3D travelling gravity waves

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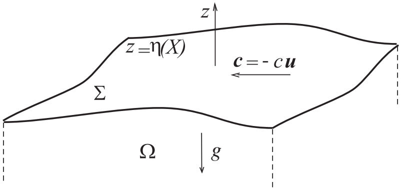
G.Iooss, P.Plotnikov. Memoirs of AMS. vol. 200, No 940, 2009 (128p.)
G.Iooss, P.Plotnikov. ARMA 200, 3 (2011), 789-880
G.Iooss, P.Plotnikov. C.R.Mécanique 2009

The Water-Wave problem



Moving frame

$$\Delta\varphi = 0 \text{ for } z < \eta(X),\ \nabla\varphi \to 0 \text{ as } z \to -\infty$$

Boundary conditions on $z = \eta(X)$

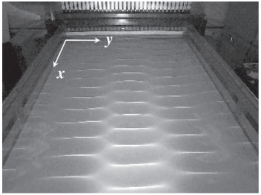
$$\nabla\eta \cdot (\mathbf{u} + \nabla_X\varphi) - \frac{\partial\varphi}{\partial z} = 0$$

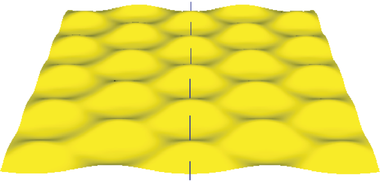
$$\mathbf{u} \cdot \nabla\varphi + \frac{(\nabla\varphi)^2}{2} + \mu\eta = 0$$

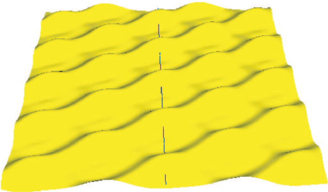
$$\mu = gL/c^2,$$

Basic solution: (flat free surface) $\varphi = 0, \quad \eta = 0.$

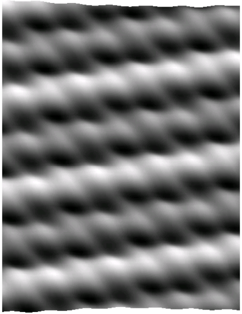
3D periodic travelling waves







τ_1=0.5, τ_2 =1, ε_1 =0.1, $\varepsilon_2/\varepsilon_1$=0.5



Bi-periodic functions

lattice Γ of periods: $\boldsymbol{\lambda}_j \in \mathbb{R}^2$, $j = 1, 2$

$$\Gamma = \{\boldsymbol{\lambda} = m_1 \boldsymbol{\lambda}_1 + m_2 \boldsymbol{\lambda}_2 : m_j \in \mathbb{Z}\}$$

lattice Γ' of wave vectors: $K_j \in \mathbb{R}^2$, $j = 1, 2$

$$\Gamma' = \{\mathbf{k} = n_1 K_1 + n_2 K_2 : n_j \in \mathbb{Z}, \quad \boldsymbol{\lambda}_j \cdot K_l = 2\pi \delta_{jl}\}$$

η bi-periodic in $X \in \mathbb{R}^2$

$$\eta(X) = \sum_{\mathbf{k} \in \Gamma'} u_{\mathbf{k}} e^{i\mathbf{k} \cdot X}$$

Basic formulation

Dirichlet-Neumann linear operator $\mathcal{G}_\eta$

$$\psi \mapsto \mathcal{G}_\eta \psi = \left(1 + (\nabla\eta)^2\right)^{1/2} \frac{d\varphi}{dn}|_{z=\eta(X)}$$

n normal exterior to Ω, and φ solution of the Dirichlet problem

$$\Delta\varphi = 0 \text{ in } z < \eta(X),\ \varphi|_{z=\eta(X)} = \psi,\ \nabla\varphi \to 0 \text{ as } z \to -\infty.$$

