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The linear operator $\mathcal{L}_{\rm im}$ associated with the second equation has also a simple zero eigenvalue with eigenfunction A^* which is even. Therefore, $\mathcal{L}_{\rm im}$ is also invertible on odd functions in $C_{1,2}$. All works in $C_{1,2}$ so $A_{\rm p} \in C^2_{1,2}$, hence $\widetilde{A}$ and $\widetilde{B}$ are in $C_{1,2}$. Tracing back to the form of the free surface $\eta = Z(\xi)$, we have after a careful examination

$$Z_\xi = \tan\alpha(x) = \mathcal{O}\left(\frac{\mu\sqrt{\mu}}{1+\mu x^2}\right) \quad \text{at infinity}$$

and since $\frac{\partial \xi}{\partial x} \approx 1$ one has finally

$$Z(\xi) = \mathcal{O}\left(\frac{\sqrt{\mu}}{|\xi|}\right) \quad \text{for } |\xi| \to \infty.$$

Summarizing we have proved

Theorem 2 *There exists a pair of reversible solitary wave solutions for the system (1.1) (1.2) (1.3) such that* $U - 1 = \mathcal{O}\left(\mu^{3/2}|\ln\mu|/(1+\mu^{1/2}|\xi|)\right), V = \mathcal{O}\left(\mu\sqrt{\mu}/(1+\mu\xi^2)\right)$ *and* $Z = \mathcal{O}\left(\sqrt{\mu}/|\xi|\right)$ *as* $|\xi| \to \infty$.

Remark. It has been pointed out to the authors by J.C. Saut that these solitary waves do not decay exponentially. This results from the nonsmoothness of the Fourier symbol at the origin, while the Fourier transform of an exponentially decaying function is analytic in a strip containing the real axis. Because of (1.8) α and β cannot both decay exponentially at infinity, hence none of them decays exponentially due to (1.5).

Let us mention that in the papers of Amick [1] and Sun [16] (the problem is different, but a similar method should work) it is found that the decay is like $1/x^2$. The problem of the true rate here is different, since the principal part of the solitary wave coming from the normal form has an exponential decay. Here the non exponential decay comes from high order terms.

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with the asymptotic expressions

$$\left\{\begin{array}{rcl} r_0(x) &=& \pm\sqrt{\dfrac{2\mu Q_{1,00}}{(-Q_{0,10})}}\dfrac{1}{\cosh\sqrt{\mu Q_{1,00}}x}, \\ r_1(x) &=& r_0'(x), \\ \theta_0(x) &=& P_{1,00}\,\mu x + \dfrac{2P_{0,10}}{(-Q_{0,10})}\sqrt{\mu Q_{1,00}}\tanh\left(\sqrt{\mu Q_{1,00}}x\right), \\ \theta_1(x) &=& \theta_0(x), \quad \theta_1(x) = \theta_0(x)+\pi. \end{array}\right. \tag{5.2}$$

It is shown in the sequel that these two reversible solitary wave solutions persist under nonlocal reversible perturbations of the normal form vector field. We cannot directly apply the results of Iooss & Pérouème [10] because the perturbations are nonlocal in the present case. We shall exploit an argument given by Kirchgässner in [11] which works also for nonlocal reversible perturbations.

The truncated equation (4.13) with right hand side zero, i.e. the equation

$$\widetilde{A}'' - Q_{1,00}\widetilde{A} - Q_{0,10}\widetilde{A}|\widetilde{A}|^2 = 0$$

has a real homoclinic solution $\widetilde{A} = A^* = r_0^*$ which is even. We are looking for a reversible homoclinic solution of the full equation (4.13) which is close to A^*. The replacement $\widetilde{A} = A^* + A_{\rm p}$ leads to a nonautonomous equation for the perturbation term $A_{\rm p}$:

$$\left\{\begin{array}{l} A_{\rm p}'' - Q_{1,00}A_{\rm p} - Q_{0,10}\left[(A^*(\tilde{x}))^2\,\overline{A_{\rm p}} + 2\,|A^*(\tilde{x})|^2\,A_{\rm p}\right] = \\ \qquad\qquad = N\left(\tilde{x}; A_{\rm p}, A_{\rm p}', \overline{A_{\rm p}}, \overline{A_{\rm p}'}\right) + R\left(\mu; \tilde{x}; A_{\rm p}, A_{\rm p}', \overline{A_{\rm p}}, \overline{A_{\rm p}'}\right) \end{array}\right. \tag{5.3}$$

The aim is to prove by Banach's fixed point theorem the existence of a small solution $A_{\rm p}$ of (5.3) which decays at $\tilde{x} = \pm\infty$. Therefore, the linear operator $\mathcal{L}(\tilde{x})$ on the left hand side of (5.3) has to be inverted in some suitable function space. Since this operator represents the variational equation around the homoclinic solution of the truncated system, it has a zero eigenvalue with eigenfunction $\mathrm{d}A^*/\mathrm{d}\tilde{x}$. Decomposing $A_{\rm p} = A_{\rm p,re} + \mathrm{i}A_{\rm p,im}$ into real and imaginary parts we obtain from (5.3) two linearly decoupled equations.

