2D Standing waves on an infinitely deep perfect fluid, under gravity

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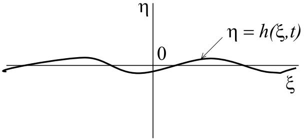
collaboration with P.Plotnikov (Novosibirsk) and J.Toland (Bath)

Basic formulation (time t and ξ-periodic)



Time scale: $\frac{T}{2\pi}$, Length scale: $\frac{\lambda}{2\pi}$, parameter: $1+\mu = \frac{gT^2}{2\pi\lambda}$

velocity potential $\phi(\xi,\eta,t)$ $\quad\Delta\phi = 0 \quad -\infty < \eta < h(\xi,t)$

Boundary conditions on $\eta = h(\xi,t)$

$$\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial \xi}\frac{\partial h}{\partial \xi} - \frac{\partial \phi}{\partial \eta} = 0$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left\{ \left(\frac{\partial \phi}{\partial \xi}\right)^2 + \left(\frac{\partial \phi}{\partial \eta}\right)^2 \right\} + (1+\mu)h = 0$$

Basic solution: (flat free surface)

$$h = 0, \quad \phi = 0.$$

Linearized problem

look for functions ϕ and h, $2\pi-$ periodic in ξ and t, h even,

$$\Delta\phi = 0 \qquad -\infty < \eta < 0$$

Boundary conditions on $\eta = 0$

$$\frac{\partial h}{\partial t} - \frac{\partial \phi}{\partial \eta} = 0, \quad \frac{\partial \phi}{\partial t} + (1+\mu)h = 0$$

$$h(\xi, t) = \sum h_p^{(q)} \cos p\xi \cos qt$$

Dispersion relation:

$$(1+\mu)p - q^2 = 0, \;\; p, q \in \mathbb{N}$$

For any rational value $(1+\mu) = r/s$
take $(q, p) = (kr, k^2 rs), \; k = 1, 2, 3, ...$

∞ dim Kernel: "*completely resonant system*"

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Assume μ near 0 (otherwise rescale (ξ, t))

$$Kernel = span\{\cos q^2\xi \cos qt; q \in \mathbb{N}\}.$$

Problem of infinitely many resonances

$$\mathcal{L}_0 X + \mu \mathcal{J} X + \mathcal{N}(X) = 0$$

$Ker(\mathcal{L}_0)$ is ∞ dim and $Range(\mathcal{L}_0)$ is ∞ codim

Formal expansions $X = \sum_{n\geq 1} \varepsilon^n X_n, \quad \mu = \sum_{n\geq 1} \varepsilon^n \mu_n$

$$\begin{aligned}
0 &= \mathcal{L}_0 X_1 \\
0 &= \mathcal{L}_0 X_2 + \mu_1 \mathcal{J} X_1 + \mathcal{N}_2(X_1, X_1) \\
&\ldots\ldots \\
0 &= \mathcal{L}_0 X_n + \mu_1 \mathcal{J} X_{n-1} + 2\mathcal{N}_2(X_1, X_{n-1}) + \\
&\quad + \mu_{n-1} \mathcal{J} X_1 + R(X_j, \mu_j, j \leq n-2)
\end{aligned}$$

Classical solution: impose X_1 !

$$h = \varepsilon \cos \xi \cos t + O(\varepsilon^2)$$

all compatibility cond. are satisfied at all orders, for $\mu = \varepsilon^2/4$ (Amick-Toland 1987)

Formal result

Bryant, Stiassnie (1994): other choices **numerically valid up to order** ε^{100}

$$h(\xi,t)=\mu^{1/2}\{2\cos\xi\cos t+\frac{1}{2}\cos 4\xi\cos 2t\}+O(\mu)$$

$$h(\xi,t)=\mu^{1/2}\{2\cos\xi\cos t+\frac{1}{2}\cos 4\xi\cos 2t+\frac{2}{9}\cos 9\xi\cos 3t\}+O(\mu)$$

Theorem (formal result) (G.I. 2002)

$$h=\varepsilon h_1+\varepsilon^2 h_2+\ldots.\ \mu=\varepsilon^2/4$$

$$h_1=\sum_{q\in I}\frac{\pm 1}{q^2}\cos q^2\xi\cos qt$$

I finite or infinite subset of $\mathbb{N}$

All orders ε^n may be computed, satisfying all compatibility conditions.