Basic formulation: $\mu > 0, \mathbf{u} \in \mathbb{S}_1$

$$\mathcal{F}(U, \mu, \mathbf{u}) = 0, \quad \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$$

$$U = (\psi, \eta), \quad \psi = \varphi(X, \eta(X))$$

$$\begin{aligned} \mathcal{F}_1(U, \mathbf{u}) &=: \mathcal{G}_\eta(\psi) - \mathbf{u} \cdot \nabla\eta, \\ \mathcal{F}_2(U, \mu, \mathbf{u}) &=: \mathbf{u} \cdot \nabla\psi + \mu\eta + \frac{(\nabla\psi)^2}{2} - \frac{1}{2(1 + (\nabla\eta)^2)}\{\nabla\eta \cdot (\nabla\psi + \mathbf{u})\}^2 \end{aligned}$$

$$U \in \mathbb{H}^m(\mathbb{R}^2/\Gamma) =: H_0^m(\mathbb{R}^2/\Gamma) \times H^m(\mathbb{R}^2/\Gamma)$$

References for 3D travelling water waves

R.Fuchs 1952, L.Sretenskii 1953 Formal expansion of 3D periodic travelling gravity waves

J.Reeder,M. Shinbrot 1981, W.Craig, D.Nicholls 2000 . existence of 3D periodic travelling gravity-capillary waves

using spatial dynamics

M.Groves, M.Haragus, A.Mielke 2001-2003. Thms on 3D travelling gravity-capillary waves localized in one direction

Existence and regularity results without surface tension

G.Iooss, P.Plotnikov. Memoirs of A.M.S. 2009, No 940 (128p.) (Existence of diamond waves)

T.Alazard, G.Métivier. Com. Part. Diff. Equ., 34 (10-12) 1632 - 1704, 2009. (Regularity of diamond waves)

G.Iooss, P.Plotnikov. A.R.M.A. 200, 3 (2011), 789-880 (Existence of non symmetric 3-D travelling waves)

Linearized system at the origin

$$\begin{aligned} \mathcal{G}^{(0)}\psi - \mathbf{u}\cdot\nabla\eta &= 0, \quad \mathcal{G}^{(0)} = (-\Delta)^{1/2} \\ \mathbf{u}\cdot\nabla\psi + \mu\eta &= 0 \end{aligned}$$

Dispersion relation: $\mu|\mathbf{k}| - (\mathbf{k}\cdot\mathbf{u})^2 = 0$

Assume the basis (K_1, K_2) of Γ' is solution of the dispersion equation

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_0 = (1,0), \\ K_1 &= (1,\tau_1), \quad \tau_1 = \tan\theta_1 \\ K_2 &= \lambda(1,-\tau_2), \quad \tau_2 = \tan\theta_2 \end{aligned}$$

Then τ_1, τ_2 and λ are linked:

$$\begin{aligned} \mu_c &= \cos\theta_1 = \frac{1}{|K_1|} = \frac{\lambda^2}{|K_2|} \\ \lambda &= \frac{\cos\theta_1}{\cos\theta_2} \end{aligned}$$

For $\lambda = 1$ the lattice Γ is a *diamond pattern*

Kernel of the linearized operator

linearized operator (acts on $U = (\psi, \eta)$)

$$\mathcal{L}_0 = \begin{pmatrix} \mathcal{G}^{(0)} & -\mathbf{u}_0 \cdot \nabla \\ \mathbf{u}_0 \cdot \nabla & \mu_c \end{pmatrix}$$

Non resonant assumption: (OK for (τ_1, τ_2) in a full measure set)
$\pm K_1$ and $\pm K_2$ are the only solutions in Γ' of the dispersion equation for $\mathbf{u} = \mathbf{u}_0$, $\mu = \mu_c$:

$$\mu_c |\mathbf{k}| - (\mathbf{k} \cdot \mathbf{u}_0)^2 = 0$$

4-dim kernel:

$$\begin{aligned} \zeta_{K_1} &= (i, \frac{1}{\mu_c}) e^{iK_1 \cdot X}, \quad \zeta_{-K_1} = \bar{\zeta}_{K_1} \\ \zeta_{K_2} &= (i, \frac{\lambda}{\mu_c}) e^{iK_2 \cdot X}, \quad \zeta_{-K_2} = \bar{\zeta}_{K_2} \end{aligned}$$

