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define the vector $\Xi = (\xi_1, \xi_2) = (t_1, t_2) + \tau(n_1, n_2)$, and the matrix "adjoint" to $\pounds(P, \Xi) : \mathbb{L}(P, \Xi) = \det(\pounds(P, \Xi).\pounds^{-1}(P, \Xi)$, then the complementing boundary condition may be expressed in terms of the product of matrices: $B(P, \Xi) \cdot \mathbb{L}(P, \Xi)$ which is a 4x6 matrix, whose coefficients are polynomials of τ. *The rows of this matrix are required to be linearly independent, modulo* $(\tau - i)^4$. In fact this matrix is the restriction of the matrix $\mathbb{L}(P, \Xi)$ where we suppress the 3rd and 6th lines. It can be written as follows:

$$B \cdot \mathbb{L} = \begin{pmatrix} \mathbb{L}_{11} & a\xi_1\xi_2 & d\xi_1 & c\xi_1^2 & c\xi_1\xi_2 & e\xi_1 \\ a\xi_1\xi_2 & \mathbb{L}_{22} & d\xi_2 & c\xi_1\xi_2 & c\xi_2^2 & e\xi_2 \\ -c\xi_1^2 & -c\xi_1\xi_2 & -e\xi_1 & \mathbb{L}_{11} & a\xi_1\xi_2 & d\xi_1 \\ -c\xi_1\xi_2 & -c\xi_2^2 & -e\xi_2 & a\xi_1\xi_2 & \mathbb{L}_{22} & d\xi_2 \end{pmatrix} \qquad (24)$$

where

$$\begin{aligned} \mathbb{L}_{11} &= -a\xi_2^2 - b(1+\tau^2), \mathbb{L}_{22} = -a\xi_1^2 - b(1+\tau^2), \\ a + b &= \nu(1+\tau^2)^2\{[\lambda_r(m+\nu) + K]^2 + \Lambda_i^2(m+\nu)^2\}, \\ b &= \nu^2(1+\tau^2)^2\{\lambda_r K + (\lambda_r^2 + \Lambda_i^2)(m+\nu)\}, \\ c &= \nu^2(1+\tau^2)^2\Lambda_i K, \;\; e = \nu^2(1+\tau^2)^3\Lambda_i K(m+\nu), \\ d &= \nu^2(1+\tau^2)^3 K[\lambda_r(m+\nu) + K]. \end{aligned}$$

Notice that all the coefficients have $(1+\tau^2)^2$ in factor, and that their degree in τ is 6 or 7 (don't forget that ξ_j has degree 1 in τ). It is not difficult to show that the columns 1, 2, 4, 5 of the matrix $B \cdot \mathbb{L}$ are linearly independent modulo $(\tau - i)^4$, thanks to the fact that $(1+\tau^2)^2$ and $(1+\tau^2)^3$ are not 0 modulo $(\tau - i)^4$, $a + b$ is never 0, and that b and c cannot be both zero together except if λ_r and Λ_i are both 0 (because of the hypothesis made at lemma 5). Now, in the case when λ_r and Λ_i are both 0, the matrix simplifies sufficiently to let us show easily that its rank is still 4. The proof is then completed.

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$\mathcal{A}_\perp^n \otimes \mathcal{A}_2^n$ acting in the 2 last components is Fredholm. In the Appendix, we show that this operator satisfies all properties required by Agmon-Douglis-Nirenberg [1] for systems of elliptic operators, including the complementing boundary conditions. This will finish the proof of lemma 5, thanks to a theorem due to Geymonat [4] (see theorem 3.4) insuring that $\mathcal{A}_\perp^n \otimes \mathcal{A}_2^n$ is then a Fredholm operator.

5 Appendix

We prove in this appendix, in the dimension 3 case, i.e, the case where Σ has dimension 2, and for $\lambda \notin \Lambda = \{z \in \mathbb{C}; \mathrm{Re}\, z = -K(l+2\nu)^{-1}\}$, that the operator $\mathcal{A}_\perp^n \otimes \mathcal{A}_2^n$ acting on the 2 last components $(\mathbf{u}_\perp, \sigma)$ satisfies all properties required by Agmon-Douglis-Nirenberg [1] for systems of elliptic operators, including the complementing boundary conditions.

First we define the 6 dimensional real vector $(\mathbf{u}_{\perp r}, \sigma_r, \mathbf{u}_{\perp i}, \sigma_i)$ where $(\mathbf{u}_\perp, \sigma) = (\mathbf{u}_{\perp r}, \sigma_r) + i(\mathbf{u}_{\perp i}, \sigma_i)$, and denoting by $m = l+\nu$, and $\Lambda_i = \lambda_i + nv_0$, $\lambda = \lambda_r + i\lambda_i$, we can write the 6x6 matrix $\pounds(P,\Xi)$corresponding to the symbol of the differential operator $\mathcal{A}_\perp^n \otimes \mathcal{A}_2^n$ acting on the 6 dim real space:

$$\pounds(P,\Xi) = \begin{pmatrix} \pounds_{11} & -m\xi_1\xi_2 & K\xi_1 & 0 & 0 & 0 \\ -m\xi_1\xi_2 & \pounds_{22} & K\xi_2 & 0 & 0 & 0 \\ \xi_1 & \xi_2 & \lambda_r & 0 & 0 & -\Lambda_i \\ 0 & 0 & 0 & \pounds_{11} & -m\xi_1\xi_2 & K\xi_1 \\ 0 & 0 & 0 & -m\xi_1\xi_2 & \pounds_{22} & K\xi_2 \\ 0 & 0 & \Lambda_i & \xi_1 & \xi_2 & \lambda_r \end{pmatrix} \qquad (22)$$

where $\pounds_{11} = -\nu(\xi_1^2 + \xi_2^2) - m\xi_1^2$, and $\pounds_{22} = -\nu(\xi_1^2 + \xi_2^2) - m\xi_2^2$. The determinant is easily computed:

$$\det(\pounds(P,\Xi)) = \nu^2(\xi_1^2 + \xi_2^2)^4\{[\lambda_r(m+\nu) + K]^2 + \Lambda_i^2(m+\nu)^2\}. \qquad (23)$$