Remark: this is not a result on the convergence of the series

Existence result

$G.I., P.Plotnikov, J.Toland, 2003-2004$

Define ε for $\mu > 0$ by $\mu = \varepsilon^2/4$, and any finite subset of integers I, then 0 is a *Lebesgue point* of *a set* $\mathcal{M}_I$ of amplitudes ε, where the standing wave with the following principal part exists:

$$h = \varepsilon \sum_{q \in I} \frac{\pm 1}{q^2} \cos q^2 \xi \cos qt + O(\varepsilon^2), \quad \varepsilon \in \mathcal{M}_I$$

$$\frac{1}{r} meas\{\mathcal{M}_I \cap [0, r]\} \to 1 \text{ as } r \to 0.$$

(i.e. the set of ε such that the solution exists is asymptotically of full measure)

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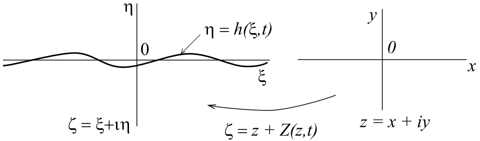
New formulation - Zakharov, Dyachenko, Toland, G.I.

$$\begin{aligned} h'_t + u h'_\xi - v &= 0, \ \eta = h(\xi, t) & (1) \\ \phi'_t + (1+\mu)h + \frac{1}{2}(u^2 + v^2) &= 0, \ \eta = h(\xi, t) & (2) \end{aligned}$$



$$\xi = x + Z_r(x,t), \quad h(\xi, t) = Z_i(x,t)$$

complex velocity potential $f(\zeta, t) = F(z,t), \quad f'_\zeta = u - iv$

$$\begin{aligned} Z'_i &= h'_\xi(1 + Z'_r), \ \dot{Z}_i = h'_t + h'_\xi \dot{Z}_r \\ F' &= (u - iv)(1 + Z') \\ f'_t &= \dot{F} - f'_\zeta \dot{Z} = \dot{F} - \frac{F'\dot{Z}}{1 + Z'} \end{aligned}$$

New formulation continued

$$\Im\{F' - (1+Z')\overline{\dot{Z}}\}|_{y=0} = 0 \tag{3}$$

$$0 = \Re\left\{\dot{F} - \frac{F'\dot{Z}}{1+Z'} - i(1+\mu)Z\right\}_{y=0} + \frac{1}{2}\left|\frac{F'}{1+Z'}\right|^2_{y=0} \tag{4}$$

Periodic Hilbert transform $\mathcal{H}$

$$\mathcal{H}\cos nx = -\sin|n|x \quad \mathcal{H}\sin nx = sgn(n)\cos nx, \quad n \neq 0.$$

F and Z holomorphic in $y < 0$, and 2π-periodic in x, hence

$$F|_{y=0} = \varphi + i\mathcal{H}\varphi, \quad Z|_{y=0} = \mathcal{H}w - iw$$

where

$$\varphi(x,t) = \phi(\xi, h(x,t), t)$$

equ (3) gives

$$w'\mathcal{H}\dot{w} - \dot{w}(1+\mathcal{H}w') + \mathcal{H}\varphi' = 0 \quad (K) \tag{5}$$

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New formulation - end

$$\text{Define } P = \Re\left(\dot{F} - (1+\mu)iZ - \frac{F'\dot{Z}}{1+Z'}\right) + \frac{1}{2}\left|\frac{F'}{1+Z'}\right|^2,$$
$$A = w'\dot{\varphi} - \dot{w}\varphi' - (1+\mu)ww',$$
$$B = (1+\mathcal{H}w')(\dot{\varphi} - (1+\mu)w) - \varphi'\mathcal{H}\dot{w}.$$

Using (5) deduce the following identities on $y = 0$

$$B = \Re(1+Z')P - \frac{1}{2}\Re\left(\frac{F'^2}{1+Z'}\right), \quad A = -\Im(Z'P) - \frac{1}{2}\Im\left(\frac{F'^2}{1+Z'}\right).$$