$$\left\{\begin{array}{l} A_{\rm p,re}'' - Q_{1,00}A_{\rm p,re} - 3Q_{0,10}\left[A^*(\tilde{x})\right]^2 A_{\rm p,re} = \\ \qquad\qquad = N_{\rm re}\left(\tilde{x}; A_{\rm p}', \overline{A_{\rm p}}, \overline{A_{\rm p}'}\right) + R_{\rm re}\left(\mu; \tilde{x}; A_{\rm p}, A_{\rm p}', \overline{A_{\rm p}}, \overline{A_{\rm p}'}\right) \\ A_{\rm p,im}'' - Q_{1,00}A_{\rm p,im} - Q_{0,10}\left[A^*(\tilde{x})\right]^2 A_{\rm p,im} = \\ \qquad\qquad = N_{\rm im}\left(\tilde{x}; A_{\rm p}, A_{\rm p}', \overline{A_{\rm p}}, \overline{A_{\rm p}'}\right) + R_{\rm im}\left(\mu; \tilde{x}; A_{\rm p}, A_{\rm p}', \overline{A_{\rm p}}, \overline{A_{\rm p}'}\right). \end{array}\right. \tag{5.4}$$

Since we are looking for reversible solutions, i.e. such that $A(-x) = \overline{A(x)}$, the real part $A_{\rm p,re}$ of $A_{\rm p}$ must be even, while the imaginary part $A_{\rm p,im}$ of $A_{\rm p}$ must be odd. According to the argument given in [11] we can invert the linear operator $\mathcal{L}_{\rm re}$ in $C_{1,2}$ on the left hand side of the first equation of (5.4) if we restrict its domain of definition to even functions. Note that this restriction eliminates the only zero eigenfunction $\mathrm{d}A^*/\mathrm{d}\tilde{x}$ of $\mathcal{L}_{\rm re}$ which is an odd function.

where $c'_\epsilon = c'(1 + |\ln \epsilon|)$.

Let us now introduce the scaling

$$A(x) = \sqrt{|\mu|}\widetilde{A}\left(\sqrt{|\mu|}x\right)\mathrm{e}^{2\mathrm{i}x}, \quad B(x) = |\mu|\widetilde{B}\left(\sqrt{|\mu|}x\right)\mathrm{e}^{2\mathrm{i}x}, \quad \tilde{x} = \sqrt{|\mu|}x, \tag{4.11}$$

and take $\epsilon = |\mu|$ in (4.10). Hence (4.10) becomes

$$\|\boldsymbol{W}^0\| \leq c''\left(1 + |\ln|\mu||\right)|\mu|^{3/2}$$

for any $\widetilde{A}(\tilde{x})$, $\widetilde{B}(\tilde{x})$ in a fixed ball of $C_{1,2}$ and $\boldsymbol{W}^0(\mu, \widetilde{\boldsymbol{V}})$ is now replaced in the $\boldsymbol{V}$ part of (3.4). This leads for $\mu > 0$ to the following reduced equation:

$$\left\{\begin{array}{lcl} \widetilde{A}' & = & \widetilde{B} + \widetilde{R}_0\left(\mu; \widetilde{A}, \widetilde{B}, \overline{\widetilde{A}}, \overline{\widetilde{B}}\right), \\ \widetilde{B}' & = & Q_{1,00}\widetilde{A} + Q_{0,10}\widetilde{A}|\widetilde{A}|^2 + \widetilde{R}_1\left(\mu; \widetilde{A}, \widetilde{B}, \overline{\widetilde{A}}, \overline{\widetilde{B}}\right). \end{array}\right. \tag{4.12}$$

Here the prime denotes differentiation w.r.t $\tilde{x}$. The remainder terms $\widetilde{R}_j$ $(j = 0, 1)$ are nonlocal functions of $(\widetilde{A}, \widetilde{B})$ with

$$\left|\widetilde{R}_0\left(\mu; \widetilde{A}, \widetilde{B}, \overline{\widetilde{A}}, \overline{\widetilde{B}}\right)\right|_{1,2} = \mathcal{O}\left(\sqrt{\mu}\right), \left|\widetilde{R}_1\left(\mu; \widetilde{A}, \widetilde{B}, \overline{\widetilde{A}}, \overline{\widetilde{B}}\right)\right|_{1,2} = \mathcal{O}\left(\sqrt{\mu}\,|\ln\mu|\right), \quad \mu \to 0.$$

The first equation of (4.12) can be solved for $\widetilde{B}$. Insertion into the second equation yields a complex second order equation for $\widetilde{A}$.

$$\widetilde{A}'' - Q_{1,00}\widetilde{A} - Q_{0,10}\widetilde{A}|\widetilde{A}|^2 = \widetilde{R}\left(\mu; \widetilde{A}, \widetilde{A}', \overline{\widetilde{A}}, \overline{\widetilde{A}'}\right). \tag{4.13}$$

The remainder term $\widetilde{R}$ is of order $\mathcal{O}\left(\sqrt{\mu}\,|\ln\mu|\right)$, $\mu \to 0$. Equation (4.13) will be studied in the next section.

