Quasipatterns in the superposition of two hexagonal patterns for the Swift-Hohenberg PDE

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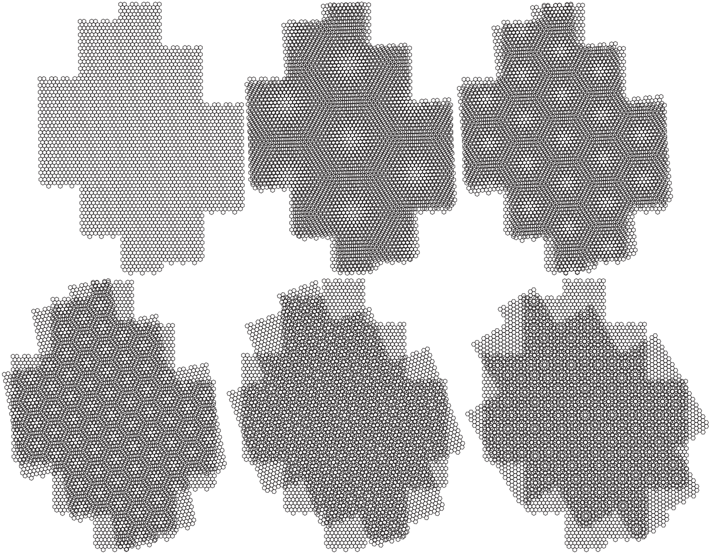
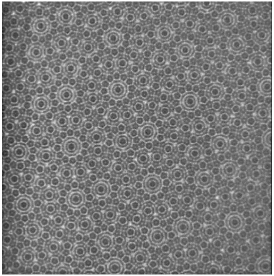
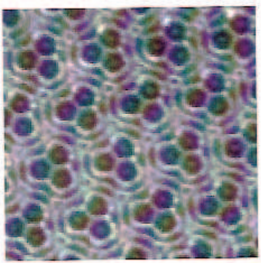
Superposition experiments



Superposition of two hexagonal patterns: $0^o, 3^o, 5^o, 10^o, 20^o, 30^o$

Quasipatterns experiments





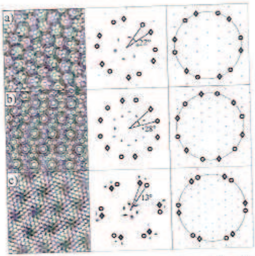


FIG. 3. (a) 3:2, (b) 4:3, and (c) 5:3 grid states. The spatial spectra (center) corresponding to the photographs (left) are spanned by grids (right) whose basis vectors, $\vec{k}_{grid}$, are stable modes. Fluid parameters are as in Fig. 2. Fluid depths: (a) and (c) 0.3 cm and (b) 0.2 cm. ○ and denote corotated sets of $\vec{K}_i$.

Experiments of Faraday type. Kudrolli, Pier, Gollub 1998, Epstein, Fineberg 2006

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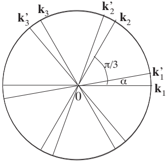
Definition of the quasilattice

$$\Gamma = \{\mathbf{k} \in \mathbb{R}^2; \mathbf{k} = \sum_{j=1,\ldots 6} m_j \mathbf{k}_j + m'_j \mathbf{k}'_j,\ \ m_j, m'_j \in \mathbb{N}\}.$$



Special angles $\mathcal{E}_p := \{\alpha \in \mathbb{R}/2\pi\mathbb{Z}; \cos\alpha \in \mathbb{Q}, \cos(\alpha + \pi/3) \in \mathbb{Q}\}$.

Lemma

The set $\mathcal{E}_p$ has a zero measure in $\mathbb{R}/2\pi\mathbb{Z}$.
(i) If the wave vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}'_1, \mathbf{k}'_2$ are not independent on $\mathbb{Q}$, then $\alpha \in \mathcal{E}_p$.
(ii) If $\alpha \in \mathcal{E}_p$ then the lattice Γ is periodic with an hexagonal symmetry, and wave vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}'_1, \mathbf{k}'_2$ are combinations of only two smaller vectors, of equal length making an angle $2\pi/3$.

Fourier series

$u(x, y)$ function under the form of a Fourier expansion

$$u = \sum_{\mathbf{k}\in\Gamma} u^{(\mathbf{k})} e^{i\mathbf{k}\cdot\mathbf{x}}, \;\; u^{(\mathbf{k})} = \overline{u}^{(-\mathbf{k})} \in \mathbb{C}. \tag{1}$$

$\mathbf{k} \in \Gamma$ may be written as

$$\mathbf{k} = z_1\mathbf{k}_1 + z_2\mathbf{k}_2 + z_3\mathbf{k}_1' + z_4\mathbf{k}_2', \;\; (z_1, z_2, z_3, z_4) \in \mathbb{Z}^4,$$

For $\alpha \in \mathcal{E}_{qp} = \mathcal{E}_p^c,$, Γ spans a 4-dimensional vector space on $\mathbb{Q}$.

$$N_{\mathbf{k}} = |\mathbf{z}| = \sum_{j=1,\ldots,4} |z_j| \text{ is a norm of } \mathbf{k}(\mathbf{z})$$

Hilbert spaces $\mathcal{H}_s$, $s \geq 0$:

$$\mathcal{H}_s = \left\{ u = \sum_{\mathbf{k}\in\Gamma} u^{(\mathbf{k})} e^{i\mathbf{k}\cdot\mathbf{x}};\, u^{(\mathbf{k})} = \overline{u}^{(-\mathbf{k})} \in \mathbb{C},\;\; \sum_{\mathbf{k}\in\Gamma} |u^{(\mathbf{k})}|^2 (1 + N_{\mathbf{k}}^2)^s < \infty \right\},$$

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Lemma

If $\alpha \in \mathcal{E}_{qp}$, a function defined by a convergent Fourier series as above represents a quasipattern, i.e. is quasiperiodic in all directions.

Lemma

For nearly all $\alpha \in (0, \pi/6)$, in particular for $\alpha \in \mathbb{Q}\pi \cap (0, \pi/6]$, the only solutions of $|\mathbf{k}(\mathbf{z})| = 1$ are $\pm \mathbf{z}_j, \pm \mathbf{k}'_j$ $j = 1, 2$ and $\mathbf{k} = \pm \mathbf{k}_3$, or $\pm \mathbf{k}'_3$, i.e. corresponding to $\mathbf{z} = (\pm 1, \mp 1, 0, 0)$ or $(0, 0, \pm 1, \mp 1)$.

