Bifurcation Theory

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Introduction

Consider the following equation

$$F(X, \mu) = 0, \tag{1}$$

where X is the variable, μ is a parameter, and X, μ, F belong to appropriate (finite or infinite dimensional) spaces. The problem of *bifurcation theory* is to *describe the singularities of the set of solutions*

$$S_\mu = \{X; (X, \mu) \text{ satisfies } F(X, \mu) = 0\}.$$

The word "bifurcation" was introduced by H. Poincaré (1885) in his study of equilibria of rotating liquid masses, but "bifurcation phenomena" have already been understood by C. Jacobi (1834).

The simplest example is the study of the real roots x of a quadratic polynomial

$$x^2 + bx + c = 0, \tag{2}$$

where μ is represented by the pair of parameters $(b, c) \in \mathbb{R}^2$. As it is well known, real roots are determined by the sign of

$$\Delta \stackrel{def}{=} b^2 - 4c = 0.$$

For $\Delta < 0$, there is no real solution of (2), while there are two solutions $x_\pm$ in the region $\Delta > 0$, which merge when the distance between the point (b, c) and the parabola $\Delta = 0$ tends towards 0. It is then clear that a *singularity occurs in the structure of the set of solutions of (2) at the crossing of the parabola* $\Delta = 0$, or in other words, a *bifurcation occurs in the parameter space* (b, c) *on the parabola* $\Delta = 0$. A point $(\mu_0, x_0) \in \mathbb{R}^3$ is then called a *bifurcation point* if $\mu_0 = (b, c)$ satisfies $\Delta = 0$, and $x_0 = -b/2$.

In the theory of differential equations, $F(X, \mu)$ often represents a vector field. This study is then concerned with the existence of equilibrium solutions to the differential equation

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X, \mu), \tag{3}$$

and is therefore referred to as *static bifurcation theory.* In addition, *dynamic bifurcation theory* is concerned here with "changes" in the dynamic properties of the solutions of the differential equation as μ varies. A widely used way to characterize these "changes" is to say that the vector field $F(\cdot, \mu_0)$ is *structurally stable* if the sets of orbits of the differential equation are homeomorphic for μ close to μ_0, with homeomorphisms which preserve the orientation of the orbits in time t. Then a bifurcation occurs at $\mu = \mu_0$ if $F(\cdot, \mu_0)$ is not structurally stable. It turns out that there is a close link between the stability properties of equilibrium solutions of the differential equation and the type of the bifurcation in static theory.

The tools developed in bifurcation theory are extensively used to solve concrete problems arising in physics and natural sciences. These problems may be modelled by ordinary or partial differential equations, integral equations, but also delay equations or iteration maps, and in all these cases the presence of parameters naturally leads to bifurcation phenomena. They can be regarded as problems of the form (1) or (3), in suitable function spaces, and bifurcation theory allows to detect solutions and to describe their qualitative properties. During the last decades, a class of problems in which the use of bifurcation theory led to significant progress is concerned with nonlinear waves in partial differential equations, including hydrodynamic problems, nonlinear water waves, elasticity, but also pattern formation, front propagation, or spiral waves in reaction-diffusion type systems.

Examples in one and two dimensions

The most complete results in bifurcation theory are available in one and two dimensions. The study in one dimension is concerned with scalar equations

$$f(x,\mu) = 0, \tag{4}$$

where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$, and the function f is supposed to be regular enough with respect to (x,μ). When $f(x_0,\mu_0) = 0$ and the derivative of f with respect to x satisfies $\partial_x f(x_0,\mu_0) \neq 0$ the implicit function theorem gives a unique branch of solutions $x(\mu)$ for μ close to μ_0, and shows the absence of bifurcation points near (μ_0, x_0). Bifurcation theory intervenes when

$$\partial_x f(x_0,\mu_0) = 0, \tag{5}$$

and one cannot apply the implicit function theorem for solving with respect to x near x_0. A complete description of the set of solutions near (x_0,μ_0) can be obtained by looking at the partial derivatives of f with respect to x and μ.