Formal Asymptotic expansion

$$\mathcal{L}_0 U + \widetilde{\mu}\mathcal{L}_1 U + \mathcal{L}_2(\omega, U) + \mathcal{N}_2(U,U) + \mathcal{N}_3(U,U,U) + .. = 0$$

$$U = (\psi, \eta), \quad \widetilde{\mu} = \mu - \mu_c, \quad \omega = \mathbf{u} - \mathbf{u}_0$$

Equivariance under symmetries:

$$\mathcal{T}_{\mathbf{v}}: \text{ shift } X \mapsto X + \mathbf{v},\ \mathbf{v} \in \mathbb{R}^2,\ \mathcal{S}_0: \ \mathcal{S}_0(\psi,\eta)(X) = (-\psi,\eta)(-X)$$

Formal Lyapunov-Schmidt method

$$U = W + V, \quad W \in \ker \mathcal{L}_0,\ W = A\zeta_{K_1} + B\zeta_{K_2} + \bar{A}\bar{\zeta}_{K_1} + \bar{B}\bar{\zeta}_{K_2}$$

needs to invert $\mathcal{L}_0$ on the complement of its 4-dimensional kernel

small divisors

$$\mu_c|\mathbf{k}| - (\mathbf{k}\cdot\mathbf{u}_0)^2, \quad \mathbf{k} \in \Gamma' \backslash \{\pm K_1, \pm K_2\}$$

in denominator of $\mathcal{L}_0^{-1} V_{\mathbf{k}} e^{i\mathbf{k}.X}$(no such problem with surface tension)

Asymptotic expansion of non-symmetric 3-dim waves

$$U = (\psi, \eta) = \sum_{p+q \geq 1} \varepsilon_1^p \varepsilon_2^q U_{pq}, \ \psi \text{ odd}, \ \eta \text{ even in } X$$

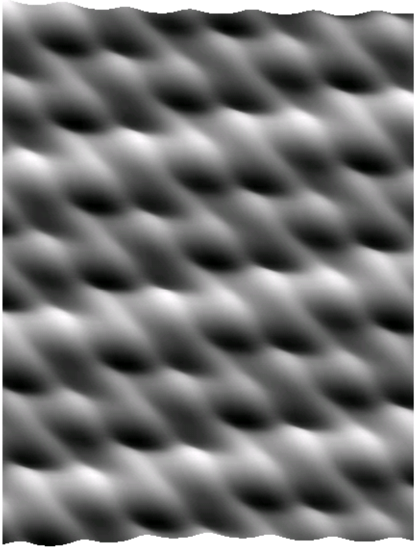
$$U_{10} = \frac{1}{2}(\zeta_{K_1} + \zeta_{-K_1}) = (-\sin K_1 \cdot X, \frac{1}{\mu_c} \cos K_1 \cdot X), \ U_{01} = \frac{1}{2}(\zeta_{K_2} + \zeta_{-K_2})$$

$\{\mathcal{T}_{\mathbf{v}} U; \mathbf{v} \in \mathbb{R}^2\}$ torus family of solutions

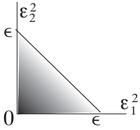
$$\begin{aligned} \mu - \mu_c &= \alpha_1 \varepsilon_1^2 + \alpha_2 \varepsilon_2^2 + O(\varepsilon_1^2 + \varepsilon_2^2)^2 \\ \mathbf{u} - \mathbf{u}_0 &= \boldsymbol{\omega} = (\omega_1, \omega_2), \ \omega_1 = -\frac{\omega_2^2}{2} + .. \\ \omega_2 &= \beta_1 \varepsilon_1^2 + \beta_2 \varepsilon_2^2 + O(\varepsilon_1^2 + \varepsilon_2^2)^2 \end{aligned}$$