$\mathcal{E}_0$ is the set of α's such that Lemma above applies.

Lemma

For nearly all $\alpha \in \mathcal{E}_{qp} \cap (0, \pi/6)$, and for $\varepsilon > 0$, there exists $c > 0$ such that, for all $|\mathbf{z}| > 0$ such that $|\mathbf{k}(\mathbf{z})| \neq 1$, $(|\mathbf{k}(\mathbf{z})|^2 - 1)^2 \geq \frac{c}{|\mathbf{z}|^{12+\varepsilon}}$ holds.

Steady Swift-Hohenberg equation in $\mathbb{R}^2$

$$(1+\Delta)^2 u = \mu u - u^3, \; \mathbf{x} \in \mathbb{R}^2 \rightarrow u(\mathbf{x}) \in \mathbb{R}$$

$$e^{i\mathbf{k}.\mathbf{x}} \in Ker\{(1+\Delta)^2 - \mu\} \text{ in } \mathcal{H}_s$$

iff Dispersion equation holds: $(1-|\mathbf{k}|^2)^2 = \mu, \; \mathbf{k} \in \Gamma$
For $\mu = 0$ all wave vectors $\mathbf{k}$ with $|\mathbf{k}| = 1$ are **critical**
We choose to look for solutions in $\mathcal{H}_s$, for $\alpha \in \mathcal{E}_{qp} \cap \mathcal{E}_0$, i.e. quasiperiodic in $\mathbb{R}^2$, moreover invariant under rotations of angle $\pi/3$ and bifurcating for μ close to 0.
define $\mathbf{L}_0 = (1+\Delta)^2$
For $\alpha \in \mathcal{E}_0$ $Ker\mathbf{L}_0$ is 2-dimensional spanned by

$$v = \sum_{j=1,2,..,6} e^{i\mathbf{k}_j \cdot \mathbf{x}}, \quad w = \sum_{j=1,2,..,6} e^{i\mathbf{k}'_j \cdot \mathbf{x}}.$$

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Symmetries

$\mathbf{S}$ represents the imparity symmetry: $\mathbf{S}u = -u$

$$\mathbf{S}\mathbf{L}_0 = \mathbf{L}_0\mathbf{S}, \quad \mathbf{S}u^3 = (\mathbf{S}u)^3.$$

$\mathbf{R}_\theta$ rotation of angle θ, centered at the origin

$$(\mathbf{R}_\theta u)(\mathbf{x}) = u(\mathbf{R}_{-\theta}\mathbf{x}),$$

$$\mathbf{R}_\theta\mathbf{L}_0 = \mathbf{L}_0\mathbf{R}_\theta, \quad \mathbf{R}_\theta u^3 = (\mathbf{R}_\theta u)^3.$$

τ represents the symmetry with respect to the bisectrix of wave vectors $\mathbf{k}_1$ and $\mathbf{k}_1'$.

$$\tau\mathbf{L}_0 = \mathbf{L}_0\tau, \quad \tau u^3 = (\tau u)^3. \qquad (2)$$

Then

$$\mathbf{R}_{\pi/3}v = v, \ \mathbf{R}_{\pi/3}w = w, \ \tau v = w, \ \tau w = v$$

Formal series

We look for a formal solution of SHE as

$$u = \sum_{n\geq 1} \varepsilon^n u_n, \ \mu = \sum_{n\geq 1} \varepsilon^n \mu_n, \ \varepsilon \text{ defined by the choice of } u_1$$

$$\text{order } \varepsilon : \mathbf{L}_0 u_1 = 0, \ u_1 \text{ lies in the kernel of } \mathbf{L}_0$$

$$u_1 = w + \beta_1 v$$

the coefficient in front of w fixes the choice of the scale ε,
provided that we choose to impose $\langle u_n, w\rangle_0 = 0, \quad n = 2, 3, \ldots$

$$\text{order } \varepsilon^2\text{: } \mathbf{L}_0 u_2 = \mu_1 u_1,$$

and the compatibility condition gives

$$\mu_1 = 0, \ u_2 = \beta_2 v.$$

Formal series-continued

Order ε^3: $\mathbf{L}_0 u_3 = \mu_2 u_1 - u_1^3.$

$$\text{Compatib:}\ a\mu_2 - c - 3b\beta_1^2 = 0,$$
$$a\beta_1\mu_2 - 3b\beta_1 - c\beta_1^3 = 0,$$

where $a = 6,\ b = 36,\ c = 90,$
$\langle v^2 w, v\rangle = \langle w^2 v, w\rangle = \langle v^3, w\rangle = \langle w^3, v\rangle = 0.$

This gives $(c - 3b)(\beta_1^3 - \beta_1) = 0,$

$$\mu_2 = \frac{c}{a} + 3\frac{b}{a}\beta_1^2,\ u_3 = \beta_3 v + \widetilde{u_3},\ \langle\widetilde{u_3}, v\rangle = \langle\widetilde{u_3}, w\rangle = 0.$$

$\widetilde{u_3}$ only contains Fourier modes $e^{i\mathbf{k}\cdot\mathbf{x}}$ with $\mathbf{k} = m_1'\mathbf{k}_1' + m_2'\mathbf{k}_2'$

First case: $\beta_1 = 0$, then $\mu_2 = 15$

Second case: $\beta_1 = \pm 1$, then $\mu_2 = 33$, $\tau u_1 = \beta_1 u_1,\ \tau\widetilde{u_3} = \beta_1\widetilde{u_3}$

For $\beta_1 = 0$ we obtain the classical bifurcating hexagonal-symmetric expansion (u_n is orthogonal to v for all n).
For $\beta_1 = \pm 1$ the expansions are uniquely determined.
$u_1 = w + \beta_1 v, \; \tau u_1 = \beta_1 u_1$

$$\begin{aligned} \beta_1 = 1 \qquad & \text{leads to } \tau u = u, \\ \beta_1 = -1 \qquad & \text{leads to } \tau u = -u. \end{aligned}$$