5 Solitary waves with damped oscillations

The normal form system (3.1) is an integrable system which admits many different types of solutions, e.g. periodic solutions, quasiperiodic solutions, homoclinic solutions, etc., see Iooss & Pérouème [10]. The forementioned paper also treats the subtle problem of persistence of normal form solutions under (reversible) perturbations.

It was shown in [8] and [10] that the normal form system (3.1) has – under certain sign conditions for the coefficients – a pair of solitary waves (homoclinic solutions) with damped oscillations which are reversible and which persist under reversible perturbations of the vector field. It follows from (3.10) that these sign conditions are fulfilled for the present problem ($Q_{1,00} = 4 > 0, Q_{0,10} = -22 < 0$). For $\mu > 0$ a pair of reversible solitary waves exists for the normal form system (3.1) and has the explicit representation

$$A(x) = r_0(x)\mathrm{e}^{\mathrm{i}(2x+\theta_0(x))}, \quad B(x) = r_1(x)\mathrm{e}^{\mathrm{i}(2x+\theta_1(x))}, \tag{5.1}$$

This shows that the estimate in the above lemma has a constant c_ϵ which is independent of $\epsilon(\leq 1)$ as far as ξ_0, ξ_1 and η_1 are concerned. Let us now study the convolution product of a $C_{1,1}$ function with a $C_{\epsilon,2}$ function, for estimating $|\eta_0|_{\sqrt{\epsilon},1}$.

$$\begin{aligned}
\int_{I\!R} \frac{(1+\sqrt{\epsilon}x)\,\mathrm{d}t}{(1+|t|)\left[1+\epsilon(x-t)^2\right]} &= \frac{2\left(1+\sqrt{\epsilon}|x|\right)\left(1+\epsilon+\epsilon x^2\right)}{\left(1+\epsilon+\epsilon x^2\right)^2-4\epsilon^2x^2}\times \\
&\times\left\{-\frac{1}{2}\ln\epsilon+\frac{1}{2}\ln\left(1+\epsilon x^2\right)+\sqrt{\epsilon}\left(\frac{\pi}{2}+|x|\ \mathrm{Arctg}\left(\sqrt{\epsilon}|x|\right)\right)\right\} \\
&-\frac{4\epsilon\sqrt{\epsilon}|x|\left(1+\sqrt{\epsilon}|x|\right)}{\left(1+\epsilon+\epsilon x^2\right)^2-4\epsilon^2x^2}\left(\frac{\pi}{2}|x|+\mathrm{Arctg}\sqrt{\epsilon}|x|\right) \\
&< \left(\frac{\sqrt{2}+1}{1-3\epsilon}\right)\left\{-\frac{1}{2}\ln\epsilon+\frac{\pi}{2}\sqrt{\epsilon}\right\}+\frac{2}{1-3\epsilon}c_1,\ \text{for } \epsilon<\frac{1}{3},
\end{aligned}$$

and where

$$c_1=\sup_{u>0}\frac{1+u}{1+u^2}\left(\frac{1}{2}\ln\left(1+u^2\right)+u\ \mathrm{Arctg}u\right)<\frac{4}{3}\left(1+\frac{\pi}{2}\right).$$

Notice that the divergence in $|\ln\epsilon|$ of the estimate is not unexpected because the integral diverges for ϵ tending to 0 (monotonic convergence theorem).

We are now able to solve the nonlinear equation

$$\frac{d}{dx}\boldsymbol{W}(x)=\boldsymbol{L}_1\boldsymbol{W}(x)+\boldsymbol{H}\left(\boldsymbol{V}(x)\right)+\boldsymbol{R}^1\left(\mu;\boldsymbol{V}(x),\boldsymbol{W}(x)\right) \tag{4.7}$$

locally in a small zero neighbourhood. The central component $\boldsymbol{V}=(A,B,\overline{A},\overline{B})$ is treated as a parameter in $(C_{\epsilon,2})^4$, and we observe that the dependency of $\boldsymbol{R}^1$ in $\boldsymbol{W}$ is only through the trace $\boldsymbol{W}^0$. So, we use the linear operator $\mathcal{K}$ defined in (4.5), to solve (4.7) in the form

$$\boldsymbol{W}^0=\mathcal{K}\mathcal{F}\left(\mu,\boldsymbol{V};\boldsymbol{W}^0\right) \tag{4.8}$$

with respect to $\boldsymbol{W}^0$. A useful observation is now that the non linear term $\boldsymbol{F}(\mu;\boldsymbol{u}^0)$ defined by (2.3) is analytic in $(\mu;\boldsymbol{u}^0)$ for

$$\boldsymbol{u}^0=\left(\alpha_0^0,\alpha_1^0,\beta_0^0,\beta_1^0\right)\in C_{\epsilon,2}\times C_{\epsilon,2}\times C_{\sqrt{\epsilon},1}\times C_{\epsilon,2}\to\boldsymbol{F}(\mu;\boldsymbol{u}^0)\in C_{\epsilon,2},\ \text{and for } |\mu|<\delta.$$