We notice that, precisely, the hypothesis made at lemma 5 insures that the coefficient in brackets, which is a regular function of the point in Σ does not cancel on $\overline{\Sigma}$. We can then verify the supplementary condition (see [1]) on $\det(\pounds(P,\Xi))$. Its degree is 8, and for every pair of linearly independent vectors Ξ and Ξ' , the polynomial (in τ) $[(\xi_1 + \tau\xi_1')^2 + (\xi_2 + \tau\xi_2')^2]^4$ has one quadruple root τ of positive imaginary part. This quadruple root is just i for any orthonormal vectors Ξ and Ξ'. In addition, our system is *uniformly elliptic* (see equ. (1.7) of [1]).

The complementing boundary conditions are a little harder to check. Here we have Dirichlet boundary conditions for $(\mathbf{u}_{\perp r}, \mathbf{u}_{\perp i})$, i.e. on 4 components only of our 6 dim vector. The corresponding matrix $B(P,\partial)$ has 4 lines and 6 columns, with only 1 or 0. Now consider any unitary tangent vector(t_1,t_2) to the boundary $\partial\Sigma$, and the unit normal (n_1,n_2), and

$\mathcal{A}_\lambda^n$ is invertible, while for dimension 3, we are only able to check that $\mathcal{A}_\lambda^n$ is an elliptic operator in the sense of Agmon-Douglis-Nirenberg [1].

In the case of dimension 2, Σ is reduced to the interval (0,1), then we show for $\lambda \notin \Lambda = \{z \in \mathbb{C}; \mathrm{Re}\, z = -K(l+2\nu)^{-1}\}$ that the operator $\mathcal{A}_\lambda^n$ is invertible, which implies the result of lemma 5. Indeed, let us study the following equation:

$$\mathcal{A}_\lambda^n U = \Phi \in [L^2(\Sigma)]^2 \times L_*^2(\Sigma);\ U \in \mathcal{D}_\Sigma, \tag{20}$$
$$\text{where } \Phi = (\phi_x, \phi_\perp, \phi_2) \text{ is given.}$$

In fact we have

$$\mathcal{A}_\perp^n U \equiv -(l+2\nu)\mathbf{u}"_\perp + K\sigma'$$

where the primes denote derivatives with respect to y, and where $\mathbf{u}_\perp$ is a scalar function. It results easily that we obtain

$$\{(l+2\nu)\left(\lambda + inv_0(y)\right) + K\}\,\sigma = \int_0^y \phi_\perp(s)ds + (l+2\nu)\phi_2 \ + \ C \tag{21}$$

where the coefficient of σ does not cancel in the interval $[0,1]$ thanks to the assumption, and where the constant C is uniquely determined by the following condition

$$\int_0^1 \phi_2 dy = \int_0^1 \left(\lambda + inv_0(y)\right)\sigma(y)dy$$

which comes from the boundary conditions on $\mathbf{u}_\perp$. This determines σ in $L_*^2(\Sigma)$, hence one obtains

$$\mathbf{u}_\perp(y) = (l+2\nu)^{-1}\left\{K\int_0^y \sigma(s)ds - Cy - \int_0^y\int_0^s \phi_\perp(\tau)d\tau\right\}$$

which satisfies the boundary condition at $y=1$ and belongs to $H_o^1(\Sigma)$. It is then obvious that one can solve the first component of (20) with respect to $u_x \in H^2 \cap H_o^1(\Sigma)$. This ends the proof of the bounded invertibility of the operator $\mathcal{A}_\lambda^n$ in the case of dimension 2. Let us observe in addition that when $\mathrm{Re}\,\lambda = -K(l+2\nu)^{-1}$, and $\mathrm{Im}\lambda \in \underset{y\in(0,1)}{range}\{-nv_0(y)\}$ then the coefficient of σ in (21) cancels at some point of $(0,1)$, hence such a λ belongs to the essential spectrum of $\mathcal{L}^n$ (invariant under a relatively compact perturbation).

In the case of dimension 3, Σ is then a 2-dimensional regular connected open set of $\mathbb{R}^2$, let us first observe that the operator $\mathcal{A}_x^n$ acting in the first component, is invertible with respect to u_x, as soon as $\mathbf{u}_\perp \in \{H_o^1(\Sigma)\}^2$, and $\sigma \in L_*^2(\Sigma)$, and that the component u_x of U does not appear in the other components of the operator $\mathcal{A}_\lambda^n$. An obvious consequence of this fact is that it is equivalent to prove that $\mathcal{A}_\lambda^n$ is Fredholm, or to prove that the operator

