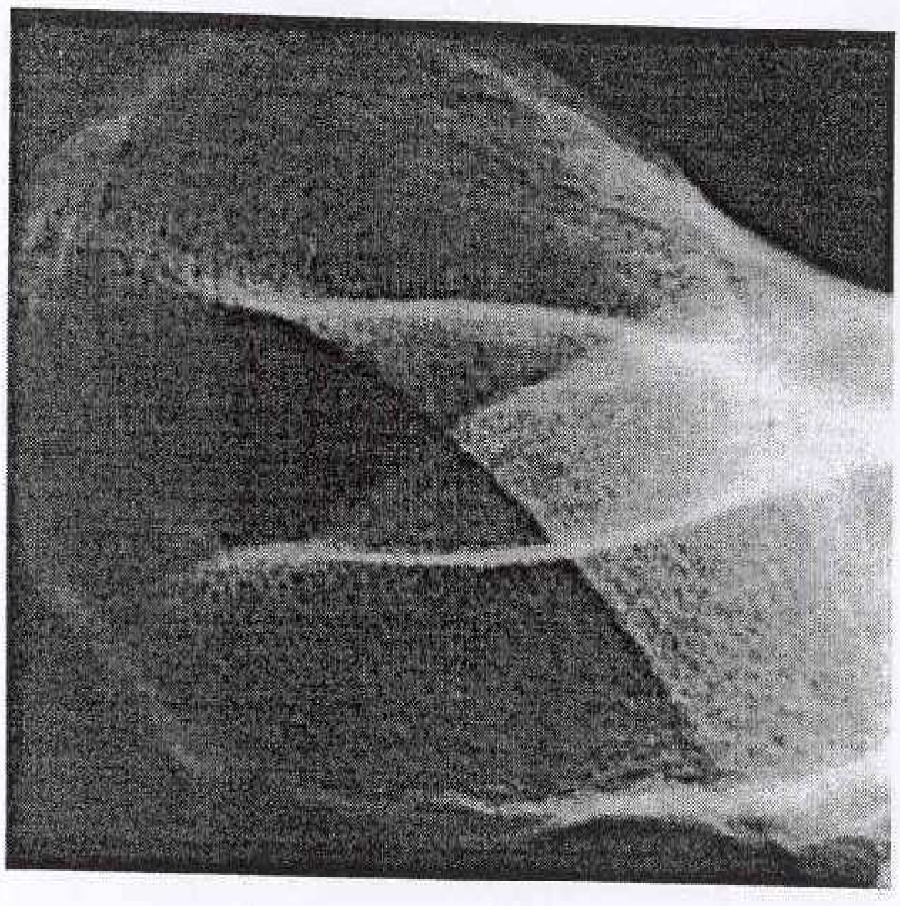


Image processing

We give two illustrations in Image processing of what we called "singularization".

• The first one is very simple and general, it is just replacing a smooth curve by a polygon which approximates it, because it is easier to represent and transform.

• The second one is contour detection, a preliminary step, after segmentation, of almost any further processing.

One of the difficulties is to detect so called triple junction points and quadruple ones ; the latest are generally degeneracies of near-by tangent level curves.

Forcing these curves to touch, i.e. “singularizing” the observed data, allows to better separate the domains of colors and provide a more efficient shape determination process.