This results in particular from the fact that the product of two continuous functions, one decaying as $1/\left(1+\epsilon x^2\right)$, the other as $1/\left(1+\sqrt{\epsilon}|x|\right)$ decays at least as $1/\left(1+\epsilon x^2\right)$. Hence $\boldsymbol{f}$ and $\boldsymbol{g}$ in (4.1) depend analytically on $\left(\mu,A,B,\overline{A},\overline{B},\boldsymbol{W}^0\right)$, which gives the dependency of $\mathcal{F}\left(\mu,\boldsymbol{V};\boldsymbol{W}^0\right)$ in (4.8). A direct estimation leads to

$$\begin{aligned}
\left|\mathcal{F}\left(\mu,\boldsymbol{V};\boldsymbol{W}^0\right)\right|_{\epsilon,2} \leq\ & c\left\{|A|_{\epsilon,2}|B|_{\epsilon,2}+|B|_{\epsilon,2}^2+|A|_{\epsilon,2}^3\right. \\
&\left.+\left(|\mu|+\|\boldsymbol{W}^0\|\right)\left(|A|_{\epsilon,2}+|B|_{\epsilon,2}+\|\boldsymbol{W}^0\|\right)\right\}
\end{aligned} \tag{4.9}$$

where $\|\boldsymbol{W}^0\|\equiv|\xi_0|_{\epsilon,2}+|\xi_1|_{\epsilon,2}+|\eta_0|_{\sqrt{\epsilon},1}+|\eta_1|_{\epsilon,2}$. For (A,B,μ) in a sufficiently small ball in $(C_{\epsilon,2})^2\times(-\delta,\delta)$ one can solve (4.8) by the implicit function theorem, with respect to $\boldsymbol{W}^0$, and find an estimate

$$\|\boldsymbol{W}^0\|\leq c_\epsilon'\left\{|A|_{\epsilon,2}|B|_{\epsilon,2}+|B|_{\epsilon,2}^2+|A|_{\epsilon,2}^3+|\mu|\left(|A|_{\epsilon,2}+|B|_{\epsilon,2}\right)\right\}, \tag{4.10}$$

For instance, behaviour for $|x|\to\infty$ follows from integration by parts

$$\int_{I\!R} \frac{e^{-ikx}}{(2+|k|)^2}dk = \frac{4}{x}\int_0^\infty \frac{\sin kx}{(2+k)^3}dk = \frac{1}{2x^2} + \mathcal{O}\left(\frac{1}{x^4}\right),$$

$$\lim_{A\to\infty}\int_{-A}^{A} \frac{ike^{-ikx}}{(2+|k|)^2}dk = 4\int_0^\infty \left(\frac{\sin kx}{x^2} - \frac{k\cos kx}{x}\right)\frac{dk}{(2+k)^3} = \frac{1}{x^3} + \mathcal{O}\left(\frac{1}{x^5}\right),$$

$$\lim_{A\to\infty}\int_{-A}^{A} \frac{|k|e^{-ikx}}{(2+|k|)^2}dk = 4\int_0^\infty \left(\frac{k\sin kx}{x} - \frac{2\sin^2\left(\frac{kx}{2}\right)}{x^2}\right)\frac{dk}{(2+k)^3} = -\frac{1}{2x^2} + \mathcal{O}\left(\frac{1}{x^4}\right),$$

$$\int_{I\!R} \frac{-i\,\mathrm{sign}(k)e^{-ikx}}{(2+|k|)^2}dk = -\frac{1}{2x} + \mathcal{O}\left(\frac{1}{x^3}\right).$$

Notice that the second and third functions are defined as L^2 limits in the sense of Fourier–Plancherel transform. The result is that these L^2 limits may be represented by continuous functions.

Continuity in x follows directly from dominated convergence theorem, except near $x=0$ in the second and third integral, where one has to cut into two integrals over $(0,1/x)$ and $(1/x,\infty)$ before applying the dominated convergence theorem. All other multipliers occuring in (4.4) may be treated in the same way.