We easily see that $\mathcal{A}_\lambda^n$ is closed, and that $\mathcal{B}_\lambda^n$ is *relatively compact with respect to* $\mathcal{A}_\lambda^n$ (see[7] chapterIV). As a consequence, for values of λ such that $\mathcal{A}_\lambda^n$ is a *Fredholm operator* [closed range with finite codimension and finite dimensional kernel (see for instance [7] chapter IV)], then the operator $\mathcal{A}_\lambda^n + \mathcal{B}_\lambda^n$ is closed and is also a Fredholm operator, with the same index and the reverse is also true ($\mathcal{A}_\lambda^n + \mathcal{B}_\lambda^n$ Fredholm implies $\mathcal{A}_\lambda^n$ Fredholm). Since we know that $\mathcal{A}_\lambda^n + \mathcal{B}_\lambda^n$ is Fredholm of index 0 for $\lambda \notin \Theta_0(\mathcal{L}) \cup \Theta_0(\mathcal{L}^*)$ (in fact this is even true outside of a sector depending on n, as what we showed for small Fourier modes at section 3), it results that this operator is Fredholm of index 0 on the connected component $\mathbb{F}_0^n$ of the $+\infty$ direction of $\mathbb{C}$ in the complementary set of $\{\lambda \in \mathbb{C}; \mathcal{A}_\lambda^n + \mathcal{B}_\lambda^n = \lambda\mathbb{I} - \mathcal{L}^n \text{ is not Fredholm}\}$. Moreover, we know that for λ real and large enough ($\in \mathbb{F}_0^n$), $\lambda\mathbb{I} - \mathcal{L}^n$ is invertible, so a general result on Fredholm operators (see [7] p.242-243) says that in $\mathbb{F}_0^n$ all points are in the resolvent set of $\mathcal{L}^n$, *except a discrete set of isolated points, corresponding to eigenvalues of finite multiplicities for* $\mathcal{L}^n$.

Our aim now is to show

Lemma 8 *For $\lambda \notin \Lambda = \{z \in \mathbb{C}; \operatorname{Re} z = -K(l + 2\nu)^{-1}\}$ the operator $\mathcal{A}_\lambda^n$ is Fredholm.*

This implies that $\lambda\mathbb{I} - \mathcal{L}^n$ is also Fredholm, hence $\mathbb{F}_0^n \supseteq \{z \in \mathbb{C}; \operatorname{Re} z > -K(l + 2\nu)^{-1}\}$. Finally, the only values of λ such that $\lambda\mathbb{I} - \mathcal{L}^n$ is not invertible, are isolated points corresponding to eigenvalues of $\mathcal{L}$ with finite multiplicities, possibly accumulating (for fixed n) on the line Λ. It then results that the *essential spectrum* of $\mathcal{L}$ (defined as the complementary set in $\mathbb{C}$ of the set where the operator $(\lambda\mathbb{I} - \mathcal{L})$ is Fredholm), consists in the union of essential spectra of $\mathcal{A}_\lambda^n$ for all integers n , this set being located on Λ, and the set of accumulation points of eigenvalues for growing values of n. We can now sum up all results above by the following

Theorem 9 *The spectrum of $\mathcal{L}$ is formed with an essential spectrum located on the left of the line of abscissa $\limsup_{N\to\infty}(-\gamma_{\alpha v_0}^N) < 0$, and a discrete set of eigenvalues of finite multiplicities possibly accumulating at points of the essential spectrum. The part of the spectrum lying on the right of a certain line of negative abscissa, is only formed with a finite number of eigenvalues (located in a triangle given by lemma 4).*

Remark 3 *The important consequence of such a result is that for the linear stability study of parallel flows such as the one we consider here, it will be sufficient to consider eigenvalues of the largest real part, as it is usual in the incompressible Navier-Stokes case.*

Proof of the lemma: We shall distinguish between dimension 2 and dimension 3, because for dimension 2, we can prove the stronger result that

Corollary 7 *Let* $\xi_0 = \sup\{\mathrm{Re}\,\lambda; \lambda \in spectrum(\mathcal{L})\}$*,then if* $\xi_0 > -\gamma^N_{\alpha v_0}$ *(true in general for* N *large enough), the growth rate of* $exp(\mathcal{L}t)$ *when* $t \to \infty$ *is equal to* ξ_0*.*

Indeed, the corollary follows from the above splitting of the space $E_0 = E_N + E^N$. When $\xi_0 > -\gamma^N_{\alpha v_0}$ the behavior at ∞ is dominated by the component in E_N. By a classical result on analytic semi-groups, it follows that for any $\epsilon > 0, \exists\ C_\varepsilon$ such that $\| \exp(\mathcal{L}t) \| \leq C_\varepsilon \exp(\xi_0 + \varepsilon)t$, for $t > 0$.

4 Structure of the spectrum of $\mathcal{L}$

We consider now, in more details, the structure of the spectrum of $\mathcal{L}$; this will be achieved by studying the resolvent equation

$$(\lambda\mathbb{I} - \mathcal{L})U = \Phi \in E_0,\ U \in \mathcal{D}. \tag{19}$$

We already know that for $\lambda \notin \Theta_0(\mathcal{L}) \cup \Theta_0(\mathcal{L}^*)$, for instance λ real and large enough, then λ belongs to the resolvent set of $\mathcal{L}$. We also notice that we have not here the opportunity to use a compactness property of the resolvent because the embedding $\mathcal{D} \subset E_0$ is not compact.