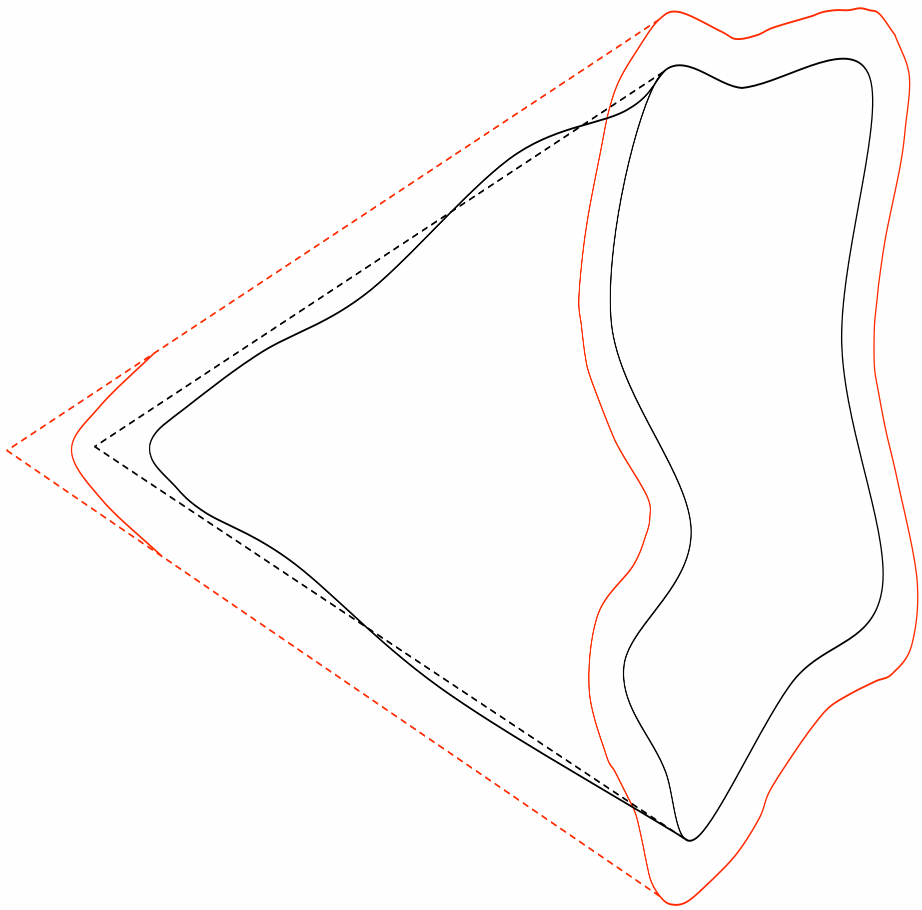
An "almost" conic surface

Designing molders is an important activity of Solid Modeling.

For an "almost" conic surface delimiting a solid (see the figure below), it is easier and more efficient to first consider a near-by conic patch.

Then, compute an offset of this conic patch, which is easier.

Then, eventually perturb this offset surface, to get a new surface better suited for the considered task.



Mechanical components

Some mechanical components have either clear singularities or singularities which have been smoothed. This is achieved by performing a blending.

In the next figures, on can see first a rocker-arm then a master cylinder, then an oil pomp.

All of them present singularities kept by the designer that, after appearing in the design, have been removed by the corresponding blending.

Singularization and approximation

- The first examples (catastrophes, shape from shadow) were direct applications of the mathematical exact classification of singularities and their stability.

- The second ones require an extra ingredient : the consideration of a level of precision, as we force (up to some approximation) a smooth situation to become singular.

- The general mathematical theory needed to analyze these developments is not yet done. However some elements exist, starting with linear algebra and univariate polynomials.

Numerical rank of a matrix

• Consider an ill-conditioned system of linear equations with approximate data (floats), say for instance $AX = B$,

$$A := \begin{bmatrix} 0.99999 & 0.00001 & 1.00001 \\ 1.00002 & -0.00001 & 0.99999 \\ 0.00003 & 1.00003 & 0.00001 \end{bmatrix}$$

$$B := [2.003, 1.915, 3.967]$$

The exact solution of this problem has rather unbalanced entries :

$$X = [-1757.619107, 4.002012644, 1759.586894]).$$

• With a SVD, one can perform a judicious change of coordinates and compute an approximate solution with more balanced coordinates.

Numerical rank of a matrix II, SVD

The notion of numerical rank is well formalized through the singular values and SVD. These notions are well presented e.g. in the book of Golub and Van Loan. See also the web page of J. Demmel.

I learned them from **Henri Lombardi**, and we used them in several papers.

One writes $A = UDV$. Here U and V are unitary square matrices, and the only non zero entries of D are real non negative and on the diagonal.

Once ordered, these elements are called the singular values. They describe how the map associated to A deform the objects from the source space to the target space.

Numerical rank III

There is a renewed interest on the numerical rank of a matrix, in the Signal processing community as well as in the Computer Algebra one.