$\frac{F'^2}{1+Z'}$ is a periodic holomorphic function in $y < 0$, hence

$$\mathcal{H}A + B = \mathcal{H}(w'P) + (1+\mathcal{H}w')P,$$

$P|_{y=0} = 0$ by (4), we obtain $\mathcal{H}A + B = 0$, i.e.

$$\mathcal{H}[w'\dot{\varphi} - \dot{w}\varphi' - (1+\mu)ww'] + (1+\mathcal{H}w')(\dot{\varphi} - (1+\mu)w) - \varphi'\mathcal{H}\dot{w} = 0 \quad (D)$$

Second order non local PDE

(K), (D) give $\mathcal{H}L_{w'}\dot{w} = -\varphi'$ and $M_{w'}(\dot{\varphi} - (1+\mu)w) - J(\varphi', \dot{w})$

$$\begin{aligned} \text{where } L_{w'}f &= f + f\mathcal{H}w' - w'\mathcal{H}f \\ J(f,g) &= f\mathcal{H}g + \mathcal{H}(fg) \\ M_{w'}f &= f + J(f, w') \end{aligned}$$

Then, new equation $\mathcal{F}(w,\mu) = 0$

$$\mathcal{F}(w,\mu) \equiv \partial_t(L_{w'}\dot{w}) - (1+\mu)\mathcal{H}w' + \mathcal{H}\partial_x M_{w'}^{-1} J(\mathcal{H}L_{w'}\dot{w}, \dot{w})$$

linear part: $\mathcal{L}_\mu w \equiv \ddot{w} - (1+\mu)\mathcal{H}w'$

Difficulties: $\ker \mathcal{L}_0 = \{\cos q^2 x \cos qt; \quad q \in \mathbb{N}\}$

$\widetilde{\mathcal{L}_0}^{-1}$ not regularizing on $(\ker \mathcal{L}_0)^\perp$

w'' and $\dot{w}' \in$ nonlinear terms, not in the linear part

Useful identities

$$\mathcal{H}f(x) = -\frac{p.v.}{2\pi}\int_{-\pi}^{\pi}\frac{f(s)}{\tan\frac{1}{2}(x-s)}ds$$

$$\mathcal{H}(\cos nx) = -\sin|n|x \ , \ \mathcal{H}(\sin nx) = sgn(n)\cos nx$$

$$\mathcal{H}^2 f = -(\mathbb{I}-\pi_0)f \ , \ \pi_0 f = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$\begin{aligned}
\mathcal{H}(f\mathcal{H}g) + \mathcal{H}(g\mathcal{H}f) &= (\mathcal{H}f)(\mathcal{H}g) - fg + (\pi_0 f)(\pi_0 g) \\
L_{w'}^{-1}f &= (1+\mathcal{H}w')(\frac{f}{D}) + w'\mathcal{H}(\frac{f}{D}) \\
M_{w'}^{-1}f &= (1+\mathcal{H}w')(\frac{f}{D}) - \frac{1}{D}\mathcal{H}(w'f) \\
M_{w'}^{-1}J(f,g) &= \frac{1}{D}\mathcal{H}(fL_{w'}g) + f\mathcal{H}(\frac{1}{D}L_{w'}g) \\
D &= (1+\mathcal{H}w')^2 + w'^2
\end{aligned}$$

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Formal study of the bifurcation problem

$$\mathcal{F}(w,\mu) \stackrel{def}{=} \mathcal{L}_0 w - \mu \mathcal{H} w' + \mathcal{N}(w) = 0$$

$$\mathcal{L}_0 w \stackrel{def}{=} \ddot{w} - \mathcal{H} w'$$

$$\begin{aligned} \mathcal{N}_2(w,w) &= \partial_t(\dot{w}\mathcal{H}w' - w'\mathcal{H}\dot{w}) + \\ &+ \partial_x\{\mathcal{H}(\dot{w}^2) - 3\dot{w}\mathcal{H}\dot{w}\} \end{aligned}$$

$$w = X + Y, \quad X = P_0 w \in \ker \mathcal{L}_0 \quad Y \perp \ker \mathcal{L}_0$$

$$X = \sum_{q>0} A_q \cos q^2 x \cos qt$$

$$P_0 \mathcal{N}_2(X,X) \equiv 0$$

since for p, q, r non 0 integers

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \{\exp i[(q^2 \pm p^2 \pm r^2)x + (\pm q \pm p \pm r)t\} dxdt = 0$$