Let us decompose (19) into Fourier components in x. We split the operator $\lambda\mathbb{I} - \mathcal{L}^n$ into the sum of two linear operators $\mathcal{A}^n_\lambda = (\mathcal{A}^n_x, \mathcal{A}^n_\perp, \mathcal{A}^n_2)$ and $\mathcal{B}^n_\lambda = (\mathcal{B}^n_x, \mathcal{B}^n_\perp, \mathcal{B}^n_2)$ defined below, where $U = (u_x, \mathbf{u}_\perp, \sigma)$:

$$\begin{aligned}
\mathcal{A}^n_x U &= -\nu\Delta_\perp u_x - in(l+\nu)\nabla_\perp \cdot \mathbf{u}_\perp + inK\sigma \\
\mathcal{A}^n_\perp U &= -\nu\Delta_\perp \mathbf{u}_\perp - (l+\nu)\nabla_\perp\nabla_\perp \cdot \mathbf{u}_\perp + K\nabla_\perp\sigma \\
\mathcal{A}^n_2 U &= (\lambda + inv_0)\sigma + \nabla_\perp \cdot \mathbf{u}_\perp
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}^n_x U &= \lambda u_x + n^2(l+2\nu)u_x + inv_0 u_x + \mathbf{u}_\perp \cdot \nabla_\perp V_0 \\
\mathcal{B}^n_\perp U &= \lambda \mathbf{u}_\perp + n^2\nu\mathbf{u}_\perp - in(l+\nu)\nabla_\perp u_x + inv_0\mathbf{u}_\perp \\
\mathcal{B}^n_2 U &= inu_x
\end{aligned}$$

where for $n = 0$, one has to add the condition

$$\int_\Sigma \sigma = 0.$$

Let us define $L^2_*(\Sigma)$ as $L^2(\Sigma)$ for $n \neq 0$, and as the quotient of $L^2(\Sigma)$ with respect to constants for $n = 0$. The basic space is now $[L^2(\Sigma)]^3 \times L^2_*(\Sigma)$, and the domain of $\mathcal{A}^n_\lambda$ becomes

$$\mathcal{D}_\Sigma = \left\{ \begin{array}{c} U = (u_x, \mathbf{u}_\perp, \sigma); u_x \in H^2 \cap H^1_o(\Sigma), \mathbf{u}_\perp \in \{H^1_o(\Sigma)\}^2, \sigma \in L^2_*(\Sigma), \\ -\nu\Delta_\perp\mathbf{u}_\perp - (l+\nu)\nabla\nabla_\perp \cdot \mathbf{u}_\perp + K\nabla_\perp\sigma \in L^2(\Sigma) \end{array} \right\}.$$

Notice that the linear stability condition has been obtained here by standard elementary inequalities. In this direction the result (17) can be considered as "optimal", in the sense that it coincides with the formula deduced for incompressible fluids with variational method [12].

Lemma 5 *Let $\mathcal{D}^N$ be the subspace of $\mathcal{D}$ such that $U = \sum_{|n|>N} U_n(y,z)e^{2in\pi x/h}$, then for any V_0, there exists N large enough such that in (16) $\gamma^N_{\alpha v_0}$ (which replaces $\gamma_{\alpha v_0}$) is positive.*

This lemma follows easily from (17) once we notice that in the above subspace $\mathcal{D}^N$ of $\mathcal{D}$, the Poincaré constant γ^N_P which replaces γ_P is such that $\gamma^N_P \leq h/2\pi N$.

From this lemma, follows the exponential decay of high Fourier modes of U and a better estimate of the spectrum of $\mathcal{L}$ on the closed subspace $\mathcal{D}^N$ of "high" Fourier modes.

Let $\mathcal{D}_N$ be the subspace of $\mathcal{D}$ such that $U = \sum_{|n|\leq N} U_n(y,z)e^{2in\pi x/h}$, then for $U \in \mathcal{D}_N$,we have

$$\|\sigma_{,x}\| \leq 2N\pi/h \, \|\sigma\|$$

hence, we can estimate in $\mathcal{D}_N$

$$\begin{aligned}
\mathrm{Im}((\mathcal{L}U,U)) \leq & \| \nabla V_0 \|_{L^\infty} \| \mathbf{u} \|^2 + \| v_0 \|_{L^\infty} \| \nabla \mathbf{u} \| \| \mathbf{u} \| \\
& +2K \| \nabla \cdot \mathbf{u} \| \| \sigma \| + 2\pi N K h^{-1} \| v_0 \|_{L^\infty} \| \sigma \|^2 \\
+\alpha\{(l+2\nu) & \| \nabla \cdot \mathbf{u} \| + \nu b \| \nabla \times \mathbf{u} \| + b\gamma_P \| v_0 \|_{L^\infty} \| \nabla \mathbf{u} \| \\
& + b\gamma_P \| \nabla V_0 \|_{L^\infty} \| \mathbf{u} \| \} \| \sigma \| + \alpha b_1 \| \mathbf{u} \| (\| \mathbf{u} \| + \| \sigma \| \| v_0 \|_{L^\infty}).
\end{aligned}$$

Combining this estimate with (15), and using classical inequalities, we obtain for any $U \in \mathcal{D}_N$ the new estimate:

$$|\,\mathrm{Im}((\mathcal{L}U,U))| + \mathrm{Re}((\mathcal{L}U,U)) \leq \Gamma_{\alpha N v_0} \|U\|^2 . \tag{18}$$

where $\Gamma_{\alpha N v_0}$ is a number depending on N, v_0 and α. This estimate gives the following

Lemma 6 *The numerical range of $\mathcal{L}$ restricted to $\mathcal{D}_N$ is located in a sector of the complex plane: $\mathbb{S} = \{z = x + iy \in \mathbb{C}; |y| \leq \Gamma_{\alpha N v_0} - x\}$.*