- In SIAM J. of Matrix Anal. and Appl. (2005), T.Y. Li and Z. Zeng propose a new rank-revealing method. It is based on a globally convergent Gauss-Newton iterative scheme and a deflation technique.

The method calculates approximate singular values below a given threshold, one by one, and returns the approximate rank of the matrix along with an orthonormal basis for the approximate null space. It is updated when a row or column is inserted or deleted.

• In 2004 and 2005, the SNC and Computer algebra community produced several papers of SVD, and so-called Fast SVD,

applied to Hankel matrices (Issac'05), Sylvester matrices (SNC'05), more generally to matrices with a displacement structure,

and also on approximate factorization problems (around Kaltofen and Gao), GCD (around L. Zhi and Z. Zeng).

They continue the approach that we had initiated with Henri Lombardi and Ioannis Emiris in 1996.

Approximate algebra

In several applications, one encounters overdetermined polynomial systems of constraints expected to have solutions.

But by the use of measures or floating point calculations the coefficients have been corrupted and are only imperfectly known.

The usual algebraic operations, such as factoring, solving and root clustering are not always well defined in this context.

So one decide to accept an answer that is exact for a perturbed input set. The approach generalizes the usual notion of backward error.

Approximate GCD

In Emiris-Galligo-Lombardi paper in JPAA (1997), a certification theorem is given for the ϵ-gcd of two univariate polynomials.

It is expressed in terms of the SVD of the subresultants matrices attached to the two polynomials.

This has been generalized to the case of several univariate polynomial and implemented by D. Rupprecht in his thesis and published in JPAA.

This problem received a renewed interest, with new implementations, and extension to the study of multiple roots. See e.g. the work of Z. Zeng and a recent paper by Diaz-Toca and Gonzalez-Vega.

Ill-conditioned polynomial systems

The zero-dimensional case generalizes the previous univariate case.

Homotopy (also called continuation, or Newton) methods follow a deformation from a (somehow similar) system with known solutions. There exist several good implementations (Li, Verschelde, ...), and rigorous mathematical treatment and complexity analysis has been

Singular or near singular solutions, slow down the convergence and create ill-conditioned behaviors. In the last years, Ojika et al. (1983) deflation method received new developments. It is also an instance of what I called "singularization".

1. Exact setting

   There is rigorous presentation and implementation, inspired by Ojika'approach, with Hensel liftings done by G. Lecerf.

2. Approximate setting with floats

   There is a new analysis and implementation by Verschelde, which uses a SVD to guess a numerical rank of a jacobian matrix.

3. Decomposition of curves

   A similar approach was followed, in a paper by Galligo and Rupprecht, on decomposition of space curves without using a projection.

- Strategy

The idea is to guess, via a SVD, that at some near x_0 (to be determined) the numerical rank of the Jacobian matrix drops, which indicates that we are near a multiple point.

One replace, recursively, the given system of equations by a new one (with more variables) augmented with derivatives, having a smaller multiplicity. Eventually we get a well conditioned system.

Other techniques :

- Approximate Groebner bases : see e.g. the works of C. Traverso, D. Lazard (together with the RUR of F.Rouille).
- Resultant : Galaad team, Gonzalez-Vega, Kapur.
- The work of Galligo-Lombardi-GonzalezVega.

Multiple roots of analytic systems

• For the case of simple roots, Smale, Shub and their co-authors have, in the last decade, improved the analysis of Newton algorithm by their so-called α-theory and majorant series.

Given an initial point x_0, Newton algorithm converges quadratically, provided the quantity $\alpha(F, x_0)$ is sufficiently small, moreover the distance from x_0 is bounded by a quantity $\beta(F, x_0)$ times a universal constant.

• This has been adapted and well tuned for polynomials, by Dedieu and Shub.

• For the case of multiple roots, the same approach started to be generalized by Giusti et al. in two papers in 2004.

They addressed the case where the Jacobian matrix is expected of corank 1.

One of these 2 papers is dedicated to the study, via a generalization of Rouché theorem and explicit majorant series, of the family of Shroeder operators in the univariate case when the multiplicity m is known :

$$x \to x - (m-l)\frac{f^{(l)}(x)}{f^{(l+1)}(x)},\ 0 \leq l \leq m-1.$$

Next step : Singularization of curves

The simplest case in positive dimension is a reduced plane curve.