α_j, β_j known analytic functions of τ_1 and τ_2

τ_1=0.5, τ_2 =1, ε_1 =0.1, ε_2 /ε_1=1



Existence Theorem



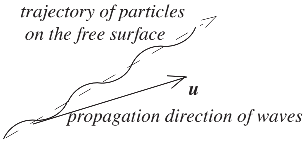
Theorem

Choose $l \geq 34$, *m even* ≥ 4, $0 < \delta < 1$. *There is a full measure set* $\mathcal{T} \subset \mathbb{R}^{+2}$ *such that for* $\boldsymbol{\tau} = (\tau_1, \tau_2) \in \mathcal{T}$, *there exists a subset* $\mathcal{E}(\boldsymbol{\tau})$ *of the quadrant* $\{(\varepsilon_1^2, \varepsilon_2^2) \in \mathbb{R}^{+2}\}$ *for which* 0 *is a Lebesgue point, i.e.*

$$\frac{2}{\epsilon^2}\mathrm{meas}(\mathcal{E}(\boldsymbol{\tau}) \cap \{\varepsilon_1^2 + \varepsilon_2^2 < \epsilon\}) \to 1 \text{ as } \epsilon \to 0.$$

Moreover, for $\delta < \varepsilon_1/\varepsilon_2 < \delta^{-1}$ *and* $(\varepsilon_1^2, \varepsilon_2^2) \in \mathcal{E}(\boldsymbol{\tau})$, *the nonlinear system has a unique solution* $(U, \mu, \mathbf{u}) \in \mathbb{H}^l_{(S)} \times \mathbb{R} \times \mathbb{S}_1$ *of the form*

$$U = U_{2m} + |\boldsymbol{\varepsilon}|^m \breve{U}(\boldsymbol{\varepsilon}),\ \mu = \mu_{2m} + |\boldsymbol{\varepsilon}|^m \breve{\mu}(\boldsymbol{\varepsilon}),\ \mathbf{u} = \mathbf{u}_{2m} + |\boldsymbol{\varepsilon}|^m \breve{\mathbf{u}}(\boldsymbol{\varepsilon})$$



Theorem

In the frame moving with the velocity of the waves, the horizontal projection of the asymptotic direction taken by fluid particles makes an angle

$$4(1+\tau_1^2)[-\tau_1\varepsilon_1^2+\lambda^4\tau_2\varepsilon_2^2]+h.o.t.$$

with the direction of propagation of the waves. There is a special value $\frac{\varepsilon_1^2}{\varepsilon_2^2}=\lambda^4\tau_2/\tau_1+h.o.t.$ *for which both directions are identical.*

Differential of $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)(U, \mu, \mathbf{u})$

$$\partial_U \mathcal{F}[\delta U] = \mathcal{L}_{\mu,\mathbf{u}}(U)[\delta\phi, \delta\eta] + \mathcal{R}(\mathcal{F}, U, \mu, \mathbf{u})[\delta U]$$

$$\mathcal{L}_{\mu,\mathbf{u}}(U) = \begin{pmatrix} \mathcal{G}_\eta & \mathcal{J}^* \\ \mathcal{J} & \mathfrak{a} \end{pmatrix}$$

$$\mathcal{J} = V \cdot \nabla(\cdot), \quad \mathcal{J}^* = -\nabla \cdot ((\cdot)V)$$

$$V = \mathbf{u} + \nabla\psi - \mathfrak{b}\nabla\eta \text{ (proj of particles velocity)}$$

$$\delta\phi = \delta\psi - \mathfrak{b}\delta\eta$$

$$\mathfrak{a} = \mu + V \cdot \nabla\mathfrak{b}, \quad \mathfrak{b} = \frac{1}{1 + \nabla\eta^2}\{\nabla\eta \cdot (\mathbf{u} + \nabla\psi)\} = V \cdot \nabla\eta$$

$$\partial_U \mathcal{F}[\delta U] = \mathcal{L}_{\mu,\mathbf{u}}(U)[\delta\phi, \delta\eta] + \mathcal{R}(\mathcal{F}, U, \mu, \mathbf{u})[\delta U]$$

Inverting $\mathcal{L}_{\mu,\mathbf{u}}(U)$ is equivalent to solve in $\delta\phi$

$$-\mathcal{J}^* \left(\frac{1}{\mathfrak{a}} \mathcal{J}(\delta\phi) \right) + \mathcal{G}_\eta(\delta\phi) = h \in H^s_{odd}(\mathbb{R}^2/\Gamma)$$

$$\mathcal{G}_\eta = \mathcal{G}_1 + \mathcal{G}_0 + \mathcal{G}_{-1}$$

$\mathcal{G}_j$ pseudodifferential operators of order j.