Remark 2 *Collecting the two last lemmas, we have a better estimate of the spectrum of $\mathcal{L}$, which is included in the union of $\mathbb{S}$ and the half plane $\{z \in \mathbb{C}; \mathrm{Re}\, z \leq -\gamma^N_{\alpha v_0}\}$. Moreover, in each of the closed subspaces E_N and E^N of E_0, we have a better estimate for the resolvent operator, than (11) of lemma 1. On E_N the semi-group $\exp(\mathcal{L}t)$ is analytic [7], while on E^N it is a contraction semi-group.*

$$|(\mathcal{L}_1 U;\mathbf{z}) - K \parallel \sigma \parallel^2 | \leq [(l+2\nu) \parallel \nabla\cdot\mathbf{u} \parallel + \nu b \parallel \nabla\times\mathbf{u} \parallel$$
$$+ b\gamma_P ||v_0||_{L^\infty} \parallel \nabla\mathbf{u} \parallel + b\gamma_P \parallel \nabla V_0 \parallel_{L^\infty} \parallel \mathbf{u} \parallel] \parallel \sigma \parallel .$$

It remains to analyze $(\mathbf{W};\mathbf{u})$ where

$$\nabla\cdot\mathbf{W} = -v_0\frac{\partial\sigma}{\partial x} - \nabla\cdot\mathbf{u} = -\nabla\cdot(\mathbf{u}+\sigma v_0\mathbf{i}).$$

Notice that problem (12) for $\mathbf{W}$ has to be solved with $g = -\nabla\cdot(\mathbf{u}+\sigma v_0\mathbf{i})$, with $(\mathbf{u}+\sigma v_0\mathbf{i})\cdot\mathbf{n} = 0$ on the boundary. Therefore by theorem 3.3 of [3] we have the stronger estimate for $\mathbf{W}$

$$\parallel \mathbf{W} \parallel \leq b_1 \parallel \mathbf{u}+\sigma v_0\mathbf{i} \parallel$$

hence

$$|(\mathbf{W};\mathbf{u})| \leq b_1 \parallel \mathbf{u} \parallel (\parallel \mathbf{u} \parallel + \parallel \sigma \parallel \parallel v_0 \parallel_{L^\infty}).$$

Inserting these informations in (14), we deduce, for $\alpha > 0$

$$\mathrm{Re}((\mathcal{L}U;U)) \leq -\nu \parallel \nabla\mathbf{u} \parallel^2 - (l+\nu) \parallel \nabla\cdot\mathbf{u} \parallel^2 - \mathrm{Re}(\mathbf{u}\cdot\nabla V_0;\mathbf{u}) \tag{15}$$
$$-\alpha K \parallel \sigma \parallel^2 + \alpha\{(l+2\nu) \parallel \nabla\cdot\mathbf{u} \parallel + \nu b \parallel \nabla\times\mathbf{u} \parallel + b\gamma_P \parallel v_0 \parallel_{L^\infty} \parallel \nabla\mathbf{u} \parallel$$
$$+ b\gamma_P \parallel \nabla V_0 \parallel_{L^\infty} \parallel \mathbf{u} \parallel\} \parallel \sigma \parallel + \alpha b_1 \parallel \mathbf{u} \parallel (\parallel \mathbf{u} \parallel + \parallel \sigma \parallel \parallel v_0 \parallel_{L^\infty})$$

By use of Poincaré inequality and results on positiveness of quadratic forms, it is straightforward to obtain

$$\mathrm{Re}((\mathcal{L}U;U)) \leq -(\nu'\gamma_P^{-2} - \parallel \nabla V_0 \parallel_{L^\infty}) \parallel \mathbf{u} \parallel^2 - \alpha K' \parallel \sigma \parallel^2,$$

where ν' and K' can be taken arbitrarily close to ν and K (respectively) provided that we choose α small enough. We then arrive to

$$\mathrm{Re}((\mathcal{L}U;U)) \leq -\gamma_{\alpha v_0} \parallel U \parallel^2, \tag{16}$$

where $\gamma_{\alpha v_0}$ is a positive constant, once

$$\nu - \gamma_P^2 \parallel \nabla V_0 \parallel_{L^\infty} > 0 \tag{17}$$

holds, i.e. if the basic flow has not too high velocity gradient.We have now the following

Theorem 4 *The operator $\mathcal{L}$ is the infinitesimal generator of a C^0 semi-group in E_0. Moreover, if $\parallel \nabla V_0 \parallel_{L^\infty} < \nu\gamma_P^{-2}$, the semi-group $\exp(\mathcal{L}t)$ is uniformly bounded by $e^{-\gamma_{\alpha v_0} t}$ for $t \geq 0$, where $\gamma_{\alpha v_0}$ is defined by (16).*

Remark 1 *In the above theorem, $\parallel \nabla V_0 \parallel_{L^\infty}$ may be replaced by $\parallel D_0 \parallel_{L^\infty}$ where D_0 is the symmetric part of ∇V_0, as it can be shown from the form of (15).*

Lemma 3 *Let $g \in L^2_\sharp$, such that $\int_{\Omega_\sharp} g = 0$. Then there exists a unique solution $\underline{\varphi} \in H^1_{0\sharp}$ of the problem*

$$\nabla \cdot \varphi = g, \ \ in\ \Omega \tag{12}$$

satisfying the estimate

$$\| \nabla\varphi \| \leq b \| g \|,$$

where b is a positive constant.