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Formal series (end)

Theorem

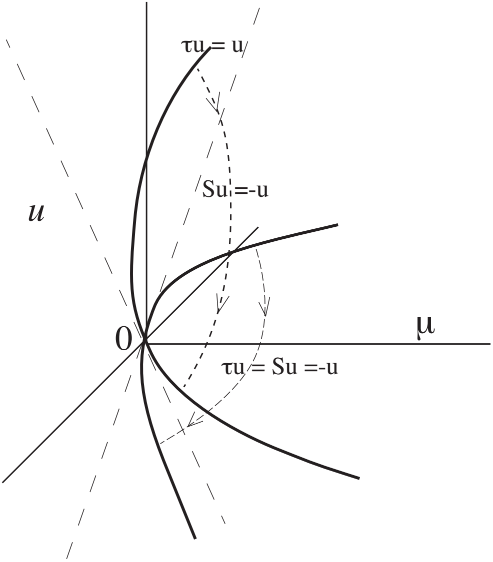
Let us consider the Swift-Hohenberg model PDE . The superposition of two hexagonal patterns, differing by a small rotation of angle $\alpha \in \mathcal{E}_0$, leads to formal expansions in powers of an amplitude ε, of new bifurcating patterns invariant under rotations of angle $\pi/3$. We obtain two new branches of patterns, with formal expansions of the form

$$
\begin{aligned}
u &= \varepsilon(w + \beta_1 v) + \varepsilon^3 \widetilde{u_3} + \ldots \varepsilon^{2n+1} \widetilde{u_{2n+1}} + .., \quad \beta_1 = \pm 1, \\
\langle \widetilde{u_{2n+1}}, v \rangle &= \langle \widetilde{u_{2n+1}}, w \rangle = 0, \ \tau \widetilde{u_{2n+1}} = \beta_1 \widetilde{u_{2n+1}}, \ \tau u = \beta_1 u, \\
\mu &= \varepsilon^2 \mu_2 + \varepsilon^4 \mu_4 + \ldots + \varepsilon^{2n} \mu_{2n} + .., \ \mu_2 > 0,
\end{aligned}
$$

$$
v = \sum_{j=1,2,..,6} e^{i\mathbf{k}_j \cdot \mathbf{x}}, \quad w = \sum_{j=1,2,..,6} e^{i\mathbf{k}'_j \cdot \mathbf{x}}, \ (\mathbf{k}_1, \mathbf{k}'_1) = \alpha.
$$

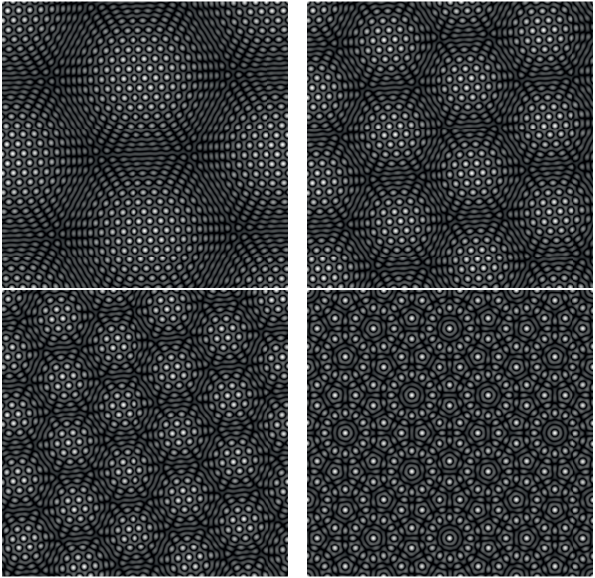
For $\alpha \in \mathcal{E}_p \cap \mathcal{E}_0$ the expansions converge, giving periodic patterns with hexagonal symmetry.

2 branches of quasipatterns



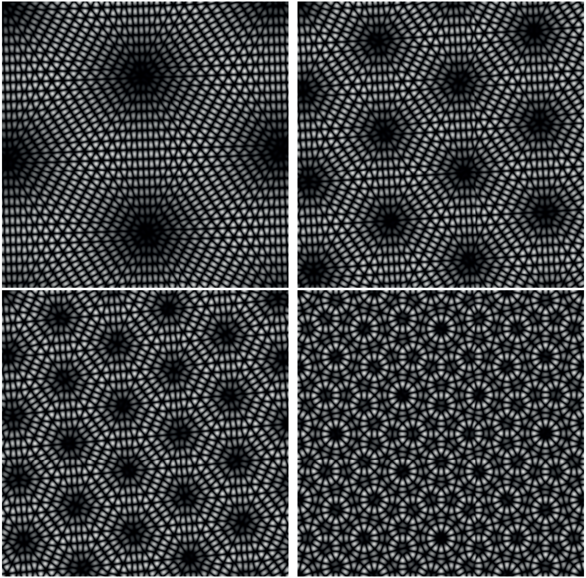
The two branches bifurcate for $\mu > 0$

Solutions of SHE, $\beta_1 = 1$



solutions of SHE for $\alpha = 4^o, 7^o, 10^o, 30^o$. Order ε and $\beta_1 = 1$.

Solutions of SHE, $\beta_1 = -1$



solutions of SHE for $\alpha = 4^o, 7^o, 10^o, 30^o$. Order ε and $\beta_1 = -1$

Existence of quasipatterns

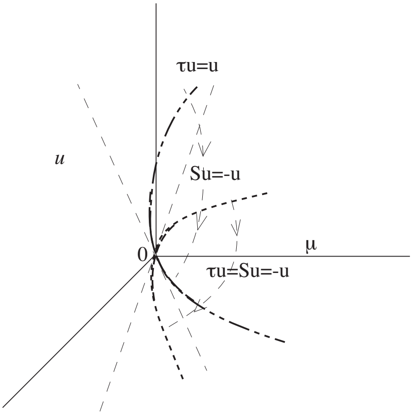
Theorem

Assume $\alpha \in \mathcal{E}_2 \cap \mathcal{E}_0 \cap \mathcal{E}_{qp}$ (full measure set). Then there exist $s_0 > 2$ and $\varepsilon_0 > 0$ such that for an asymptotically full measure set of values of $\varepsilon \in (0, \varepsilon_0)$, there exists a bifurcating quasipattern solution of SHE, invariant under rotation of angle $\pi/3$, of the form