For $(\psi, \eta) = 0$, and $\mu = \mu_c$

$$\{\mu_c^{-1}(\partial_{x_1})^2 + (-\Delta)^{1/2}\}(\delta\phi) = h$$

Idea of Strategy

Find a diffeomorphism of the torus such that main orders of the diff equ. for $\delta\phi = \delta\psi - \mathfrak{b}\delta\eta$ have constant coefficients, leading to

$$\mathfrak{L} = \nu\mathcal{D}^2 + (-\Delta)^{1/2},\ \mathcal{D} =: \partial_{y_1} + \rho\partial_{y_2}$$

where this operator (diagonal on the Fourier basis) would have a controlled inverse.

The new linear operator to invert would look like

$$\mathfrak{L} + \text{perturbation of lower order}$$

It would then be possible to invert

$$(\mathfrak{L} + \text{perturbation})^{-1} = (\mathbb{I} + \mathfrak{L}^{-1}\text{perturbation})^{-1}\mathfrak{L}^{-1}$$

Unfortunately ($\mathfrak{L}^{-1}$perturbation) is unbounded

Two problems: i) find the good diffeomorphism;

ii) reduce the new operator to the sum of a diagonal operator with a controllable inverse, plus a nicely smoothing operator.

Strategy

1. The diffeomorphism of the torus $Y \in (\mathbb{R}/2\pi\mathbb{Z})^2 \mapsto X(Y)$ allowing to change into constant coefficients the main orders of the diff. equ. for $\delta\phi$, satisfies a new equation where two new constants ρ, ν occur (ρ is the rotation number of the velocity vector field on the free surface).
2. Extended system leads to a formal expansion solution ($\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$) provided that $\lambda \notin \mathbb{Q}$
Then choose parameters $(\rho - \lambda, \nu - \nu_c)$ instead of $(\varepsilon_1, \varepsilon_2)$.
3. Provided that ρ satisfies a diophantine condition, the differential of the extended system reduces to a differential equ. for $\delta\phi$ with constant main coefficients, with a linear operator of the form

$$\mathfrak{L} + \mathfrak{A}_0 \mathcal{D} + \mathfrak{B}_0 + \mathfrak{L}'_{-1}, \text{ with } \mathfrak{L} = \nu \mathcal{D}^2 + (-\Delta)^{1/2}, \ \mathcal{D} =: \partial_{y_1} + \rho \partial_{y_2}$$

4. Descent method (change of variable close to identity) leads to

$$\Pi(\mathfrak{L} + \mathfrak{V} + \mathfrak{F}_{-1})\Pi + (\mathbb{I} - \Pi)(\mathfrak{L} + \mathfrak{P}) \text{ (triangular form)}$$

with $\mathfrak{V}$ bounded and constant, $\mathfrak{P}$ bounded, $\mathfrak{F}_{-1}$ smoothing.
$\Pi(\mathfrak{L} + \mathfrak{V})^{-1}\Pi$ controllable for suitable (ρ, ν), loss of one derivative.

Diffeomorphism of the torus

$$Y \in (\mathbb{R}/2\pi\mathbb{Z})^2 \mapsto X(Y) = \mathbb{T}^{-1}Y + \mathcal{W}(Y), \quad \mathbb{T}X = (K_1 \cdot X, K_2 \cdot X)^t$$

$$\frac{dX}{dt} = V(X) \Rightarrow \frac{dY}{dt} = \sqrt{\nu}\tilde{f}(Y)\boldsymbol{\varrho}, \qquad \boldsymbol{\varrho} =: (1, \rho), \ \rho \text{ rotation number of } V$$

$$\text{define: } \mathcal{D}X =: \partial_{y_1}X + \rho\partial_{y_2}X, \ \mathbb{Q}(X) = \mathbb{I} + \nabla_X\eta \otimes \nabla_X\eta$$