For example, if

$$\partial_\mu f(x_0,\mu_0) \neq 0,$$

it is possible to solve with respect to μ and obtain a regular solution $\mu(x)$ such that $\mu(x_0) = \mu_0$ and $f(x,\mu(x)) \equiv 0$. In addition, if the second order derivative

$$\partial_x^2 f(x_0,\mu_0) \neq 0$$

the picture of the solution set in the plane (μ, x), also called *bifurcation diagram*, shows a *turning point* with a fold opened to the left or to the right depending upon the sign of the product $\partial_\mu f(x_0,\mu_0) \cdot \partial_x^2 f(x_0,\mu_0)$; see Figure 1. Notice

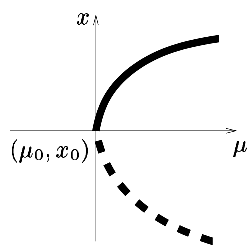

Figure 1: Turning point bifurcation in the case $\partial_\mu f(x_0,\mu_0) > 0$ and $\partial_x^2 f(x_0,\mu_0) < 0$. The solid (dashed) line indicates the branch of stable (unstable) solutions.

that here the bifurcation point $(\mu_0, x_0) \in \mathbb{R}^2$ corresponds to the appearance of a pair of solutions of (4) "from nowhere". This is the simplest example of a *one-sided bifurcation* in which the bifurcating solutions exist for either $\mu > \mu_0$ or $\mu < \mu_0$.

A particularly interesting situation arises when the equation possesses a symmetry. For example, assume that in (4) the function f is odd with respect to x. This implies that we always have the solution $x = 0$, for any value of the parameter μ. Assume now that f satisfies

$$\partial_x f(0, \mu_0) = 0, \tag{6}$$

and that

$$\partial^2_{x\mu} f(0, \mu_0) \neq 0, \quad \partial^3_x f(0, \mu_0) \neq 0. \tag{7}$$

Then the point $(\mu_0, 0)$ is a *pitchfork bifurcation point*, this denomination being related with the bifurcation diagram in the plane (μ, x); see Figure 2. Notice that here, the bifurcation point

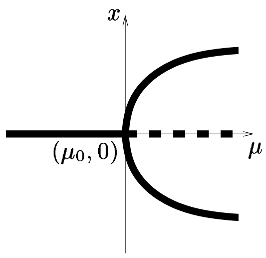

Figure 2: Supercritical pitchfork bifurcation in the case $\partial^2_{x\mu} f(0, \mu_0) > 0$ and $\partial^3_x f(0, \mu_0) < 0$. The solid (dashed) lines indicate the branch of stable (unstable) solutions.

$(\mu_0, x_0) \in \mathbb{R}^2$ corresponds to the bifurcation from the origin of a pair of solutions exchanged by the symmetry $x \to -x$, in addition to the persistent "trivial" solution $x = 0$ which is invariant under the above symmetry. Such a bifurcation is also referred to as a *symmetry-breaking bifurcation*. Similar bifurcation diagrams are found when the equation (4) has a "known" branch of solutions $x(\mu)$ for μ close to μ_0. This situation arises often in applications where usually this branch consists of trivial solutions $x(\mu) = 0$. Then at a bifurcation point (μ_0, x_0) a second branch of solutions appears forming either a one-sided bifurcation, or a two-sided bifurcation; see Figure 3.

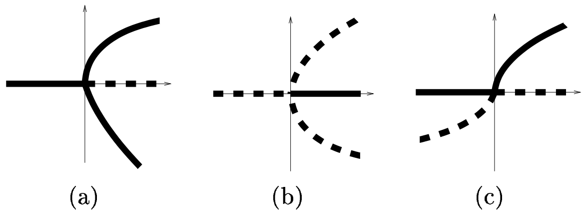

Figure 3: Typical bifurcation diagrams in the case of a branch of trivial solutions. One-sided bifurcations: (a) supercritical, (b) subcritical; two-sided bifurcation: (c) transcritical. The solid (dashed) lines indicate the branch of stable (unstable) solutions.

We can now view f as a vector field in the ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \mu), \tag{8}$$

and the study above corresponds to looking for equilibrium solutions of (8). The stability of such a solution is determined by the sign of the derivative $\partial_x f(x, \mu)$ of f at this equilibrium, and it is closely related to the type of the static bifurcation.