$$\begin{aligned} u &= U_\varepsilon + \varepsilon^{2p}\widetilde{u}(\varepsilon),\ \widetilde{u} \in \{v, w\}^\perp, \\ U_\varepsilon &= \varepsilon(w + \beta_1 v) + \varepsilon^3 \widetilde{U_\varepsilon},\ \beta_1 = \pm 1, \tau u = \beta_1 u, \\ \widetilde{U}_\varepsilon &= \widetilde{u}_3 + \ldots.\varepsilon^{2p-4}\widetilde{u}_{2p-1}, \\ \mu &= \mu_\varepsilon + \widetilde{\mu}(\varepsilon),\ \mu_\varepsilon = \varepsilon^2\mu_2 + \ldots\varepsilon^{2p}\mu_{2p}, \end{aligned}$$

where $\widetilde{u}(\varepsilon) \in \mathbf{Q}_0\mathcal{H}_{s_0}$, $w, v, \widetilde{u_{2n-1}}, \mu_{2n}$ are defined above, and functions of ε are C^1 with $\widetilde{u}(0) = 0$, $\widetilde{\mu}(\varepsilon) = \mathcal{O}(\varepsilon^{2p+2})$. $\mathbf{S}u = -u$ corresponds to change ε into $-\varepsilon$.

Bifurcation diagram



Asymptotically full measure set of good values fof $\mu > 0$ for the two bifurcating branches

Idea of Proof 1

$$\begin{aligned} u &= U_\varepsilon + \varepsilon^{2p} W, \;\; W = \widetilde{u} + \beta v, \;\; \widetilde{u} \in \{v, w\}^\perp, \\ U_\varepsilon &= \varepsilon(w + \beta_1 v) + \varepsilon^3 \widetilde{U_\varepsilon}, \;\; \beta_1 = \pm 1, \;\; \widetilde{U}_\varepsilon = \widetilde{u}_3 + \ldots.\varepsilon^{2p-4}\widetilde{u}_{2p-1}, \\ \mu &= \mu_\varepsilon + \widetilde{\mu}(\varepsilon), \;\; \mu_\varepsilon = \varepsilon^2 \mu_2 + \ldots \varepsilon^{2p}\mu_{2p}, \end{aligned}$$

$$(\mathbf{L}_0 - \widetilde{\mu})\widetilde{u} + g(\varepsilon, \beta, \widetilde{\mu}) + \mathcal{B}_{\varepsilon,\beta}\widetilde{u} + \mathcal{C}_{\varepsilon,\beta}(\widetilde{u}) = 0,$$

where $\mathcal{B}_{\varepsilon,\beta}$ is linear and $O(\varepsilon^2)$ in $\mathcal{H}_s, s \geq 0$, and $\mathcal{C}_{\varepsilon,\beta}$ is at least quadratic and $O(\varepsilon^{2p+1})$ in $\mathcal{H}_s, s > 2$.
We expect to solve this range-equation for $\widetilde{\mu} \in [-\varepsilon^{2p-2}, \varepsilon^{2p-2}]$, with respect to $\widetilde{u}$ which should be of order $O(\varepsilon)$, and put it into Bifurcation equations:

$$\begin{aligned} a\widetilde{\mu} - 6\varepsilon^{2p+1} b\beta_1\beta - 3\varepsilon^{2p+1}\langle u_1^2 \widetilde{u}, w\rangle &= \mathcal{O}(\varepsilon^{2p+2}), \\ -\beta_1 a\widetilde{\mu} + 2c\varepsilon^{2p+1}\beta + 3\varepsilon^{2p+1}\langle u_1^2\widetilde{u}, v\rangle &= \mathcal{O}(\varepsilon^{2p+2}). \end{aligned}$$

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Then solve with respect to $(\widetilde{\mu}, \beta) = (O(\varepsilon^{2p+2}), O(\varepsilon))$. Finally $\beta(\varepsilon) \equiv 0$ by a symmetry argument.

Idea of Proof (continued)

We have a small divisor problem:

$$\widetilde{\mathbf{L}_0}^{-1} e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{(|\mathbf{k}|^2-1)^2} e^{i\mathbf{k}\cdot\mathbf{x}}$$

with $(|\mathbf{k}|^2-1)^2 \geq cN_{\mathbf{k}}^{-13}$
Nash-Moser method needs to invert the differential at any V near 0: $\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I}$ where $\mathcal{L}_{\varepsilon,\beta,V}$ acts in $\mathbf{Q}_0\mathcal{H}_t, \ \ t\geq 0$ and is defined by

$$\mathcal{L}_{\varepsilon,\beta,V} = \mathbf{L}_0 - \mu_\varepsilon\mathbb{I} + 3\mathbf{Q}_0(U_\varepsilon^2\cdot) - 6\varepsilon^{2p}\mathbf{Q}_0[U_\varepsilon(V+\beta v)(\cdot)] - 3\varepsilon^{4p}\mathbf{Q}_0[(V+\beta v)^2(\cdot$$

inverse of $\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I}$

Definition

Truncation of the space. Let $s \geq 0$ and $N > 1$ be an integer, we define $E_N := \Pi_N \mathbf{Q}_0 \mathcal{H}_s$, which consists in keeping in the Fourier expansion of $\widetilde{u} \in \mathbf{Q}_0\mathcal{H}_s$ only those $\mathbf{k} \in \Gamma$ such that $N_{\mathbf{k}} \leq N$. By construction we obtain

$$||(\Pi_N \mathbf{L}_0 \Pi_N)^{-1}||_s \leq c_0(1+N)^{13}.$$

Inverse of $\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I}$ for $N < M_\varepsilon$