Suppression of quadratic terms

$$v = w + B(w, w') \overset{def}{=} F(w),\ F \text{ analytic} : H^s \to H^{s-1},\ s > 2$$

$$B(w_1, w_2) = \mathcal{H}(w_1 w_2) - w_1 \mathcal{H} w_2 - 2 w_2 \mathcal{H} w_1$$

property of B:

$$B(\ddot{w}, w') + 2B(\dot{w}, \dot{w}') - B(w', \ddot{w}) = \mathcal{N}_2(w, w)$$

$$\begin{aligned} G &= F^{-1} : H^p \to H^p, \quad p > 3 \\ \partial_v^k G(v) &: (H^{p-1})^k \to H^{p-k}, \quad p > k+3 \end{aligned}$$

New equation $\mathcal{E}(v, \mu) = 0,\ \mathcal{E} : H^p \times \mathbb{R} \to H^{p-3}$

$$\begin{aligned} \mathcal{E}(v, \mu) &= \mathcal{F}(w, \mu) + \partial_x B(w, \mathcal{F}(w, \mu)), \quad w = G(v) \\ &\equiv \mathcal{L}_0 v - \mu \mathcal{H} v' + \mathcal{E}_3(v, v, v) + O(||v||^4) \\ \mathcal{E}_3(v, v, v) &= \mathcal{N}_3(v, v, v) + \partial_x B(v, \mathcal{N}_2(v, v)) \end{aligned}$$

Bifurcation equation

$$X = \sum_{q>0} A_q \cos q^2 x \cos qt \in \ker \mathcal{L}_0$$

$$\{-\mu P_0 \mathcal{H} \partial_x X + P_0 \mathcal{E}_3(X, X, X)\}_q = \mu q^2 A_q - \frac{1}{4} q^6 A_q^3$$

all modes are uncoupled up to cubic order!

$$\begin{aligned} A_q &= 0, \text{ and } A_q = \pm\varepsilon/q^2,\ \mu = \varepsilon^2/4,\ q \in \mathbb{N} \\ &\text{simple roots of principal part} \end{aligned}$$

Theorem on formal solutions:

$$\begin{aligned} w &= \sum_{n>0} \varepsilon^n w^{(n)}, \quad I \subset \mathbb{N} \\ w^{(1)} &= \sum_{q \in I} (\pm 1/q^2) \cos q^2 x \cos qt \end{aligned}$$

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Existence problem

$$\mathcal{F}(w,\mu)=0,\ \mathcal{F}(w_\varepsilon^{(N)},\varepsilon^2/4)=\varepsilon^{N+1}Q_\varepsilon$$

$$w=w_\varepsilon^{(N)}+\varepsilon^N\underline{w},\ \ (N\geq 4)$$

$$\mathcal{G}(\underline{w},\varepsilon)\overset{def}{=}\varepsilon^{-N}\mathcal{F}(w_\varepsilon^{(N)}+\varepsilon^N\underline{w},\varepsilon^2/4)$$

$$\mathcal{G}(\underline{w},\varepsilon)=0,\ \ \mathcal{G}(0,\varepsilon)=\varepsilon Q_\varepsilon$$

Iteration process "Newton method": for $\varepsilon\in\mathcal{M}_k\subset(0,r_0)$

$$\begin{aligned}
\underline{w}_{k+1}(\varepsilon) &= \underline{w}_k(\varepsilon)+\mathcal{T}_{\rho_{k+1}(\varepsilon)}v_k(\varepsilon)\\
\Lambda(\underline{w}_k(\varepsilon),\varepsilon)v_k(\varepsilon) &= -\mathcal{G}(\underline{w}_k(\varepsilon),\varepsilon)\\
\Lambda(\underline{w},\varepsilon)-\partial_{\underline{w}}\mathcal{G}(\underline{w},\varepsilon) &= \Gamma(\mathcal{G})\ \ \text{cancels if } \mathcal{G}(\underline{w},\varepsilon)=0\\
\rho_k(\varepsilon) &= \varepsilon^{(3/2)^k},\\
\mathcal{T}_\rho &: \ \text{cuts Fourier series at } |n|\leq 1/\rho
\end{aligned}$$