Let us now introduce spaces of continuous functions on $I\!R$ with an algebraic decay at infinity depending on a positive number ϵ, and an integer p

$$C_{\epsilon,p} = \left\{ u \in C^0 : I\!R \to \mathbb{C} \mid \sup_{x\in I\!R} (1+\epsilon|x|^p)\,|u(x)| < \infty \right\},$$

and we define the corresponding norms

$$|u|_{\epsilon,p} = \sup_{x\in I\!R} (1+\epsilon|x|^p)\,|u(x)|$$

which give these spaces structures of Banach spaces. Now we study the linear mapping $\mathcal{K}$ defined by

$$(f,g,\overline{g}) \mapsto \boldsymbol{W}^0 = \mathcal{K}(f,g,\overline{g}) \qquad (4.5)$$

given by the inverse Fourier transform of (4.3) and (4.4), and where we take (f,g) in $(C_{\epsilon,2})^2$.

Lemma 1 *For any $(f,g)\in (C_{\epsilon,2})^2$, we have the following estimate for some positive number c*

$$|\xi_0|_{\epsilon,2} + |\xi_1|_{\epsilon,2} + |\eta_0|_{\sqrt{\epsilon},1} + |\eta_1|_{\epsilon,2} \le c_\epsilon\left(|f|_{\epsilon,2} + |g|_{\epsilon,2}\right), \qquad (4.6)$$

i.e. $\mathcal{K}$ is continuous from $(C_{\epsilon,2})^3$ into $(C_{\epsilon,2})^2 \times C_{\sqrt{\epsilon},1} \times C_{\epsilon,2}$. The constant c_ϵ satisfies $c_\epsilon \le c(|\ln\epsilon|+1)$, moreover $c_\epsilon \le c$ if we omit η_0.

Proof. Let us first study the convolution product of a $C_{1,2}$ function with a $C_{\epsilon,2}$ function, leading to a $C_{\epsilon,2}$ function. This proves the results for ξ_0, ξ_1 and η_1 since the convolution kernels belong to $C_{1,2}$. We have

$$\begin{aligned}\int_{I\!R} \frac{(1+\epsilon x^2)\,dt}{(1+t^2)\left[1+\epsilon(x-t)^2\right]} &= \pi\frac{(1+\epsilon x^2)\left[(1-\epsilon+\epsilon x^2)+\sqrt{\epsilon}\,(-1+\epsilon+\epsilon x^2)\right]}{\epsilon^2x^4+2\epsilon(1+\epsilon)x^2+(1-\epsilon)^2} \\ &\le \pi(1+\sqrt{\epsilon}) \le 2\pi \text{ for } \epsilon\le 1.\end{aligned}$$

There are two different types of functions appearing on the r.h.s. of (4.1).

$$
\begin{aligned}
\boldsymbol{f}_1 &= \boldsymbol{P}_1 \boldsymbol{f} = f(\mu; \boldsymbol{V}, \boldsymbol{W})\ (0, -1, 0, -2(1+2y)\mathrm{e}^{2y}, -2y\mathrm{e}^{2y}, 0)\,, \\
\boldsymbol{f}_2 &= \boldsymbol{R}_1(4\mathrm{i})\boldsymbol{P}_1\boldsymbol{g} + \boldsymbol{R}_1(-4\mathrm{i})\boldsymbol{P}_1\overline{\boldsymbol{g}} \\
&= g_{\mathrm{re}}(\mu; \boldsymbol{V}, \boldsymbol{W})\ \left(\tfrac{1}{18}, 0, \tfrac{2}{3}y\,\mathrm{e}^{2y} + \tfrac{5}{9}\,\mathrm{e}^{2y} - \tfrac{1}{2}\,\mathrm{e}^{4y}, 0, 0, -\tfrac{4}{3}y\,\mathrm{e}^{2y} - \tfrac{16}{9}\,\mathrm{e}^{2y} + 2\mathrm{e}^{4y}\right) \\
&\quad + g_{\mathrm{im}}(\mu; \boldsymbol{V}, \boldsymbol{W})\ \left(0, -\tfrac{2}{9}, 0, -\tfrac{8}{3}y\,\mathrm{e}^{2y} - \tfrac{20}{9}\,\mathrm{e}^{2y} + 2\,\mathrm{e}^{4y}, -\tfrac{4}{3}y\,\mathrm{e}^{2y} - \tfrac{4}{9}\,\mathrm{e}^{2y} + \tfrac{1}{2}\mathrm{e}^{4y}, 0\right)
\end{aligned}
$$

As we intend to study solutions decaying to zero at infinity, let us apply the Fourier transform. Our study then excludes periodic waves. The solution of the Fourier–transformed system (4.1) can be expressed with the help of the resolvent function.

$$
\widehat{\boldsymbol{W}}(k) = \widehat{\boldsymbol{W}}_1(k) + \widehat{\boldsymbol{W}}_2(k) = \boldsymbol{R}_1(\mathrm{i}k)\widehat{\boldsymbol{f}}_1(k) + \boldsymbol{R}_1(\mathrm{i}k)\widehat{\boldsymbol{f}}_2(k), \quad k \in I\!R \setminus \{0\}. \tag{4.2}
$$