Lemma

Let $S > s_0 > 2$ and $\varepsilon_0 > 0$ small enough and $\alpha \in (\mathcal{E}_1 \cap \mathcal{E}_0) \cup \mathcal{E}_{\mathbb{Q}}$. Then for $0 < \varepsilon \leq \varepsilon_0$ and $N \leq M_\varepsilon$ with $M_\varepsilon := \left[\frac{c_1}{\varepsilon^{2/13}}\right]$ and $(\varepsilon, \widetilde{\mu}, \beta, V) \in [-\varepsilon_0, \varepsilon_0] \times [-\varepsilon^{2p-2}, \varepsilon^{2p-2}] \times [-\beta_0, \beta_0] \times E_N$, the following holds for $s \in [s_0, S]$ and V such that $||V||_s \leq 1$,
$||(\Pi_N(\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I})\Pi_N)^{-1}||_s \leq 2c_0(1+N)^{13}$

Inverse of $\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I}$ for large N

define $\Lambda := \{\varepsilon, \tilde{\mu}); \varepsilon \in [-\varepsilon_0, \varepsilon_0], \tilde{\mu} \in [-\varepsilon^{2p-2}, \varepsilon^{2p-2}]\}$, and for $M > 0$, $s_0 > 2$,

$$\begin{aligned} \mathcal{U}_M^{(N)} \quad : \quad &= \{ V \in C^1(\Lambda \times [-\beta_0, \beta_0], E_N); V(0, \widetilde{\mu}, \beta) = 0, \\ &||V||_{s_0} \leq 1, ||\partial_{\varepsilon,\beta} V||_{s_0} \leq M, ||\partial_{\widetilde{\mu}} V||_{s_0} \leq (M/\varepsilon^{2p-2}) \} . \end{aligned}$$

For $V \in \mathcal{U}_M^{(N)}$, we consider the operator

$$\begin{aligned} \Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)} - \widetilde{\mu}\mathbb{I})\Pi_N \quad &= \quad \Pi_N \mathbf{L}_0 \Pi_N - \widetilde{\mu}\mathbb{I}_N + \varepsilon^2 \mathcal{B}_1^{(N)}(\varepsilon) + \\ &+\varepsilon^{2p+1} \mathcal{B}_2^{(N)}(\varepsilon, \beta, V(\varepsilon, \widetilde{\mu}, \beta)), \end{aligned}$$

$\Pi_N \mathbf{L}_0 \Pi_N$, $\mathcal{B}_1^{(N)}$, $\mathcal{B}_2^{(N)}$ selfadjoint in $\Pi_N \mathbf{Q}_0 \mathcal{H}_0$ and analytic in their arguments.

Eigenvalues of $\Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)} - \widetilde{\mu}\mathbb{I})\Pi_N$ have the form

$$\sigma_j(\varepsilon, \widetilde{\mu}, \beta) = s_j(\varepsilon) + f_j(\varepsilon, \widetilde{\mu}, \beta) - \widetilde{\mu},$$

where s_j is analytic and f_j is Lipschitz

$$|f_j(\varepsilon_2, \widetilde{\mu}_2, \beta_2) - f_j(\varepsilon_1, \widetilde{\mu}_1, \beta_1)| \leq c[\varepsilon^{2p}|\varepsilon_2 - \varepsilon_1| + \varepsilon^3 |\widetilde{\mu}_2 - \widetilde{\mu}_1| + \varepsilon^{2p+1}|\beta_2 - \beta_1|]$$

Bad set of $\tilde{\mu}$

$$B^{(N)}_{\varepsilon,\beta,\gamma}(V) = \Big\{\widetilde{\mu} \in [-\varepsilon_0, \varepsilon_0]; (\varepsilon, \beta, V) \in [-\varepsilon_0, \varepsilon_0] \times [-\beta_0, \beta_0] \times \mathcal{U}^{(N)}_M, \exists j \in \{1, ...\mathcal{N}\}, |\sigma_j(\varepsilon, \widetilde{\mu}, \beta)| < \frac{\gamma}{N^\tau}\Big\}$$

$$B^{(N)}_{\varepsilon,\beta,\gamma}(V) = \cup_{j=1}^{\mathcal{N}}(\widetilde{\mu}_j^-(\varepsilon,\beta), \widetilde{\mu}_j^+(\varepsilon,\beta)),$$

$$0 < \widetilde{\mu}_j^+(\varepsilon,\beta) - \widetilde{\mu}_j^-(\varepsilon,\beta) \leq \frac{4\gamma}{N^\tau},$$

$$\mathrm{meas}(B^{(N)}_{\varepsilon,\beta,\gamma}(V)) \leq \frac{4b\gamma}{N^{\tau-4}},$$

Good set of $\tilde{\mu}$: $G^{(N)}_{\varepsilon,\beta,\gamma}(V) := [-\varepsilon_0, \varepsilon_0] \backslash B^{(N)}_{\varepsilon,\beta,\gamma}(V)$.

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Inverse of $\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I}$ for large N (continued 2)

Lemma

Assume $\gamma \leq \widetilde{\gamma} = (2^{13/2+1}c_0)^{-1}$ *and* $\tau > 33 + 26p$. *For* $V \in \mathcal{U}_M^{(N)}$ *and* $(\varepsilon, \beta) \in [-\varepsilon_0, \varepsilon_0] \times [-\beta_0, \beta_0]$ *fixed, then if* $\widetilde{\mu} \in G_{\varepsilon,\beta,\gamma}^{(N)}(V) \cap [-\varepsilon^{2p-2}, \varepsilon^{2p-2}]$, $N > 1$

$$||(\Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)} - \widetilde{\mu}\mathbb{I})\Pi_N)^{-1}||_0 \leq \frac{N^\tau}{\gamma}.$$

Moreover, for $N > M_\varepsilon$, *the measure of the "bad set"* $B_{\varepsilon,\beta,\gamma}^{(N)}(V)$ *is bounded by* $4b\gamma/N^{\tau-4}$, *while it is* 0 *for* $N \leq M_\varepsilon$.

This estimate is in $\mathcal{L}(\mathbf{Q}_0\mathcal{H}_0)$. In fact, we need to obtain a tame estimate for $(\Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)} - \widetilde{\mu}\mathbb{I})\Pi_N)^{-1}$ in $\mathcal{L}(\mathbf{Q}_0\mathcal{H}_s)$ for $s > 0$, with an exponent on N not depending on s.
We use Bourgain 1995, Craig 2000, Berti-Bolle 2010 with a suitable adaptation.