Let $U = (\mathbf{u}, \sigma), V = (\mathbf{v}, \tau)$, we denote by $\mathbf{z}$ and $\mathbf{w}$ solutions to (12) respectively corresponding to $g = \sigma$ and τ. Let α be a suitable real number which will be defined below, then we introduce the following new scalar product in E_0

$$((U;V)) = (U;V)_{E_0} - \alpha[(\mathbf{u};\mathbf{w}) + (\mathbf{z};\mathbf{v})].$$

It is easy to see that $\omega : (U,V) \to ((U;V))$ is a sesquilinear symmetric form. Moreover, the following estimate holds

$$\begin{aligned} ((U;U)) &\geq \| \mathbf{u} \|^2 + K \| \sigma \|^2 - 2|\alpha| \| \mathbf{u} \| \| \underline{z} \| \\ &\geq \| \mathbf{u} \|^2 + K \| \sigma \|^2 - 2|\alpha| b\gamma_P \| \mathbf{u} \| \| \sigma \| . \end{aligned}$$

Therefore, the sesquilinear form ω is positive definite when

$$|\alpha| b\gamma_P < \sqrt{K}, \tag{13}$$

which also insures the equivalence of the two norms in E_0. We can study the numerical range of $\mathcal{L}$ using the new scalar product. To simplify the formalism, we introduce the following notations

$$\begin{aligned} \mathcal{L}_1 U &= \nu\Delta\mathbf{u} + (l+\nu)\nabla(\nabla\cdot\mathbf{u}) - v_0\frac{\partial \mathbf{u}}{\partial x} - \mathbf{u}\cdot\nabla V_0 - K\nabla\sigma \\ \mathcal{L}_2 U &= -v_0\frac{\partial\sigma}{\partial x} - \nabla\cdot\mathbf{u} \end{aligned}$$

so, for $U \in \mathcal{D}$, we have

$$((\mathcal{L}U;U)) = (\mathcal{L}_1 U;\underline{u}) + K(\mathcal{L}_2 U;\sigma) - \alpha[(\mathcal{L}_1 U;\mathbf{z}) + (\mathbf{W};\mathbf{u})] \tag{14}$$

where $\mathbf{W}$ solves problem (12) with $g = \mathcal{L}_2 U$. Let us now estimate the two new last terms of the right hand side of (14).

We observe that

$$(\mathcal{L}_1 U;\mathbf{z}) = K \| \sigma \|^2 - (l+2\nu)(\nabla\cdot\mathbf{u};\sigma) - \nu(\nabla\times\mathbf{u};\nabla\times\mathbf{z}) - (v_0\mathbf{u}_{,x} + \mathbf{u}\cdot\nabla V_0;\mathbf{z})$$

for any $u \in H^1_{0\sharp}$. We have at once, for any $U \in \mathcal{D}$

$$\mathrm{Re}(\mathcal{L}U;U)_{E_0} \leq (-\nu\gamma_P^{-2}+ \| \nabla V_0 \|_{L^\infty}) \| \mathbf{u} \|^2 \leq C \| U \|^2_{E_0}, \qquad (9)$$
$$where \quad C = \max(0, -\nu\gamma_P^{-2}+ \| \nabla V_0 \|_{L^\infty}).$$

Let us define the adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$, since we shall use the property that the spectrum of $\mathcal{L}$ is included in $\Theta_0(\mathcal{L}) \cup \Theta_0(\mathcal{L}^*)$. For $U \in \mathcal{D}$ and $V \in E_0$ and sufficiently regular, we have

$$(\mathcal{L}U;V)_{E_0} = (\mathbf{u}; \nu\Delta\mathbf{v} + (l+\nu)\nabla(\nabla\cdot\mathbf{v}) + v_0\mathbf{v}_{,x} - \mathbf{v}\cdot(\nabla V_0)^T + K\nabla\tau) +$$

$$+K(\sigma; \nabla\cdot\mathbf{v} + v_0\tau_{,x}) + B = (U, \mathcal{L}^*V)_{E_0} \qquad (10)$$

where

$$B := \nu\int_{\partial\Sigma} \frac{\partial\mathbf{u}}{\partial n}\cdot\overline{\mathbf{v}} + (l+\nu)\int_{\partial\Sigma} \nabla\cdot\mathbf{u}(\overline{\mathbf{v}}\cdot\underline{n}) - K\int_{\partial\Sigma}\sigma(\overline{\mathbf{v}}\cdot\underline{n}).$$

The domain of $\mathcal{L}^*$ is by definition the set of V such that $(\mathcal{L}U;V)_{E_0}$ can be extended as a bounded linear form on U in E_0. To this aim we need to have $\mathbf{v} = 0$ on $\partial\Sigma$. Then, it is easy to show (via an analogous argument as for the study of $D(\mathcal{L})$ via (8)), that

$$D(\mathcal{L}^*) = \left\{ \begin{array}{l} V = (\mathbf{v},\tau) \in E_0; \mathbf{v} \in (H^1_{0\sharp})^3, v_0\frac{\partial\tau}{\partial x} \in L^2_\sharp, \\ \nu\Delta\mathbf{v} + (l+\nu)\nabla(\nabla\cdot\mathbf{v}) + K\nabla\tau \in (L^2_\sharp)^3 \end{array} \right\} \neq D(\mathcal{L}).$$

Concerning the numerical range of $\mathcal{L}^*$, in the wake of what is proved for the operator $\mathcal{L}$, we can see that (9) also holds for $\mathcal{L}^*$. As a consequence, when λ is not in the closure of $\Theta_0(\mathcal{L}) \cup \Theta_0(\mathcal{L}^*)$ it belongs to the resolvent set of $\mathcal{L}$ (see Kato [7], chapter 5). So, we have the following

Lemma 1 *For $Re(\lambda) > C$, λ belongs to the resolvent set of $\mathcal{L}$. Moreover, we have the following upper bound:*

$$\| (\lambda\mathbb{I} - \mathcal{L})^{-1} \| \leq (Re(\lambda) - C)^{-1} \qquad (11)$$

where C is defined in (9).