In view of the fact that the nonlinearity depends only on the traces of the components of $\boldsymbol{W}$, it suffices to derive expressions for these traces. The following notation is introduced (cf. (3.2)):

$$
\xi_0 = \theta_0|_{y=0},\ \xi_1 = \theta_1|_{y=0}, \quad \eta_0 = \tau_0|_{y=0},\ \eta_1 = \tau_1|_{y=0},
$$
$$
\boldsymbol{W}^0 = (\xi_0, \xi_1, \eta_0, \eta_1)\,.
$$

System (4.1) can then be formulated as a system for these variables. The full vector $\widehat{\boldsymbol{W}}$ is completely determined by the traces, therefore, it suffices to calculate $(\xi_0, \xi_1, \eta_0, \eta_1)$. Explicit expressions for $\widehat{\boldsymbol{W}}_1(k)$ and $\widehat{\boldsymbol{W}}_2(k)$ are obtained from (4.2).

$$
\widehat{\boldsymbol{W}}{}^0_1(k):\quad
\begin{aligned}
\widehat{\xi}_0(k) &= (2+|k|)^{-2}\,\widehat{f}(k) \\
\widehat{\xi}_1(k) &= \mathrm{i}k\,(2+|k|)^{-2}\,\widehat{f}(k) \\
\widehat{\eta}_0(k) &= -\mathrm{i}\,\mathrm{sign}\,(k)\widehat{\xi}_0(k) \\
\widehat{\eta}_1(k) &= |k|\widehat{\xi}_0(k)
\end{aligned}
\tag{4.3}
$$

$$
\widehat{\boldsymbol{W}}{}^0_2(k):\quad
\begin{aligned}
\widehat{\xi}_0(k) &= \frac{1}{18}\frac{(8+|k|)(-\mathrm{i}k\widehat{g}_{\mathrm{re}}(k) + 4\widehat{g}_{\mathrm{im}}(k))}{(4+|k|)(2+|k|)^2} \\
\widehat{\xi}_1(k) &= \frac{2}{9}\frac{-(4+5|k|)\widehat{g}_{\mathrm{re}}(k) + \mathrm{i}k(8+|k|)\widehat{g}_{\mathrm{im}}(k)}{(4+|k|)(2+|k|)^2} \\
\widehat{\eta}_0(k) &= \frac{1}{18}\frac{4(1-|k|)\widehat{g}_{\mathrm{re}}(k) - \mathrm{i}(k|k| + 8k + 36\,\mathrm{sign}\,(k))\widehat{g}_{\mathrm{im}}(k)}{(4+|k|)(2+|k|)^2} \\
\widehat{\eta}_1(k) &= \frac{2}{9}\frac{\mathrm{i}k(1-|k|)\widehat{g}_{\mathrm{re}}(k) - 4(1-|k|)\widehat{g}_{\mathrm{im}}(k)}{(4+|k|)(2+|k|)^2}
\end{aligned}
\tag{4.4}
$$

We observe from (4.3) and (4.4) that the multipliers on the right hand sides are bounded and continuous for *all* $k \in I\!R$ except at $k = 0$ and are of order $\mathcal{O}(1/|k|)$ for $|k| \to \infty$.

Hence, all multipliers are in $L^2(I\!R)$, and they represent Fourier transforms of $L^2(I\!R)$ functions with which one has to make a convolution product with f or g. In fact all these functions are continuous on $I\!R$ with an algebraic decay at infinity, in $1/x^2$ for all of them except the one acting into the formulas for $\eta_0(x)$, in (4.3) as well as in (4.4) for which the decay is in $1/x$.

Notice that *a very nice and important fact* arises in our problem, since there is *no term in* $|A|^2$ (as well as in $|B|^2$ and AB) *in* $\boldsymbol{N}_{0,2}(\boldsymbol{V},\boldsymbol{V})$. This avoids the problem of resonant terms at the lowest order.

Resolution of system (3.8), (3.9) is classical (see for instance [7]); this allows the determination of the polynomials P and Q appearing in $\boldsymbol{G}(\mu,\boldsymbol{V})$ (see (3.1)). More precisely, the coefficient $Q_{0,10}$ is given by the following formula

$$Q_{0,10} = \left(\boldsymbol{3N}_{03}\left(\boldsymbol{\varphi}_+^0,\boldsymbol{\varphi}_+^0,\boldsymbol{\varphi}_-^0\right) + \boldsymbol{2N}_{0,2}\left(\boldsymbol{\varphi}_-^0,\boldsymbol{\Phi}_{0,2}\left(\boldsymbol{\varphi}_+^0,\boldsymbol{\varphi}_+^0\right)\right),\boldsymbol{\psi}_1^-\right)$$

where

$$\boldsymbol{\Phi}_{0,2}(\boldsymbol{\varphi}_+^0,\boldsymbol{\varphi}_+^0) = (4i-\boldsymbol{L})^{-1}\boldsymbol{N}_{0,2}\left(\boldsymbol{\varphi}_+^0,\boldsymbol{\varphi}_+^0\right).$$