Inverse of $\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I}$ in $\mathbf{Q}_0\mathcal{H}_s$ for large N (continued 3)

Singular set in $\mathbb{Z}^d$: $S(N) := \{\mathbf{z} \in \Gamma(N);\ (1 - |\mathbf{k}(\mathbf{z})|^2)^2 < \rho\}$ with

$$\Gamma(N) := \{\mathbf{z} \in \mathbb{Z}^4;\ 0 \leq |\mathbf{z}| \leq N,\ \mathbf{k}(\mathbf{z}) \in \Gamma \backslash \{\mathbf{k}_j, \mathbf{k}'_j, j = 1, ..., 6\}\}.$$

Useful lemma (uses Bourgain 1995, Craig 2000, Berti-Bolle 2010)
There exists $\rho_0 > 0$ independent of N such that if $\rho \in]0, \rho_0]$ then $S(N) = \bigcup_{\alpha \in \mathcal{A}} \Omega_\alpha$ is a union of disjoint clusters Ω_α satisfying :

- $(H1)$, for all $\alpha \in \mathcal{A}$, $M_\alpha \leq 2m_\alpha$ where $M_\alpha = \max_{\mathbf{z} \in \Omega_\alpha} |\mathbf{z}|$ and $m_\alpha = \min_{\mathbf{z} \in \Omega_\alpha} |\mathbf{z}|$;
- $(H2)$, there exists $\delta = \delta(d) \in]0, 1[$ independent of N such that if $\alpha, \beta \in \mathcal{A}, \alpha \neq \beta$ then $\mathrm{dist}(\Omega_\alpha, \Omega_\beta) := \min_{\mathbf{z} \in \Omega_\alpha, \mathbf{z}' \in \Omega_\beta} |\mathbf{z} - \mathbf{z}'| \geq \frac{(M_\alpha + M_\beta)^\delta}{2}$

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Basic ingredient for the Lemma above

Define the positive definite matrix **A** in $\mathbb{Z}^d$:

$$|\mathbf{k}(\mathbf{z})|^2 = \langle \mathbf{z}, \mathbf{A}\mathbf{z} \rangle = q_1 + q_2 \cos\alpha + q_3\sqrt{3}\sin\alpha$$

where q_1, q_2, q_3 are quadradic forms of $\mathbf{z}$ with integer coefficients.
Then, for any $\mathbb{Q}$-linearly independent family $\{\mathbf{e}_j, j = 1, ..., d \leq 4\}$ in $\mathbb{Z}^4$, let consider the matrix **M** such that $M_{l,m} = \langle \mathbf{e}_l, \mathbf{A}\mathbf{e}_m \rangle$.
We have
$det\mathbf{M} = \frac{1}{2^4}[a_0 + \sum_{1\leq n\leq d} a_{n0}\cos^n\alpha + a_{n-1,1}\cos^{n-1}\alpha(\sqrt{3}\sin\alpha)]$,
with integers a_{nm} bounded by $6^d max_m\{|\mathbf{e}_m|^{2d}\}$.
Then, we can prove: for $\alpha \in \mathcal{E}_2$ (full measure set), there exists $C > 0$ such that for all $\mathbf{a} \in \mathbb{Z}^{(2d+1)} \backslash \{0\}$,

$$|det\mathbf{M}| \geq \frac{C}{|\mathbf{a}|^l}, \ l = 2d(2d+1),$$

with $|\mathbf{a}| = |a_0| + \sum_{1\leq n\leq d} |a_{n0}| + |a_{n-1,1}|$

Lemma

Assume $\alpha \in \mathcal{E}_2$. Let γ and τ be as in Lemma for $\mathcal{H}_0$, and choose $s_0 \geq 3 + \frac{\tau+4}{\delta}$ where δ is the number introduced in previous Lemma, and define

$$m = 2\tau + 6.$$

Assume $(\varepsilon, \widetilde{\mu}, \beta, V) \in [-\varepsilon_1, \varepsilon_1] \times [-\varepsilon^2, \varepsilon^2] \times [-\beta_0, \beta_0] \times \mathcal{U}_M^{(N)}$, with ε_1 small enough and $\widetilde{\mu} \in \mathcal{G}^{(N)}_{\varepsilon,\beta,\gamma}(V)$, where

$$\mathcal{G}^{(N)}_{\varepsilon,\beta,\gamma}(V) = \cap_{M_\varepsilon < K \leq N} G^{(K)}_{\varepsilon,\beta,\gamma}(V) \cup [-\varepsilon^{2p-2}, \varepsilon^{2p-2}].$$

Let $S > s_0$. Then for all $s \in [s_0, S]$ there exists $K(s) > 0$ such that for any $\widetilde{u} \in \Pi_N \mathbf{Q}_0 \mathcal{H}_s$, we have for any $N > 1$

$$||(\Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)} - \widetilde{\mu}\mathbb{I})\Pi_N)^{-1}\widetilde{u}||_s \leq K(s)\frac{N^m}{\gamma}(||\widetilde{u}||_s + ||V||_s||\widetilde{u}||_{s_0}).$$

The proof follows Berti-Bolle 2010

We set $\tilde{\mu} = \varepsilon^{2p-2}\widehat{\mu}$
Nash-Moser method, following Berti-Bolle-Procesi 2010 leads to:

Theorem

Assume $\alpha \in \mathcal{E}_2 \cap \mathcal{E}_0 \cap \mathcal{E}_{qp}$ and let s_0 be as in Lemma above. Then for all $0 < \gamma \leq \widetilde{\gamma}$ there exist $\varepsilon_2(\gamma) \in (0, \varepsilon_0)$ and a C^1- map V : $(0, \varepsilon_2(\gamma)) \times [-1, 1] \rightarrow \mathcal{H}_{s_0+4}$ such that $V(0, \widehat{\mu}, \beta) = 0$ and if $\varepsilon \in (0, \varepsilon_2(\gamma))$, $\widehat{\mu} \in [-1, 1] \backslash C_{\varepsilon,\beta,\gamma}$, the function $\widetilde{u} = V(\varepsilon, \widehat{\mu}, \beta)$ is solution of the range equation. Here $C_{\varepsilon,\beta,\gamma}$ is a subset of $[-1, 1]$, which is a Lipschitz function of (ε, β) and has a Lebesgue measure less than $C\gamma|\varepsilon|^3$ for some constant $C > 0$, independent of $(\varepsilon, \beta, \gamma)$.