In the case of a *turning point bifurcation*, when $\partial^2_x f(x_0, \mu_0) \neq 0$, the sign of $\partial_x f(x, \mu)$ is different for the two bifurcating solutions. This means that one solution is attracting (i.e. stable), the other one being repelling (i.e. unstable); see Figure 1. In the case of a *pitchfork bifurcation* as above, the stability of the trivial solution $x = 0$ changes when μ crosses μ_0, and the stability of both bifurcating nonzero solutions is the opposite from the stability of the origin on the side of the bifurcation. The bifurcation is called *supercritical* if the bifurcating solutions lie on the side of the bifurcation point where the basic solution $x = 0$ is unstable and *subcritical* otherwise; see Figure 2. The situation is the same in the case of one-sided bifurcations for an equation which has a "known" branch of solutions. In the case of a two-sided bifurcation there is an *exchange of stability* at the bifurcation point (μ_0, x_0), solutions on the two branches having opposite stability for $\mu > \mu_0$ and $\mu < \mu_0$, which changes at (μ_0, x_0). Such a bifurcation is also referred to as *transcritical*; see Figure 3.

The analysis of bifurcations in two dimensions leads to more complicated scenarios. Consider the differential equation (8) in which now $x \in \mathbb{R}^2$ and $f(x, \mu) \in \mathbb{R}^2$, and assume that $f(x_0, \mu_0) = 0$. The behavior of solutions near (x_0, μ_0) is determined by the differential $D_x f(x_0, \mu_0) =: L$ of f with respect to x, which can be identified with a 2×2 matrix. For steady solutions, the implicit function theorem insures the existence of a unique branch of solutions $x(\mu)$ provided L is invertible, or in other words, zero does not belong to the spectrum of L. Consequently, the study of bifurcations of steady solutions is concerned with the case when zero belongs to the spectrum of L, and can be performed following the strategy described for one dimension, provided that the zero eigenvalue of L is simple. For example, assuming that the second eigenvalue is negative, leads in general to a *saddle-node bifurcation*, where an additional dimension is added to the previous picture of a turning point bifurcation, in which one of the two bifurcating steady solutions is a stable node, while the other one is a saddle. If, in addition, there is a symmetry S commuting with f, i.e., such that $f(Sx, \mu) = Sf(x, \mu)$, and if, for example, x_0 is invariant under S, $Sx_0 = x_0$, and the eigenvector ζ_0 associated to the zero eigenvalue of L is antisymmetric, $L\zeta_0 = -\zeta_0$, then there is again a *pitchfork bifurcation.* The equation possesses a branch of symmetric steady solutions the stability of which changes when crossing the value μ_0 of the parameter, node on one side and saddle on the other side, and a pair of solutions is created in a one-sided bifurcation which are exchanged by the symmetry S and have stability opposite to the one of the symmetric solution, just as in the one-dimensional pitchfork bifurcation above.

A new type of bifurcation which arises for vector fields in two dimensions is the so-called *Hopf bifurcation.* This bifurcation has been first understood by Poincaré, and then proved in two dimensions by Andronov (1937) using a Poincaré map, and later in n dimensions by Hopf (1948) by means of a Liapunov-Schmidt type method. For the differential equation, the absence of the zero eigenvalue in the spectrum of L is not enough to insure that the vector field $f(\cdot, \mu_0)$ is structurally stable in a neighborhood of x_0. This only holds when the spectrum of L does not contain purely imaginary eigenvalues, as asserted by the Hartman-Grobman theorem. We are then left with the case when L has a pair of purely imaginary eigenvalues $\pm i\omega$, $\omega \in \mathbb{R}^*$. Static bifurcation theory gives that the system has a unique branch of equilibria $(x(\mu), \mu)$ for μ close to μ_0, and typically their stability changes as μ crosses μ_0. For the differential equation a *Hopf bifurcation* occurs in which a branch of *periodic orbits* bifurcates on one side of μ_0, and their stability is opposite to that of the steady solution on this side; see Figure 4. A convenient way for the study of this bifurcation is through "normal form theory" which is briefly described below.

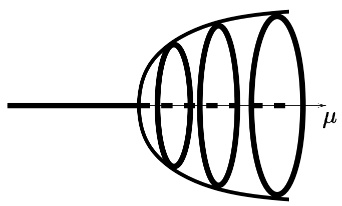

Figure 4: Supercritical Hopf bifurcation.

Local bifurcation theory

There are two aspects of bifurcation theory, *local* and *global* theory. As this designation suggests, local theory is concerned with (local) properties of the set of solutions in a neighborhood of a "known" solution, while global theory investigates solutions in the entire space.