In that case the singularities must be isolated.

So we are led to consider a zero-dimensional system of equations.

The algorithms will depend on the representation of the input data : parametric curve or defined by an implicit equation, exact or approximate data, polynomials or analytic functions.

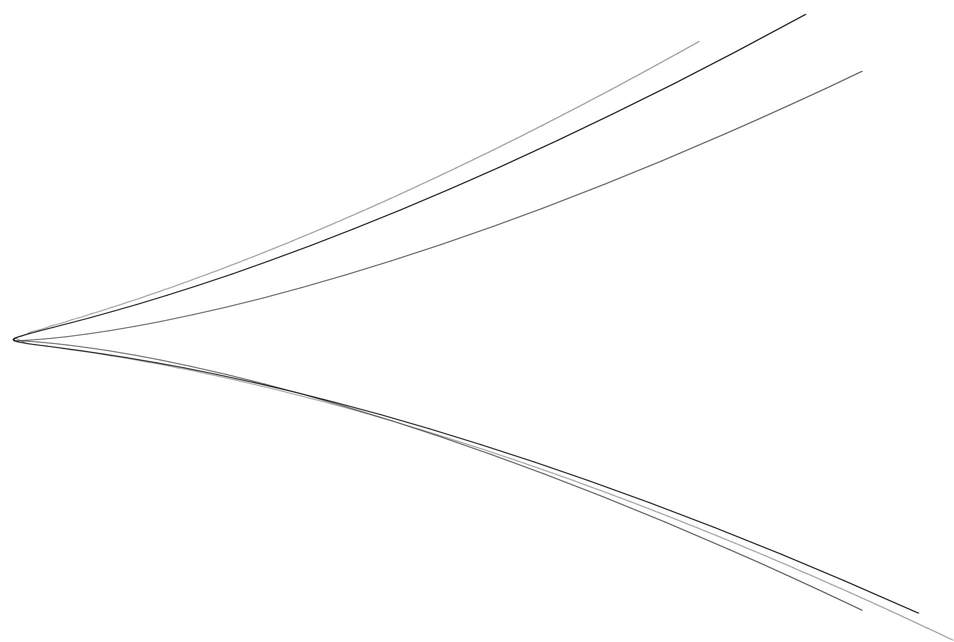
This is also related to the study of curvature.

To my knowledge, in the approximate setting this has not been done yet, even for simple singularities.

Example The curve given "locally" by the implicit equation :

$$-0.9*x^3 - 0.0185*x^2 - .12*x*y - .00110*x - 0.00220*y + 0.8*y^2 = 0$$

We approximate it with $x = t^2 + 0.1*t$; $y = t^3 + 0.02*t^2 - 0.01*t$.



Next step II : Singularization of surfaces

For 3D- surfaces, usual singularity and stable mappings theory “a la Thom-Mather” is not well suited for many applications. as it focuses on isolated singularities.

Consider a surface with lines of singularities, with one, two or three edges. This is rather common in many geometric applications. Then in Thom-Mather model this cannot be the graph of a generic map, because in their theory it has infinite co-dimension.

With Yosi Yomdin (Weizmann Institute), we started preliminary discussions on alternative models of genericity, which support more efficient data representations.

Conclusion

We have seen, that in Solid modeling and other fields of applied sciences, many volumes are delimited by surfaces with singularities.

These singularities are either apparent or smoothed. Therefore geometric models able to represent these patches of surfaces in a simple, robust and efficient way would be really useful.

The corresponding needed mathematical tools and developments should be able to mix :

geometry, genericity, simple algebraic representations, approximation and create efficient algorithms.

This scientific program just started. It opens a promising field of research.

My team, Galaad directed by B. Mourrain, is a joint project INRIA-UNSA-CNRS developed on these questions many collaborations, in particular through :

– The European projects GAIA I and GAIA II (2001-2005),
– The European Net of Excellence Aim@Shape (2004-2009)
– The future French A.N.R. project GECKO (2006-2009).

And we are open to new collaborations.