Resolution of the Bifurcation equations

$$a\widehat{\mu} - 6\varepsilon^3 b\beta_1\beta - 3\varepsilon^3\langle u_1^2 V(\varepsilon,\widehat{\mu},\beta), w\rangle = \mathcal{O}(\varepsilon^4),$$
$$-\beta_1 a\widehat{\mu} + 2c\varepsilon^3\beta + 3\varepsilon^3\langle u_1^2 V(\varepsilon,\widehat{\mu},\beta), v\rangle = \mathcal{O}(\varepsilon^4).$$

Implicit function theorem gives

$$\widetilde{\mu} = \varepsilon^{2p-2}\widehat{\mu} = \varepsilon^{2p+2}h(\varepsilon),\ \ \beta = \varepsilon g(\varepsilon), \text{ (H)}$$

$\varepsilon h(\varepsilon)$ and $\varepsilon g(\varepsilon)$ are C^1 functions of $\varepsilon \in [0,\varepsilon_1]$.

Define "bad layers" of degree N: $BS_N(V) := \{(\varepsilon,\widetilde{\mu},\beta) \in \Lambda \times [-\beta_0,\beta_0]; \exists j; \widetilde{\mu} \in (\widetilde{\mu}_j^-(\varepsilon,\beta), \widetilde{\mu}_j^+(\varepsilon,\beta))\}$.
In the 3-dimensional space $(\varepsilon,\widetilde{\mu},\beta)$ we need to check that the curve (H) crosses transversally the bad set formed by the infinitely many thin layers $\cup_{n\in\mathbb{N}} B_n S_{N_n}(V_{n-1})$, where $N_n = [N_0(\gamma)]^{2^n}$, and V_n are the successive points obtained in the Newton iteration of the Nash-Moser method.

The intersection of the surface $\beta = \varepsilon g(\varepsilon)$ with $\cup_{n\in\mathbb{N}} B_n S_{N_n}(V_{n-1})$ is a set of bad strips, bounded by curves of the form

$$\begin{aligned}
\widetilde{\mu}_j^{\pm(N_n)}(\varepsilon) &= s_j^{(N_n)}(\varepsilon) + g_j^{\pm(N_n)}(\varepsilon), \\
s_j^{(N_n)}(\varepsilon) &= s_j^{(N_n)}(0) + 3\varepsilon^2 + \mathcal{O}(\varepsilon^4) \\
|g_j^{\pm(N_n)}(\varepsilon_2) - g_j^{\pm(N_n)}(\varepsilon_1)| &\leq c\varepsilon^4|\varepsilon_2 - \varepsilon_1|
\end{aligned}$$

Then for any of the limiting curves,

$$|\widetilde{\mu}(\varepsilon + h) - \widetilde{\mu}(\varepsilon)| \geq c|\varepsilon||h|$$

Which is sufficient for having a transverse intersection with (H).

measure of "bad" $\widetilde{\mu}$'s $< C\gamma|\varepsilon|^{2p+1}$ hence
measure of "bad" ε's $< \frac{C\gamma|\varepsilon|^{2p+1}}{min|\text{slope}|} < \frac{C\gamma|\varepsilon|^{2p+1}}{c|\varepsilon|} \leq C'\gamma\varepsilon^{2p}$.

The complementary subset in $(0,\varepsilon)$, is the good set of $|\varepsilon|$, which is of asymptotic full measure since $[|\varepsilon| - C'\gamma\varepsilon^{2p}]/|\varepsilon| \to 1$ as $\varepsilon \to 0$.

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final result 2: $\beta(\varepsilon) \equiv 0$

Use a symmetry argument

$\tau U_\varepsilon = \beta_1 U_\varepsilon$,

For $\beta_1 = 1$,

$$\tau(\widetilde{u}(\varepsilon) + \beta(\varepsilon)v) = \tau\widetilde{u}(\varepsilon) + \beta(\varepsilon)w = \widetilde{u}(\varepsilon) + \beta(\varepsilon)v$$

by the uniqueness of the solution u. Hence, $\beta(\varepsilon) \equiv 0$.

For $\beta_1 = -1$,

$$\tau\widetilde{u}(\varepsilon) + \beta(\varepsilon)w = -\widetilde{u}(\varepsilon) - \beta(\varepsilon)v$$

by the uniqueness of the solution $-u$. This implies that in all cases

$$\tau\widetilde{u}(\varepsilon) = \beta_1\widetilde{u}(\varepsilon), \ \beta(\varepsilon) \equiv 0$$

Existence of quasipatterns

Theorem

Assume $\alpha \in \mathcal{E}_2 \cap \mathcal{E}_0 \cap \mathcal{E}_{qp}$ which is a full measure set. Then there exist $s_0 > 2$ and $\varepsilon_0 > 0$ such that for an asymptotically full measure set of values of $\varepsilon \in (0, \varepsilon_0)$, there exist two branches of bifurcating quasipattern solutions of SHE, invariant under rotation of angle $\pi/3$, of the form

$$
\begin{aligned}
u &= U_\varepsilon + \varepsilon^{2p}\widetilde{u}(\varepsilon),\ \widetilde{u} \in \{v, w\}^\perp, \\
U_\varepsilon &= \varepsilon(w + \beta_1 v) + \varepsilon^3 \widetilde{U_\varepsilon},\ \beta_1 = \pm 1, \tau u = \beta_1 u, \\
\mu &= \mu_\varepsilon + \widetilde{\mu}(\varepsilon),\ \mu_\varepsilon = \varepsilon^2 \mu_2 + ...\varepsilon^{2p}\mu_{2p}
\end{aligned}
$$

where $\widetilde{u}(\varepsilon) \in \mathbf{Q}_0\mathcal{H}_{s_0}$, $w, v, \widetilde{U_\varepsilon}, \mu_{2n}$ are defined above, and functions of ε are C^1 with $\widetilde{u}(0) = 0$, $\widetilde{\mu}(\varepsilon) = \mathcal{O}(\varepsilon^{2p+2})$. $\mathbf{S}u = -u$ corresponds to change ε into $-\varepsilon$.