Major problem: invert the approximate linear equation ($\underline{w}\neq 0$ in some ball of $H^s_{\natural\natural}$)

$$\Lambda(\underline{w},\varepsilon)u=f$$

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Theorem Inversion of $\Lambda(\underline{w},\varepsilon)$

For any finite subset $I \subset \mathbb{N}$ such that property

$$H(I) : \text{for any } p \in I, \sum_{q\in I,\ \ q>p} p^2 q^{-4}(q^2-p^2) \neq 1/2$$

holds, there exists $\varepsilon_0 > 0$, such that for $0 < \varepsilon < \varepsilon_0$ and $||\underline{w}||_{17} \leq M_{17}$ and if the following diophantine condition on the scalars $\beta^{(0)}(\underline{w},\varepsilon)$ and $\kappa^{(0)}(\underline{w},\varepsilon)$ (defined later) holds

$$|q^2-(1+\beta^{(0)})p-\kappa^{(0)}| \geq c/q^2, \quad p \neq q^2 \neq 0,$$

then for any $f \in H_{\natural\natural}^{s,ee}$, $s \geq 2$, $\langle f,1\rangle = 0$, the linear equation

$$\Lambda(\underline{w},\varepsilon)u = f, \quad \langle u,1\rangle = 0,$$

has a unique solution $u \in H_{\natural\natural}^{s-2,ee}$ which moreover satisfies

$$||u||_{s-2} \leq \frac{c_s(M_{17})}{\varepsilon^2}\{||\underline{w}||_{s+15}||f||_0 + ||f||_s\}.$$

Main Theorem

For any finite subset $I \subset \mathbb{N}$ such that property

$$H(I) : \text{for any } p \in I, \sum_{q \in I,\ q > p} p^2 q^{-4}(q^2 - p^2) \neq 1/2$$

holds, there exists a measurable set $\mathcal{M}_I \subset [0, \varepsilon_0]$ which is dense at 0 (0 is a Lebesgue point) such that for any $\varepsilon \in \mathcal{M}_I$, there exists a solution $w \in H_{\natural\natural}^{17,ee}$ Lipschitz continuous in ε, with 0 average, $\mu = \varepsilon^2/4$, and

$$w = w_\varepsilon^{(N)} + o(\varepsilon^N),\ N \geq 4$$

$$w_\varepsilon^{(N)} \sim \varepsilon \sum_{q \in I} \frac{(\pm 1)_q}{q^2} \cos q^2 x \cos qt.$$

Inverse of $\mathbf{\Lambda}(\underline{w}, \varepsilon)$ (1)

Theorem 1 Assume $\underline{w}$ smooth enough

$$D_{\underline{w}}\mathcal{G}(\underline{w}, \varepsilon)L_{w'}^{-1}v \equiv$$

$$\partial_t\{\dot{v} - \partial_x(av)\} + \mathcal{H}\partial_x\left(a\mathcal{H}\{\dot{v} - \partial_x(av)\}\right) - \mathcal{H}\partial_x\{bv\} + \Gamma(\mathcal{G})v$$

$$a = \mathcal{H}(\frac{1}{D}L_{w'}\dot{w}) + \frac{1}{D}\mathcal{H}L_{w'}\dot{w}, \quad b \quad \text{smooth} \quad \text{functions of } (\underline{w}, \varepsilon)$$

$$\Gamma(\mathcal{G}) = 0 \text{ for } \mathcal{G} = 0$$

Approximate linear operator to be inverted

$$\Lambda(\underline{w}, \varepsilon)u = D_{\underline{w}}\mathcal{G}(\underline{w}, \varepsilon)u - \Gamma(\mathcal{G})L_{w'}u$$

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Useful property: Assume ω smooth, then

$$\mathcal{H}(\omega f) = \omega \mathcal{H} f + \mathcal{S}_\omega f$$

$\mathcal{S}_\omega$ smoothing operator (in x)

$$(\mathcal{S}_\omega f)(x) = -\frac{p.v.}{2\pi} \int_{-\pi}^{\pi} \frac{(\omega(s) - \omega(x))}{\tan \frac{1}{2}(x-s)} f(s) ds$$