For the calculation of the normal form coefficients and the polynomials $\boldsymbol{\Phi}$, we implemented a symbolic algebra program (MAPLE V). The normal form coefficients are found to be

$$\left\{\begin{array}{llll} P_{1,00}=0, & P_{2,00}=-\frac{1}{4}, & P_{0,10}=\frac{31}{6}, & P_{0,01}=-4, \\ Q_{1,00}=4, & Q_{2,00}=-1, & Q_{0,10}=-22, & Q_{0,01}=\frac{149}{3}. \end{array}\right. \tag{3.10}$$

The fact $Q_{1,00}Q_{0,10} < 0$ is crucial for the existence of solitary waves for the normal form system (for $\mu > 0$ here).

4 Reduction to a system of ordinary differential equations involving nonlocal terms

The normal form algorithm yields a system of differential equations in $(\boldsymbol{V},\boldsymbol{W})$ which is partially decoupled: in the equation for the central component $\boldsymbol{V}$ the hyperbolic component $\boldsymbol{W}$ only enters in the higher order terms. It is desirable to derive an equation for $\boldsymbol{V}$ which is entirely independent of $\boldsymbol{W}$. Such a reduced equation can be obtained by solving the equation for $\boldsymbol{W}$ with $\boldsymbol{V}$ as a parameter and inserting $\boldsymbol{W}=\boldsymbol{W}(\mu;\boldsymbol{V})$ into the equation for $\boldsymbol{V}$. This is reminiscent of a Lyapunov–Schmidt reduction. One drawback of this reduction is the fact that the reduced equation is a nonlocal equation.

Our goal is to solve the hyperbolic part of (3.4) for $\boldsymbol{W}=\boldsymbol{W}(\mu;\boldsymbol{V})$ where the central component is regarded as a parameter. An observation concerns the special shape of the nonlinearity $\boldsymbol{N}(\mu;\boldsymbol{u})$ which depends on the traces of $\boldsymbol{u}$ at $y=0$ only. Moreover, all components of $\boldsymbol{N}$ except the second one vanish. For that purpose we investigate the linearized equation and make use of the specific structure of the inhomogeneous terms. From the normal form algorithm we can show that it takes the form

$$\frac{d}{dx}\boldsymbol{W}(x) = \boldsymbol{L}_1\boldsymbol{W}(x) + \boldsymbol{P}_1\boldsymbol{f}(x) + \boldsymbol{R}_1(4\mathrm{i})\boldsymbol{P}_1\boldsymbol{g}(x) + \boldsymbol{R}_1(-4\mathrm{i})\boldsymbol{P}_1\overline{\boldsymbol{g}}(x), \tag{4.1}$$

where we can restrict our analysis to functions $\boldsymbol{f}$ and $\boldsymbol{g}$ of the following form

$$\boldsymbol{f} = (0,f,0,0,0,0)\,,\ \boldsymbol{g} = (0,g,0,0,0,0)\,,$$

with scalar functions $f(\mu;\boldsymbol{V},\boldsymbol{W})$ and $g(\mu;\boldsymbol{V},\boldsymbol{W}) = g_{\mathrm{re}} + \mathrm{i}g_{\mathrm{im}}$ which *only depend on the trace* $\boldsymbol{W}^0$ *of* $\boldsymbol{W}$*, and on* $\boldsymbol{V}$*, hence do not depend on* y. The symbol $\boldsymbol{R}_1$ denotes the resolvent function, $\boldsymbol{R}_1(\mathrm{i}k) = (\mathrm{i}k-\boldsymbol{L}_1)^{-1}$. Notice that the last two terms are due to the coefficients $\boldsymbol{\Phi}_{0,2000}$ and $\boldsymbol{\Phi}_{0,0020}$ which have nonzero components in the hyperbolic subspace $\boldsymbol{X}_1$.

Proof. According to (3.3) we make the ansatz

$$\begin{aligned} \boldsymbol{u}(x) &= A(x)\boldsymbol{\varphi}_0^+ + B(x)\boldsymbol{\varphi}_1^+ + \overline{A(x)}\boldsymbol{\varphi}_0^- + \overline{B(x)}\boldsymbol{\varphi}_1^- \\ &\quad + \boldsymbol{W} + \boldsymbol{\Phi}(\mu; A(x), B(x), \overline{A(x)}, \overline{B(x)}), \end{aligned} \tag{3.6}$$

where $\boldsymbol{\Phi}^1 = (\boldsymbol{I} - \boldsymbol{P}_0)\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}^0$ represents $\boldsymbol{P}_0\boldsymbol{\Phi}$. The transformation $\boldsymbol{\Phi}$ is written in the form