Choose $\tilde{f}$ for having proportional the two main coefficients of the differential $-\mathcal{J}^*\left(\frac{1}{\mathfrak{a}}\mathcal{J}(\delta\phi)\right) + \mathcal{G}_\eta(\delta\phi)$

$$\tilde{f}^2 = \frac{\widetilde{\mathfrak{a}}}{|\det X'|}(\mathbb{Q}\mathcal{D}X \cdot \mathcal{D}X)^{1/2}, \ \widetilde{\mathfrak{a}}(Y) = \mathfrak{a}(X(Y))$$

$$\mathcal{F}_3(U, X, \mu, \mathbf{u}, \rho, \nu) =: \partial_{y_1}X + \rho\partial_{y_2}X - \left(\frac{|\det X'|}{\nu\widetilde{\mathfrak{a}}}\right)^{1/3} \frac{\widetilde{V}}{(\widetilde{\mathbb{Q}}\widetilde{V} \cdot \widetilde{V})^{1/6}} = 0$$

Extended system

$$\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)(U, X, \mu, \mathbf{u}, \rho, \nu) = 0, \quad U = (\psi, \eta)$$

$$\mathcal{G}_\eta(\psi) - \mathbf{u} \cdot \nabla\eta = 0$$

$$\mathbf{u} \cdot \nabla\psi + \mu\eta + \frac{(\nabla\psi)^2}{2} - \frac{1}{2(1 + (\nabla\eta)^2)}\{\nabla\eta \cdot (\nabla\psi + \mathbf{u})\}^2 = 0$$

$$\partial_{y_1} X + \rho\partial_{y_2} X - \left(\frac{|\det X'|}{\nu\widetilde{\mathfrak{a}}}\right)^{1/3} \frac{\widetilde{V}}{(\widetilde{\mathbb{Q}}\widetilde{V} \cdot \widetilde{V})^{1/6}} = 0$$

where

$\widetilde{\mathfrak{a}}(Y) = \mathfrak{a}(X(Y))$, $\widetilde{V}(Y) = V(X(Y))$, $\widetilde{\mathbb{Q}}(Y) = \mathbb{Q}(X(Y))$

$$V = \mathbf{u} + \nabla\psi - \mathfrak{b}\nabla\eta, \quad \mathfrak{a} = \mu + V \cdot \nabla\mathfrak{b}, \quad \mathbb{Q}(X) = \mathbb{I} + \nabla\eta \otimes \nabla\eta$$

$$\mathfrak{b} = \frac{1}{1 + \nabla\eta^2}\{\nabla\eta \cdot (\mathbf{u} + \nabla\psi)\} = V \cdot \nabla\eta$$

Sketch of the method

For $\boldsymbol{\tau} = (\tau_1, \tau_2)$ fixed such that $\lambda \notin \mathbb{Q}$ and non resonance is satisfied we obtain a formal asymptotic expansion for $\widetilde{U}(Y), X(Y), \mu, \mathbf{u}, \rho, \nu$ solution of Extended system at order $|(\varepsilon_1, \varepsilon_2)|^m$, chosen with ψ_m odd, η_m even in Y

$$U_m(Y, \boldsymbol{\varepsilon}), X_m(Y, \boldsymbol{\varepsilon}), \mu_m(\boldsymbol{\varepsilon}), \mathbf{u}_m(\boldsymbol{\varepsilon}), \rho_m(\boldsymbol{\varepsilon}), \nu_m(\boldsymbol{\varepsilon})$$

with

$$\rho = \lambda + \rho_1 \varepsilon_1^2 + \rho_2 \varepsilon_2^2 + ..., \ \nu = \nu_c + \nu_1 \varepsilon_1^2 + \nu_2 \varepsilon_2^2 + ...$$