An important class of tools in *local* bifurcation theory consists of *reduction methods*, among which the *Liapunov-Schmidt reduction* and the *center manifold reduction* are often used to investigate static and dynamic bifurcations, respectively. The basic idea to replace the bifurcation problem by an equivalent problem in lower dimensions, for example, a one- or a two-dimensional problem as the ones above.

Consider again the equation (1) in which $F : \mathcal{X}\times\mathcal{M} \to \mathcal{Y}$ is sufficiently regular, and $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{M}$ are Banach spaces. Assume, without loss of generality, that $F(0,0) = 0$, or, in other words, that one solution is known. The equation can be then written as

$$LX + G(X, \mu) = 0,$$

in which $L = D_X F(0,0)$ represents the differential of F with respect to X at $(0,0)$, and is assumed to have a closed range. The implicit function theorem shows absence of bifurcation if L has a bounded inverse, so that bifurcations are related to the existence of a non trivial kernel of L. The *Liapunov-Schmidt reduction* then goes as follows.

Let $N(L)$ and $R(L)$ denote the kernel and the range of L, respectively, and consider continuous projections $P : \mathcal{X} \to N(L)$ and $Q : \mathcal{Y} \to R(L)$. Then there exists a bounded linear operator $B : R(L) \to (\mathrm{id} - P)\mathcal{X}$, the right inverse of L, satisfying $LB = \mathrm{id}$ on $R(L)$ and $BL = \mathrm{id} - P$ on $\mathcal{X}$. For $X \in \mathcal{X}$ one may write

$$X = X_0 + X_1, \quad X_0 = PX,\ X_1 = (\mathrm{id} - P)X,$$

and then by projecting with $\mathrm{id} - Q$ and Q the equation becomes

$$(\mathrm{id} - Q)G(X_0 + X_1, \mu) = 0,$$
$$X_1 + BQG(X_0 + X_1, \mu) = 0.$$

The implicit function theorem allows to solve the second equation for $X_1 = \psi(X_0, \mu)$ in a neighborhood of the origin. Substitution into the first equation leads to the equation in $(\mathrm{id} - Q)\mathcal{Y}$ for X_0 in $P\mathcal{X}$,

$$(\mathrm{id} - Q)G(X_0 + \psi(X_0, \mu), \mu) = 0,$$

also called *bifurcation equation.* This equation completely describes the set of solutions to (1) in a neighborhood of $(0,0)$, and this problem is then posed in a space of dimension much smaller than the dimension of $\mathcal{X}$.

The basic principle of the Liapunov-Schmidt method has been discovered and used independently by different authors. E. Schmidt (1908) used this method for integral equations, while Liapunov used it to study the stability of the zero solution of nonlinear partial differential equations when the linear part has zero eigenvalues (1947), and later in 1960 for the bifurcation problem studied by Poincaré (1885). In working in a Banach space of t-periodic functions, the Liapunov-Schmidt method may be used to solve the Hopf bifurcation problem, as did Hopf himself in 1948.

The analogue of this reduction procedure for the differential equation (3) is the *center manifold reduction*. Assuming that $F(0,0) = 0$, we obtain the differential equation

$$\frac{\mathrm{d}X}{\mathrm{d}t} = LX + G(X,\mu).$$

Since dynamic bifurcations are related to the existence of purely imaginary spectral values of L, the kernel of L alone is not enough to describe this situation. One has to consider the spectral space $\mathcal{Y}_c$ of L associated to the purely imaginary spectrum of L. A spectral gap is needed between this part of the spectrum and the rest (always true in finite dimensions), so that the spectral projection P onto $\mathcal{Y}_c$ is well-defined. One writes

$$X = X_c + X_h, \quad X_c = PX, \ X_h = (\mathrm{id} - P)X,$$

and obtains the decomposed system

$$\frac{\mathrm{d}X_c}{\mathrm{d}t} = LX_c + PG(X_c + X_h, \mu)$$
$$\frac{\mathrm{d}X_h}{\mathrm{d}t} = LX_h + (\mathrm{id} - P)G(X_c + X_h, \mu).$$