References on Quasipatterns

G.I., A.M. Rucklidge. On the existence of quasipattern solutions of the Swift-Hohenberg equation. J. Nonlinear Science 20, 3, 361-394, 2010.

B.Braaksma, G.I., L.Stolovitch. Existence of quasipatterns solutions of the Swift-Hohenberg equation. A.R.M.A. 209, 1 (2013), 255-285. Erratum ARMA 211, 3 (2014), 1065.

B.Braaksma, G.I., L.Stolovitch. Existence proof of of quasipatterns solutions of the Swift-Hohenberg equation.Com. Math. Phys. 353(1), 37-67, 2017 DOI 10.1007/s00220-017-2878-x

B.Braaksma, G.I. Existence of bifurcating quasipatterns in steady Bénard-Rayleigh convection. Arch. Rat. Mech. Anal. 231(3), 1917-1981 (2019)DOI: 10.1007/s00205-018-1313-6

G.I. Existence of quasipatterns in the superposition of two hexagonal patterns. Nonlinearity (to appear in 2019)

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Proof of the first diophantine Lemma

Let us define

$$\begin{aligned} P &= (q_1-1)^2,\ Q = q_2^2+3q_3^2, \\ \theta(\mathbf{z}) &\in [0,2\pi];\quad \cos\theta(\mathbf{z}) = q_2/\sqrt{Q},\ \sin\theta(\mathbf{z}) = \sqrt{3}q_3/\sqrt{Q}. \end{aligned}$$

$$|\mathbf{k}(\mathbf{z})|^2 - 1 = q_1 - 1 + \sqrt{Q}\cos(\alpha-\theta(\mathbf{z})). \tag{3}$$

Choose $\varepsilon > 0$, for nearly all $\Omega \notin \mathbb{Q}$, there exists $C > 0$ such that (classical diophantine estimate)

$$|P/Q - \Omega| \geq \frac{C}{Q^{2+\varepsilon}},\ \text{for all } Q \in \mathbb{Z}\backslash\{0\}.$$

a zero measure set in Ω corresponds to a zero measure set in β, the set of angles β such that there exists $C(\beta) > 0$ such that

$$|P/Q - \cos^2\beta| \geq \frac{C(\beta)}{Q^{2+\varepsilon}},\ \text{for all } Q \in \mathbb{Z}\backslash\{0\}$$

is of full measure.

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Proof of the first diophantine Lemma - continued

For each Q there corresponds a finite set $\{\mathbf{z}_j\}$, hence a finite set $\{\theta(\mathbf{z}_j)\}$, so that the set of $\alpha \in (0, \pi/6)$ such that there exists $C'(\alpha)$ and

$$|P/Q - \cos^2(\alpha - \theta(\mathbf{z}))| \geq \frac{C'(\alpha)}{Q^{2+\varepsilon}}, \text{ for all } Q \in \mathbb{Z}\backslash\{0\}$$

is, for each Q, the intersection of the sets above for a finite number of $\theta(\mathbf{z}_j)$. This set is then also of full measure.
A simple study of hyperbolas $y^2 - \omega^2 = \pm\frac{C'}{Q^l}$ and an estimate of the distance to the asymptote for $\omega = 1$, implies that

$$\left|\sqrt{\frac{P}{Q}} - |\cos(\alpha - \theta)|\right| \geq \frac{C'}{4Q^{2+\varepsilon}}, \text{ for } Q \text{ large enough.}$$

Then

$$|q_1 - 1 + \sqrt{Q}\cos(\alpha - \theta(\mathbf{z}))| \geq \frac{C'}{4Q^{3/2+\varepsilon}},$$

and, since $Q \leq 3|\mathbf{z}|^4$, the Lemma follows.

Proof of the 2nd diophantine Lemma

$$\tau = \sqrt{3}\tan\alpha/2,$$

$$\begin{aligned} P(\mathbf{a},\alpha) &= a_{0+}\sum_{1\leq n\leq d} a_{n0}\left(\frac{3-\tau^2}{3+\tau^2}\right)^n + a_{n-1,1}\left(\frac{3-\tau^2}{3+\tau^2}\right)^{n-1}\left(\frac{6\tau}{3+\tau^2}\right) \\ &= \frac{Q(\mathbf{a},\tau)}{(3+\tau^2)^d}, \end{aligned}$$

It is sufficient to consider the "bad τ's" such that $|Q(\mathbf{a},\tau)| \leq \frac{c}{|\mathbf{a}|^l}$, $Q(\mathbf{a},\cdot)$ polynomial of degree $2d$ not identical to 0, *with integer coefficients*

$$Q(\mathbf{a},\tau) = (a_0 + \sum_{1\leq n\leq d}(-1)^n a_{n0})\Pi_{j=1,\dots 2d}(\tau-\tau_j),$$

there exists $j(\tau)$ such that

$$|\tau-\tau_{j(\tau)}|^{2d} \leq |Q(\mathbf{a},\tau)|$$

in all cases, the bad τ's satisfy $|\tau - \tau_{j(\tau)}| \leq \left(\frac{c}{|\mathbf{a}|^l}\right)^{1/2d}$. Summing for $j = 1, ...2d$, their measure $|\delta\tau|$ is bounded by

$$|\delta\tau| \leq 4d \left(\frac{c}{|\mathbf{a}|^l}\right)^{1/2d} .$$

Hence the measure of bad α , for **a** fixed:

$$|\delta\alpha| \leq \frac{2}{\sqrt{3}}|\delta\tau| \leq \frac{8dc^{1/2d}}{\sqrt{3}|\mathbf{a}|^{l/(2d)}}.$$

We now count the number of coefficients **a** of polynomials corresponding to $|\mathbf{a}|$. This number is bounded by $(2|\mathbf{a}|)^{(2d+1)}$. Hence the measure of the set of bad α's for all $\mathbf{a} \in \mathbb{Z}^{(2d+1)} \backslash \{0\}$ with a fixed norm $|\mathbf{a}|$ is bounded by

$$\frac{dc^{1/2d}2^{2-2d}}{\sqrt{3}|\mathbf{a}|^{l/(2d)-(2d+1)}}.$$