new parameters: $(\rho, \nu) : \rho = \rho_m(\boldsymbol{\varepsilon}), \nu = \nu_m(\boldsymbol{\varepsilon})$ instead of $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$
Newton iteration scheme (starts with the approximate solution) for the Nash-Moser theorem
Perturbation: $\breve{U}, \breve{\mathcal{W}}, \breve{\mu}, \breve{\mathbf{u}}$ determined at each step of the iteration, after inverting the differential

$$\partial_{(U, X, \mu, \mathbf{u})} \mathcal{F}$$

taken at every iterated point

Equation for $\widetilde{\delta\phi} = u$

$$\nu \mathcal{D}^2 u + p\mathcal{D} u + \mathfrak{G}_1 u + \mathfrak{G}_0 u + \mathfrak{L}_{-1} u = f$$

symbol of $\mathfrak{G}_1 = (\mathbb{G}_1(Y)\widetilde{\mathbf{k}} \cdot \widetilde{\mathbf{k}})^{1/2}$, $\widetilde{\mathbf{k}} = (k_1 + \rho k_2, k_2)$, coef of $k_2^2 = 1$

$$\mathbb{A} = \frac{1}{4\pi^2} \int \mathbb{G}_1(Y) dY, \quad \mathbb{A}_c = \begin{pmatrix} \nu_c^2 & -\nu_c \cos\theta_1 \\ -\nu_c \cos\theta_1 & 1 \end{pmatrix}$$ positive definite

$L(\mathbf{k}) = -\nu(k_1 + \rho k_2)^2 + (\mathbb{A}\widetilde{\mathbf{k}} \cdot \widetilde{\mathbf{k}})^{1/2}$ symbol of $\mathfrak{L} = \nu\mathcal{D}^2 + (-\Delta)^{1/2}$

ρ and ν fixed functions of ε, and $\mathbb{A}$ function of ε and of the iteration point.

Theorem

$\forall \alpha > 0, \exists \mathcal{T} \subset (\mathbb{R}^+)^2$ *with full measure, and such that for any* $\boldsymbol{\tau} = (\tau_1, \tau_2) \in \mathcal{T}$ *, the Kernel of* $\mathfrak{L}_c$ *is 4-dim:* $\{e^{\pm i y_1}, e^{\pm i y_2}\}$,

$$\exists c > 0; \qquad |k_1 + \lambda k_2| \geq \frac{c}{|\mathbf{k}|^{1+\alpha}}, \ \forall \mathbf{k} \in \mathbb{Z}^2 \backslash \{\mathbf{0}\}$$

$$|L_c(\mathbf{k})| \geq \frac{c}{|\mathbf{k}|^{1/2+\alpha}}, \qquad \forall \mathbf{k} \in \mathbb{Z}^2 \backslash S_{triv}, \ S_{triv} = \{(0,0), (\pm 1, 0), (0, \pm 1)\}$$

Descent method

successive change of variables
starts with $\mathfrak{L} + \mathfrak{A}_0\mathcal{D} + \mathfrak{B}_0 + \mathfrak{L}'_{-1}$
First step leads to

$$\mathfrak{L} + \mathfrak{B}'_0 + \mathfrak{L}''_{-1}$$

Second step introduces a projection Π satisfying

$$\mathfrak{L}^{-1}(\mathbb{I} - \Pi) \text{ and } \mathcal{D}^{-1}\Pi \text{ regularizing operators}$$

and leads to the new operator (triangular form)

$$\Pi(\mathfrak{L} + \mathfrak{V} + \mathfrak{F}_{-1})\Pi + (\mathbb{I} - \Pi)(\mathfrak{L} + \mathfrak{P})$$

with $\mathfrak{V}$ bounded and constant, $\mathfrak{P}$ bounded, $\mathfrak{F}_{-1}$ smoothing.
$\Pi(\mathfrak{L} + \mathfrak{V})^{-1}\Pi$ controllable for suitable (ρ, ν), with the loss of one derivative.