The reduction procedure works provided the nonhomogeneous linear equation

$$\frac{\mathrm{d}X_h}{\mathrm{d}t} = LX_h + f(t)$$

possesses a unique solution in suitably chosen function spaces with weak exponential growth, such that one can then solve the second equation for $X_h = \Psi(X_c)$ in a neighborhood of the origin in these function spaces. This property is always true in finite dimensions, but it has to be checked in infinite dimensions. Different results showing the solvability of this equation are available in both Banach and Hilbert spaces, relying upon additional conditions on the spectrum of L, decaying properties of the resolvent of L on the imaginary axis, and regularity properties of the nonlinearity G. The map Ψ is then used to construct a map $\psi : P\mathcal{X} \times \mathcal{M} \to (\mathrm{id} - P)\mathcal{X}$, defined in a neighborhood of the origin, which parameterizes a *local center manifold* invariant under the flow of the equation. The flow on this center manifold is governed by the *reduced equation* in $\mathcal{Y}_c$

$$\frac{\mathrm{d}X_c}{\mathrm{d}t} = LX_c + PG(X_c + \psi(X_c,\mu), \mu),$$

which completely describes the bifurcation problem.

The first proofs of this result have been given in finite dimensions by Pliss (1964) and Kelley (1967). Center manifolds in infinite dimensions have been studied in different settings determined by assumptions on the linear part L

and the nonlinear part G. One typical assumption in infinite dimensions is that the spectrum of L contains only a finite number of purely imaginary eigenvalues, so that the reduced equation above is a differential equation in a finite dimensional space.

These reduction methods work for a large class of problems and the advantage of such an approach is that one is left with a bifurcation problem in a lower dimensional space. The methods involved in solving this reduced bifurcation problem can be very different from one problem to another, and often make use of some additional structure in the problem, such as a gradient-like structure, Hamiltonian structure, or the presence of symmetries, which are preserved by the reduction procedure.

A powerful tool for the analysis of these reduced differential equations is provided by the *normal form theory* which goes back to works of Poincaré (1885) and Birkhoff (1927). The idea is to use coordinate transformations to make the expression of the vector field as simple as possible. The transformed vector field is called *normal form.* There is an extensive literature on normal forms for vector fields in many different contexts, in both finite and infinite dimensional cases. Typically the classes of normal forms are characterized in terms of the linear part of the differential equation.

For differential equations of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Lx + g(x,\mu), \tag{9}$$

in which L is a matrix and g a sufficiently regular map such that $g(0,0) = 0$, $D_x g(0,0) = 0$, as encountered in bifurcation theory, one possible characterization of normal forms makes use of the adjoint matrix L^*. Fixing any order $k \geq 2$, there exist polynomials Φ and N of degree k with $\Phi(0,0) = N(0,0) = 0$, $D_x\Phi(0,0) = D_x N(0,0) = 0$, such that by the change of variables

$$x = y + \Phi(y,\mu),$$

the equation (9) is transformed into the *normal form*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = Ly + N(y,\mu) + \mathrm{o}((\|y\| + \|\mu\|)^k), \tag{10}$$

in which the polynomial N is characterized through

$$N(\mathrm{e}^{tL^*}y,\mu) = \mathrm{e}^{tL^*}N(y,\mu),$$

for all y, μ, and t, or equivalently,

$$D_y N(y,\mu)L^*y = L^*N(y,\mu),$$

for all y and μ. This characterization allows to determine the classes of possible normal forms for a given matrix L, and also provides an efficient way to compute the normal form for a given vector field g. As for the reduction methods, normal form transformations can be make to preserve the additional structure of the problem, such as Hamiltonian structure or symmetries.

As an example, consider a differential equation of the form (9) with $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$, which supports a Hopf bifurcation so that L has simple eigenvalues $\pm\mathrm{i}\omega$, $\omega > 0$, and no other eigenvalues with zero real part. The center manifold reduction provides a two-dimensional reduced system with linear part having the simple eigenvalues

$\pm i\omega$, for which it is convenient to write the normal form in complex variables

$$\frac{dA}{dt} = i\omega A + AQ(|A|^2, \mu) + o(|A|^{2k+2}),$$

for $A(t) \in \mathbb{C}$, where Q is a complex polynomial of degree k in $|A|^2$ and μ with $Q(0,0) = 0$, or equivalently, in polar coordinates $A = re^{i\phi}$,

$$\frac{dr}{dt} = rQ_r(r^2, \mu) + o(r^{2k+2}),$$
$$\frac{d\phi}{dt} = \omega + Q_\phi(r^2, \mu) + o(r^{2k+2}),$$

Q_r and Q_ϕ being the real and imaginary part of Q, respectively. The radial equation for r truncated at order $2k + 1$ decouples and admits a pitchfork bifurcation. The bifurcating steady solutions of this equation then lead to periodic solutions for the truncated system, first, which are then shown to persist for the full equation by a standard perturbation analysis.