Now, by the Hille-Yoshida theorem, we have

Corollary 2 *The operator $\mathcal{L}$ is the infinitesimal generator of a C^0 semi-group in E_0. Moreover, if $\| \nabla V_0 \|_{L^\infty} \leq \nu\gamma_P^{-2}$, the semi-group $\exp(\mathcal{L}t)$ is uniformly bounded for $t > 0$.*

In order to obtain a better estimate on the spectrum of $\mathcal{L}$ we deem it better to introduce a new scalar product. To this aim, we first remind a classical result [2].

From now on, we omit the subscript $L^2_\sharp$ and $(L^2_\sharp)^3$ in the common scalar products and norms in $L^2_\sharp$.

Let us denote by $\mathcal{L}$ the linear operator defined by the right hand sides of (4),(5) as an operator acting in the space E_0, with domain $D(\mathcal{L})$, taking into account the Dirichlet boundary condition on $\mathbf{u}$. The system (4),(5) can be rewritten in the following form:

$$\frac{dU}{dt} = \mathcal{L}U, \qquad \text{in } E_0 \tag{7}$$
$$U|_{t=0} = U_0, \qquad \text{in } E_0.$$

Let us first show that $D(\mathcal{L}) = \mathcal{D}$.

We first notice that $D(\mathcal{L})$ is dense in E_0 and we have for any $U \in D(\mathcal{L})$, after integrations by parts

$$\begin{aligned}(\mathcal{L}U;U)_{E_0} &= -\nu \parallel \nabla\mathbf{u} \parallel^2 -(l+\nu) \parallel \nabla\cdot\mathbf{u} \parallel^2 -(\mathbf{u}\cdot\nabla V_0;\mathbf{u}) \\ &\quad - i\,\mathrm{Im}(v_0\mathbf{u}_{,x}\,;\mathbf{u}) - 2iK\,\mathrm{Im}(\nabla\cdot\mathbf{u};\sigma) - iK\,\mathrm{Im}(v_0\sigma_{,x}\,;\sigma).\end{aligned} \tag{8}$$

Then, noticing that $|(\mathbf{u}\cdot\nabla V_0;\mathbf{u})| \leq\parallel \nabla V_0 \parallel_{L^\infty}\parallel \mathbf{u} \parallel^2$, and taking the real part of identity (8) we see that $U \in (H^1_{0\sharp})^3 \times L^2_\sharp$ which proves, from the definition of $\mathcal{D}$, that $D(\mathcal{L}) = \mathcal{D}$.

The operator $\mathcal{L}$ is closed. Indeed, let us consider $U_n \to U$ and $\mathcal{L}U_n \to V$ in E_0, then by a classical argument using distributions, it follows that $\mathcal{L}U = V$. Then, by using (8) we can prove that $U \in (H^1_{0\sharp})^3 \times L^2_\sharp$,which implies that $U \in \mathcal{D}$.

We are interested in the existence, uniqueness, and asymptotic behavior as $t \to \infty$ of solutions of (7). To this end, we analyse in section 3 the numerical range of $\mathcal{L}$ with suitable scalar products in E_0. This will imply that $\mathcal{L}$ is the infinitesimal generator of a C^0- semi-group, with first estimates on the growth rate as $t \to \infty$ and a first idea of the location of the spectrum of $\mathcal{L}$. In section 4, we study the spectrum of $\mathcal{L}$, which allows to furnish a more precise behavior as $t \to \infty$ of the solutions of (7), even though there is *an essential part* in the spectrum.

3 Numerical range of $\mathcal{L}$

We begin by studying the numerical range

$$\Theta_0(\mathcal{L}) = \{(\mathcal{L}U;U)_{E_0};\ U \in \mathcal{D}, \parallel U \parallel_{E_0} = 1, \}.$$

Therefore, we denote by γ_P the Poincaré constant of $\Omega_\sharp$, i.e.

$$\parallel u \parallel \leq \gamma_P \parallel \nabla u \parallel$$

admit solutions, which will be refered below as "the basic flow", of the form:

$$V_0 = v_0(y,z)\mathbf{i}, \qquad \text{where} \left\{ \begin{array}{l} \Delta v_0 = g \text{ on } \Sigma, \\ v_0 = a \text{ on } \partial\Sigma \end{array} \right. \tag{1}$$

$$p = p_0, \qquad \rho = \rho_0 \tag{2}$$

where V_0 is the velocity field of a parallel flow, $\underline{i}$ is the unit vector in the direction of x axis and g and a are given functions respectively on Σ and $\partial\Sigma$, the basic pressure p_0 as well as the basic volumic mass ρ_0 are constant. Let us set

$$V = V_0 + \mathbf{u},\ \rho = \rho_0(1+\sigma) \tag{3}$$

then the linearized Navier-Stokes equations may be written as follows:

$$\frac{\partial \mathbf{u}}{\partial t} = \nu\Delta\mathbf{u} + (l+\nu)\nabla(\nabla\cdot\mathbf{u}) - v_0\frac{\partial \mathbf{u}}{\partial x} - \mathbf{u}\cdot\nabla V_0 - K\nabla\sigma \tag{4}$$

$$\frac{\partial \sigma}{\partial t} = -v_0\frac{\partial \sigma}{\partial x} - \nabla\cdot\mathbf{u} \tag{5}$$

where $\mathbf{u} = 0$ on $\mathbb{R}\times\partial\Sigma$, $K = c_0^2$ (square of sound velocity), and ν, l are kinematic viscosity coefficients (assumed to be constant). We only consider here periodic functions in x (period h). The periodicity cell is denoted by $\Omega_\sharp = (\mathbb{R}/h\mathbb{Z})\times\Sigma$.We have to complete the system (4,5) with initial data such as: $\mathbf{u}\mid_{t=o} = \mathbf{u}_0$ and $\sigma|_{t=o} = \sigma_0$. Moreover, we ask $\int_{\Omega_\sharp}\sigma_0 = 0$, which means that the total mass in $\Omega_\sharp$ is the same as for the basic flow.