Observation: highest order in linear operator is degenerate

$$\{\partial_t - \partial_x(a \cdot)\}^2 v$$

Change of coord. killing $\partial^2_{xt}, \partial^2_{xx}$ derivatives

$$\begin{aligned} y &= x + d(x,t) = \mathcal{U}_t(x) \\ \partial_t d &= a(1 + \partial_x d) \\ d\big|_{t=0} &= 0 \end{aligned}$$

Other smoothing operator $\mathcal{S}$

$$\{\mathcal{H}(u \circ \mathcal{U}_t)\} \circ \mathcal{U}_t^{-1} = \mathcal{H} u + \mathcal{S} u$$

New form of the linear operator

$$p(y,t) = \frac{1}{(\partial_x \mathcal{U}_t(x))} \quad v(x,t) = \frac{\varphi(y,t)}{p(y,t)}$$

$$(\dot{v} - \partial_x(av)) \circ \mathcal{U}_t^{-1} = \frac{\partial_t \varphi}{p}$$

$$q = \frac{(b \circ \mathcal{U}_t^{-1})}{p}$$

Theorem 2: linear equation in coordinate $y = \mathcal{U}_t(x)$

$$\begin{aligned} \Lambda(\underline{w}, \varepsilon)u &= f \\ \varphi &= p(L_{w'}u) \circ \mathcal{U}_t^{-1} \end{aligned}$$

$$\partial_{tt}\varphi - \partial_y(q\mathcal{H}\varphi) + \mathcal{B}_\varepsilon(\varphi) = p(f \circ \mathcal{U}_t^{-1})$$

$\mathcal{B}_\varepsilon$ smoothing in y, 1st order in ∂_t

Structure of the linear operator (2)

$$\mathcal{A}\varphi = \mathcal{L}_0\varphi + \varepsilon\mathcal{A}_1\varphi, \quad \mathcal{L}_0 = \partial_{tt} - \mathcal{H}\partial_y$$

pseudo-inverse $\widetilde{\mathcal{L}_0}^{-1}$ not regularizing

$$(\widetilde{\mathcal{L}_0}^{-1} f)_p^{(q)} = \frac{f_p^{(q)}}{p - q^2}$$

$\mathcal{A}_1$ looses one derivative in y

$\widetilde{\mathcal{L}_0}^{-1}\mathcal{A}_1$ not bounded

Averaging necessary

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First averaging

$$\begin{aligned} (\xi,\tau) &= Q(y,t) = (y + d_0(y), t + e(y,t)) \\ \theta(\xi,\tau) &= \varphi \circ Q^{-1}(\xi,\tau) \end{aligned}$$

transforms the linear equation into

$$\partial_{\tau\tau}\theta - (1+\beta^{(0)})\mathcal{H}\partial_\xi\theta + (\gamma + \delta\mathcal{H})\partial_\tau\theta + \alpha\mathcal{H}\theta + \widehat{\mathcal{B}}_\varepsilon\theta = \hat{f}$$

$$\beta^{(0)}(\underline{w},\varepsilon) = \frac{\varepsilon^2}{4} + O(\varepsilon^3)$$

defined by

$$(1+\beta^{(0)})^{-1} = 2\pi \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} q(y,t)^{1/2} dt \right)^{-2} dy$$

This averaging (giving an easily invertible principal part of $\mathcal{A}$), introduces a *small divisor problem*, with a loss of regularity. It is then necessary to still "normalize" the highest orders in the rest of the linear operator, to make the *non diagonal part of the linear operator smoothing enough*

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2^{nd} averaging (descent method)

new change of variable

$$\begin{aligned}\psi &= (1+\alpha_0+\beta_0\mathcal{H})\theta + (\alpha_1+\beta_1\mathcal{H})\partial_\tau^{-1}\theta + (\alpha_2+\beta_2\mathcal{H})\partial_\tau^{-2}\theta \\ \theta(\xi,\tau) &= \varphi(y,t) = p(\mathcal{U}_t(x),t)(L_{w'}u)(x,t) \ \ (\xi,\tau) = Q(\mathcal{U}_t(x),t)\end{aligned}$$