Inverse estimate

Thm. Let $0 < \alpha < 1/2, 1/(2+\alpha) < \gamma < 1/2$, $\boldsymbol{\tau} \in \mathcal{T}$. there exists a subset $\mathcal{E}(\boldsymbol{\tau})$ of the quadrant $\{(\varepsilon_1^2, \varepsilon_2^2) \in \mathbb{R}^{+2}\}$ for which 0 is a Lebesgue point, i.e.

$$\frac{2}{\epsilon^2}\mathrm{meas}(\mathcal{E}(\boldsymbol{\tau}) \cap \{\varepsilon_1^2 + \varepsilon_2^2 < \epsilon\}) \to 1 \text{ as } \epsilon \to 0$$

and for $\varepsilon_1^2 + \varepsilon_2^2 < \epsilon, (\varepsilon_1^2, \varepsilon_2^2) \in \mathcal{E}(\boldsymbol{\tau})$ the following estimates hold (the second being uniform in all iteration points)

$$|k_1 + \rho_m(\boldsymbol{\varepsilon})k_2| \geq \frac{c}{|\mathbf{k}|^{1+\alpha}}, \quad |L(\mathbf{k}) + V(\mathbf{k})| \geq \frac{c\epsilon^\gamma}{|\mathbf{k}|^{\alpha+1/2}}, \ \forall \mathbf{k} \in \mathbb{Z}^2 \backslash S_{triv}$$

$\Rightarrow ||\Pi(\mathfrak{L} + \mathfrak{V})^{-1}\Pi||_{s \to s-1} \leq \frac{C}{\epsilon^\gamma}$ on $(\ker \mathfrak{L}_c)^\perp$

Inverse of the differential

For $\boldsymbol{\tau} = (\tau_1, \tau_2) \in \mathcal{T}$ and $\varepsilon_1^2 + \varepsilon_2^2 < \epsilon, (\varepsilon_1^2, \varepsilon_2^2) \in \mathcal{E}(\boldsymbol{\tau})$, there is an $m > 0$ such that for any $s \geq m$, and for all iteration points staying in a certain ball

$$||\{\Pi(\mathfrak{L} + \mathfrak{V} + \mathfrak{F}_{-1})\Pi + (\mathbb{I} - \Pi)(\mathfrak{L} + \mathfrak{P})\}^{-1}||_{\mathcal{L}(H^s, H^{s-1}) \cap (\ker \mathfrak{L}_c)^{\perp}} \leq \frac{c(s)}{\epsilon}$$

Hence the Nash-Moser theorem applies

Existence Theorem

Theorem

Choose $l \geq 34$, *m even* ≥ 4, $0 < \delta < 1$. *There is a full measure set* $\mathcal{T} \subset \mathbb{R}^{+2}$ *such that for* $\boldsymbol{\tau} \in \mathcal{T}$, *there exists a subset* $\mathcal{E}(\boldsymbol{\tau})$ *of the quadrant* $\{(\varepsilon_1^2, \varepsilon_2^2) \in \mathbb{R}^{+2}\}$ *for which* 0 *is a Lebesgue point, i.e.*

$$\frac{2}{\epsilon^2}\mathrm{meas}(\mathcal{E}(\boldsymbol{\tau}) \cap \{\varepsilon_1^2 + \varepsilon_2^2 < \epsilon\}) \to 1 \text{ as } \epsilon \to 0.$$

Moreover, for $\delta < \varepsilon_1/\varepsilon_2 < \delta^{-1}$ *and* $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in \mathcal{E}(\boldsymbol{\tau})$, *the nonlinear system has a unique solution* $(U, \mu, \mathbf{u}) \in \mathbb{H}^l_{(S)} \times \mathbb{R} \times \mathbb{S}_1$ *of the form*

$$U = U_{2m} + |\boldsymbol{\varepsilon}|^m \breve{U}(\boldsymbol{\varepsilon}),\ \mu = \mu_{2m} + |\boldsymbol{\varepsilon}|^m \breve{\mu}(\boldsymbol{\varepsilon}),\ \mathbf{u} = \mathbf{u}_{2m} + |\boldsymbol{\varepsilon}|^m \breve{\mathbf{u}}(\boldsymbol{\varepsilon})$$