We now introduce the basic function spaces of our problem:

i) $H^k_\sharp$ is the Hilbert space of a.e periodic functions in x (period h), which are square summable with their kth first derivatives in their periodicity cell $\Omega_\sharp$; for $k=0$, we denote this space $L^2_\sharp$. We denote by $H^1_{0\sharp}$ the subspace of $H^1_\sharp$ such that $u = 0$ on $\mathbb{R}\times\partial\Sigma$. Moreover

ii) $E_0 = \left\{ U = (\mathbf{u},\sigma) \in (L^2_\sharp)^4; \int_{\Omega_\sharp}\sigma = 0 \right\}$,

iii) $\mathcal{D} = \left\{ \begin{array}{c} U = (\mathbf{u},\sigma)\in E_0; \mathbf{u}\in (H^1_{0\sharp})^3, v_0\frac{\partial\sigma}{\partial x}\in L^2_\sharp, \\ \nu\Delta\mathbf{u} + (l+\nu)\nabla(\nabla\cdot\mathbf{u}) - K\nabla\sigma \in (L^2_\sharp)^3 \end{array} \right\}$.

The space E_0 is a Hilbert space with the following scalar product in $(L^2_\sharp)^4$:

$$(U;V)_{E_0} = (\mathbf{u};\mathbf{v})_{(L^2_\sharp)^3} + K(\sigma;\tau)_{L^2_\sharp}, \text{ where } U = (\mathbf{u},\sigma), V = (\mathbf{v},\tau). \tag{6}$$

where the linearized problem would give a weakly (growth rate close to 0) decaying or weakly exploding solution as $t \to \infty$. This is our motivation for studying the structure of the linearized problem around a steady basic solution flow (hence corresponding to non necessarily small velocity field). A further non trivial step would be to rely the linear analysis done here, with the nonlinear behavior of the solutions as $t \to \infty$, in justifying, for instance a *center manifold reduction technique*. This is not done here. However, the *justification of linearization principle*, when this one gives linear stability, is provided by Neustupa [11] in working in Hölder spaces, for flows in bounded domains.

Existence (for all $t > 0$), uniqueness and regularity of the solution of linearized Navier-Stokes equations for barotropic fluids, in a bounded domain of $\mathbb{R}^2$ or $\mathbb{R}^3$ has been established by G.Geymonat and P.Leyland in [5]. In particular, they show that if the basic flow is steady, the linear operator is *the infinitesimal generator of a* C^0*- semi-group* in a suitable Banach space and that for "small" datas, it is a *semi-group of contraction.* It is also stressed that the difference, between this case and the incompressible Navier-Stokes case, is that the mass conservation is expressed via a transport equation, which perturbs a lot the parabolic form of the system. In the present work, we consider the linearized form of the Navier-Stokes equations governing nonsteady flows of a viscous barotropic gas filling a cylindrical domain. Precisely, assuming a periodicity condition along the generatrix direction, we show again, but in a simpler way than in [5], that the linear operator generates a C^0- semi-group in a suitable space, being a contraction semi-group for small datas. In addition we establish, in quite general situations, that *the growth rate* (in time) *of the solution is given by the greatest real part of eigenvalues of the linear operator*. Notice that a similar result is obtained by J.Neustupa [11] using very different techniques in Hölder spaces (more regular than our space). In addition, we prove that the spectrum of the above linear operator is the union of an *essential spectrum lying on the left side of a line* of the left complex plane, and of a *discrete set of isolated eigenvalues of finite multiplicities*, eventually giving the most "dangerous modes" as $t \to \infty$ (we give some precisions also on the location of this "dangerous" part of the spectrum). The technique we use here is analogous to the one used in [6], but with the additional difficulties here specific to the compressible Navier-Stokes equations.

2 Statement of the problem

We consider a viscous compressible fluid (gas) occupying the cylindrical region $\Omega = \mathbb{R} \times \Sigma$ whose cross section Σ is a regular simply connected domain of $\mathbb{R}^2$, orthogonal to the x axis. We assume that the state law for the gas is a barotropic one, i.e. $p = f(\rho)$. Then the Navier-Stokes equations

Structure of the linearized problem for compressible parallel fluid flows

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Abstract

We consider the linearized compressible Navier-Stokes equation near a parallel flow in a cylindrical domain restricting our study to perturbations periodic in the generatrix direction. For any parameter values, we show that the initial value linear evolution problem is solved by the direct sum of a (strictly) contraction semi-group and an analytic semi-group. Any unbounded in time solution of this linear problem comes from isolated eigenvalues with finite multiplicities, which have non negative real part, and whose imaginary part is bounded. In addition, we precise the structure of the spectrum of the generator of the semi-group, locating the essential spectrum strictly on the left side of the complex plane.

1 Introduction

The existence, uniqueness and regularity of non-steady solutions of compressible fluids, besides the pioneering papers of Nash, Graffi and Serrin, has recently received "global in time" results (see for example [9],[10],[8],[13]). The results in [9], [10], [13] prove existence, uniqueness and stability of regular solutions for the full nonlinear problem, in a bounded domain, assuming *smallness on the initial velocity field and on the part of the external force which is not potential.* Very few is known, on the stability of steady solutions, having large velocity field, of the full nonlinear system. One possibility for starting the study of this problem is to make the analysis of the linearized problem, which seems more affordable. Moreover, we wish to infer informations on the nonlinear problem, from those known for the linear one, and we need to study what happens near critical situations, i.e. situations