Theorem 3: Assume $\underline{w} \in H_{\natural\natural}^{m,ee}$, $m \geq 14$, then consider the linear equation

$$\Lambda(\underline{w},\varepsilon)u = f$$

Then ψ satisfies

$$\partial_\tau^2\psi - (1+\beta^{(0)})\mathcal{H}\partial_\xi\psi - \kappa^{(0)}\psi + (b_3+b_4\mathcal{H})\partial_\tau^{-2}\psi + \widetilde{\mathcal{B}}\psi = g$$

$$\begin{aligned}\beta^{(0)}(\underline{w},\varepsilon) &= \varepsilon^2/4 + O(\varepsilon^3),\ \ \kappa^{(0)}(\underline{w},\varepsilon) = c_I\varepsilon^4 + O(\varepsilon^5) \text{ const} \\ b_3(\underline{w},\varepsilon),\quad b_4(\underline{w},\varepsilon) &= O(\varepsilon^2) \ \ C^{m-8} \text{ functions of } (\xi,\tau) \\ c_I &= (1/4)(card(I)-1/2)\sum_{q\in I} q^2\end{aligned}$$

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$\widetilde{\mathcal{B}} = O(\varepsilon)$ is smoothing enough and depends smoothly on $(\underline{w},\varepsilon)$

Structure of the linear operator (3)

$$\mathcal{A}_0\psi + \widetilde{\mathcal{A}}_\varepsilon\psi = g$$

$$\begin{aligned}\mathcal{A}_0\psi &\equiv \partial_\tau^2\psi - (1+\beta^{(0)})\mathcal{H}\partial_\xi\psi - \kappa^{(0)}\psi \\ \widetilde{\mathcal{A}}_\varepsilon\psi &\equiv (b_3 + b_4\mathcal{H})\partial_\tau^{-2}\psi + \widetilde{\mathcal{B}}\psi \text{ smoothing and } O(\varepsilon)\end{aligned}$$

$$\psi = \Theta + \varepsilon\Upsilon, \quad \Theta = P_0\psi \in \ker \mathcal{L}_0$$

$$\begin{aligned}\mathcal{M}_\varepsilon\Theta + \mathcal{E}_\varepsilon\Upsilon &= \frac{1}{\varepsilon^2}P_0 g \\ (\Lambda_\varepsilon^{(0)} + \varepsilon\Lambda_\varepsilon^{(1)})\Upsilon + (\mathcal{K}_{-1} + \varepsilon\mathcal{K}_\varepsilon)\Theta &= \frac{1}{\varepsilon}(\mathbb{I} - P_0)g\end{aligned}$$

$$\mathcal{M}_\varepsilon = \frac{1}{\varepsilon^2}P_0(\mathcal{A}_0 + \widetilde{\mathcal{A}}_\varepsilon)P_0 \quad \Lambda_\varepsilon^{(0)} = (\mathbb{I} - P_0)\mathcal{A}_0(\mathbb{I} - P_0)$$

$\mathcal{M}_\varepsilon^{-1}\mathcal{E}_\varepsilon$ bounded in $\ker \mathcal{L}_0$ if $H(I)$ holds

$\mathcal{K}_{-1}\mathcal{M}_\varepsilon^{-1}\mathcal{E}_\varepsilon$ finite-dim operator

small divisor pb for inverting $\Lambda_\varepsilon^{(0)}$

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Pb: invert the "diagonal part"

$$\Lambda_\varepsilon^{(0)}\Upsilon \equiv \partial_{\tau\tau}\Upsilon - (1+\beta^{(0)})\mathcal{H}\partial_\xi\Upsilon - \kappa^{(0)}\Upsilon = f$$

in Fourier modes $\{-q^2 + (1+\beta^{(0)})p - \kappa^{(0)}\}\Upsilon_p^{(q)} = f_p^{(q)}$

Diophantine cond.: assume $(\underline{w}, \varepsilon)$ s. t. $\beta^{(0)}$ and $\kappa^{(0)}$ satisfy

$$|q^2 - (1+\beta^{(0)})p + \kappa^{(0)}| \geq \frac{c}{q^2}, \text{ for all } (p,q) \in \mathbb{N}^2, \quad p \neq q^2$$