A situation which occurs in a large class of problems is when the problem possess a reversibility symmetry, which often comes from some reflection invariance in the physical space, i.e., when the vector field $F(\cdot, \mu)$ anti-commutes with a symmetry operator S. One of the simplest example is the case of a differential equation (9) when the matrix L has a double eigenvalue in 0, no other eigenvalues with zero real part, and a one dimensional kernel which is invariant by S. In this case, the center manifold reduction provides a two-dimensional reduced system, which can be put in the normal form

$$\frac{da}{dt} = b$$
$$\frac{db}{dt} = \mu - a^2 + o((|a| + |b|)^3)$$

which anti-commutes with the symmetry $(a, b) \mapsto (a, -b)$. The above system undergoes for $\mu > 0$ a phase portrait as in Figure 5. There are two equilibria, one is a saddle, the other one being a center, and a family of periodic orbits with the zero amplitude limit at the center-equilibrium, and the infinite period limit a homoclinic orbit, originating at the saddle point.

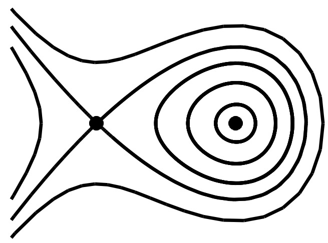

Figure 5: Phase portrait of the reduced system in a reversible Takens-Bogdanov bifurcation

Global bifurcation theory

Most of the existing results in global bifurcation theory concern the static problem (1). The analysis of *global sets* of solutions often relies upon topological methods, degree theory, but also variational methods, or analytic function theory. Significant progress in understanding global branches of solutions has been made in the 1970s, in particular, for nonlinear eigenvalue problems and the Hopf bifurcation problem (see e.g. works by Rabinowitz, Crandall, Dancer, and Alexander, Yorke, Ize, respectively).

A now-classical result in the topological theory of global bifurcations is the following theorem by Rabinowitz (1970), which gives a characterization of global sets of solutions for *eigenvalue* *problems* of the form

$$X = F(X,\mu) = \mu LX + H(X,\mu),$$

$H(X,\mu) = o(\|X\|)$, posed for $(X,\mu) \in \mathcal{X} \times \mathbb{R}$, $\mathcal{X}$ being a Banach space. In contrast to local theory where the function F is usually k-times differentiable (with a suitable k), in the global theory a typical assumption is that $F : \mathcal{X} \times \mathbb{R} \to \mathcal{X}$ is *compact*. The equation above possesses a "trivial" branch of solutions $(0,\mu)$ for any μ. The bifurcation result asserts that if for some real parameter value μ_0 zero is an eigenvalue of odd multiplicity of the operator $\mathrm{id} - \mu_0 L$, then the set S of nontrivial solutions (X,μ) possesses a maximal subcontinuum which contains $(0,\mu_0)$ and meets either infinity in $\mathcal{X} \times \mathbb{R}$ or another trivial solution $(0,\mu_1)$, $\mu_1 \neq \mu_0$. In particular, $(\mu_0, 0)$ is a bifurcation point. A local version of this result is often referred to as Krasnoselski's theorem.

Different versions and extensions of these theorems can be found in the literature, as, for example, in the case of a simple eigenvalue, or if the field F is real-analytic when the set of solutions is path-connected. More recent works address the question of lack of compactness, and a number of results are now available for problems with additional structure (gradient-like or Hamiltonian structure), but also for concrete problems, such as the water-wave problem.

See also

Bifurcation theory of periodic orbits. Central manifolds, stability theory. Stationary solutions of PDEs and heteroclinic/homoclinic connexions of dynamical systems. Bifurcation in fluid dynamics. Symmetry and symmetry breaking in dynamical systems.

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Keywords

dynamical system, partial differential equation, Hamiltonian system, symmetry, reversibility, bifurcation, stability, Hopf bifurcation, periodic orbit, homoclinic orbit, Liapunov-Schmidt method, center manifold, normal form.