$\Longrightarrow$ the inverse $(\Lambda_\varepsilon^{(0)} - \mathcal{K}_{-1}\mathcal{M}_\varepsilon^{-1}\mathcal{E}_\varepsilon)^{-1}$ is controlled, loosing two derivatives in τ

$$\Upsilon \mapsto (\Lambda_\varepsilon^{(0)} - \mathcal{K}_{-1}\mathcal{M}_\varepsilon^{-1}\mathcal{E}_\varepsilon)^{-1}\{\varepsilon\Lambda_\varepsilon^{(1)}\Upsilon - \varepsilon\mathcal{K}_\varepsilon\mathcal{M}_\varepsilon^{-1}\mathcal{E}_\varepsilon\Upsilon\}$$

bounded by $O(\varepsilon)$

$\Longrightarrow \mathcal{A}_0 + \widetilde{\mathcal{A}}_\varepsilon$ can be inverted (loss of two derivatives)

Difficulty: the diophantine condition depends on the point $\underline{w}$ where we linearize (Newton method). We need to control this condition along the iteration process

Control of the diophantine condition

$$\beta^{(0)} = \varepsilon^2/4 + \varepsilon^3\beta_1(\varepsilon) + \varepsilon^N\beta(\underline{w}_k(\varepsilon),\varepsilon)$$
$$\kappa^{(0)} = c_I\varepsilon^4 + \varepsilon^5\kappa_1(\varepsilon) + \varepsilon^{N+1}\kappa(\underline{w}_k(\varepsilon),\varepsilon)$$
$$||\kappa||_{Lip} + ||\beta||_{Lip} \leq R,\ \ N \geq 4$$

$$\text{define}\ \ \mathcal{M}(\beta,\kappa) = \left\{\varepsilon \in [0,r_0]; |d| \geq \frac{1}{2q^2}, p \neq q^2\right\}$$
$$d = q^2 - (1+\beta^{(0)}(\varepsilon))p + \kappa^{(0)}(\varepsilon)$$
$$\Longrightarrow \frac{1}{r} meas\{[0,r]\cap\mathcal{M}(\beta,\kappa)\} \geq 1 - O(r^{1/2})$$

k^{th} step of the iteration defined for $\varepsilon \in \mathcal{M}_k = \cap_{j=0}^{j=k}\mathcal{M}(\beta_j,\kappa_j)$,
$\beta_k(\varepsilon), \kappa_k(\varepsilon)$ computed as functions of $\underline{w}_k(\varepsilon)$ satisfy

$$|\kappa_{k+1} - \kappa_k| + |\beta_{k+1} - \beta_k| \leq 2^{-k}$$

$$\frac{1}{r} meas\{[0,r]\cap_1^\infty\mathcal{M}(\beta_j,\kappa_j)\} \to 1\ \ \text{as}\ r \to 0$$

Smoothing operator $\mathcal{S}$

$$\{\mathcal{H}(v \circ \mathcal{U}_t)\}\mathcal{U}_t^{-1}(y) - \mathcal{H}(v)(y) = (\mathcal{S}v)(y,t) =$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{1}{\tan\frac{1}{2}(y-\xi)} - \frac{(1-\widetilde{d}'(\xi,t))}{\tan\frac{1}{2}[\mathcal{U}_t^{-1}(y)-\mathcal{U}_t^{-1}(\xi)]}\right\}v(\xi,t)d\xi$$

$$\mathcal{U}_t^{-1}(y) = y - \widetilde{d}(y,t)$$

$$(1-\widetilde{d}'(\xi,t))\tan\frac{1}{2}(y-\xi) - \tan\frac{1}{2}[\mathcal{U}_t^{-1}(y)-\mathcal{U}_t^{-1}(\xi)] =$$

$$\begin{aligned} = \quad & \frac{1}{2}[\widetilde{d}(y,t)-\widetilde{d}(\xi,t)][1+\tan^2\frac{1}{2}(y-\xi)] + \\ & -\widetilde{d}'(\xi,t)\tan\frac{1}{2}(y-\xi) + O(|y-\xi|^2)(||\widetilde{d}'||_{L^\infty})^2 \end{aligned}$$

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