$$\boldsymbol{\Phi}\left(\mu; A, B, \overline{A}, \overline{B}\right) = \sum_{r+i+j+k+l=2}^{3} \boldsymbol{\Phi}_{r,ijkl}\, \mu^r A^i B^j \overline{A}^k \overline{B}^l$$

with coefficients $\boldsymbol{\Phi}_{r,ijkl} \in D(\boldsymbol{L})$. The nonlinearity in (2.12) is replaced by its Taylor polynomial of order three plus remainder term

$$\begin{aligned} \boldsymbol{N}(\mu; \boldsymbol{u}) &= \boldsymbol{N}_{1,1}(\mu; \boldsymbol{u}) + \boldsymbol{N}_{0,2}(\boldsymbol{u}, \boldsymbol{u}) + \boldsymbol{N}_{1,2}(\mu; \boldsymbol{u}, \boldsymbol{u}) + \boldsymbol{N}_{0,3}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) \\ &\quad + \mathcal{O}\left(|\mu| \, \|\boldsymbol{u}\|_{D(L)}^3 + \|\boldsymbol{u}\|_{D(L)}^4\right), \quad |\mu| + \|\boldsymbol{u}\|_{D(L)} \to 0. \end{aligned}$$

The terms $\boldsymbol{N}_{r,m}(\mu; \cdot, \ldots, \cdot)$ are m–linear *symmetric* mappings from $D(\boldsymbol{L})$ into $\boldsymbol{X}$.

The polynomials P and Q of the normal form depend on eight unknown coefficients which will be determined in the normal form algorithm

$$\begin{aligned} P(\mu; u, v) &= P_{1,00}\, \mu + P_{2,00}\, \mu^2 + P_{0,10} u + P_{0,01} v, \\ Q(\mu; u, v) &= Q_{1,00}\, \mu + Q_{2,00}\, \mu^2 + Q_{0,10} u + Q_{0,01} v. \end{aligned}$$

Inserting (3.6) into (2.1) and taking into account the system (3.4) to be satisfied by $(\boldsymbol{V}, \boldsymbol{W})$ we obtain, by collecting equal powers in $\left(\mu, A, B, \overline{A}, \overline{B}\right)$, linear equations for the Taylor coefficients $\boldsymbol{\Phi}_{r,ijkl}$. Since we are looking for real solutions $\boldsymbol{u}$ of (2.1), $\boldsymbol{\Phi}_{r,ijkl} = \overline{\boldsymbol{\Phi}_{r,klij}}$, so we may restrict ourselves to coefficients with indices $i + j \geq k + l$. In what follows, $\boldsymbol{\Phi}_{r,m}$ means a m-linear symmetric mapping like $\boldsymbol{N}_{r,m}$ above, and a superscript ^{0} means its projection by $\boldsymbol{P}_0$ on the space $\boldsymbol{X}_0$. We obtain the following equations for $\boldsymbol{\Phi}_{0,2}$, $\boldsymbol{\Phi}_{0,3}^0$, $\boldsymbol{\Phi}_{1,1}^0$, on which the compatibility conditions determine the coefficients of polynomials P and Q occuring in $\boldsymbol{G}_{0,3}$ and $\boldsymbol{G}_{1,1}$, as shown in (3.1)

$$2\boldsymbol{\Phi}_{0,2}(\boldsymbol{V}, \boldsymbol{L}_0\boldsymbol{V}) - \boldsymbol{L}\boldsymbol{\Phi}_{0,2}(\boldsymbol{V}, \boldsymbol{V}) = \boldsymbol{N}_{0,2}(\boldsymbol{V}, \boldsymbol{V}) - \boldsymbol{H}(\boldsymbol{V}) \tag{3.7}$$

$$\begin{aligned} &\boldsymbol{G}_{0,3}(\boldsymbol{V}, \boldsymbol{V}, \boldsymbol{V}) + 3\boldsymbol{\Phi}_{0,3}^0(\boldsymbol{V}, \boldsymbol{V}, \boldsymbol{L}_0\boldsymbol{V}) - \boldsymbol{L}_0\boldsymbol{\Phi}_{0,3}^0(\boldsymbol{V}, \boldsymbol{V}, \boldsymbol{V}) = \\ &\qquad = \boldsymbol{P}_0\boldsymbol{N}_{0,3}(\boldsymbol{V}, \boldsymbol{V}, \boldsymbol{V}) + 2\boldsymbol{P}_0\boldsymbol{N}_{0,2}[\boldsymbol{V}, \boldsymbol{\Phi}_{0,2}(\boldsymbol{V}, \boldsymbol{V})] \end{aligned} \tag{3.8}$$

$$\boldsymbol{G}_{1,1}(\mu, \boldsymbol{V}) + \boldsymbol{\Phi}_{1,1}^0(\mu, \boldsymbol{L}_0\boldsymbol{V}) - \boldsymbol{L}_0\boldsymbol{\Phi}_{1,1}^0(\mu, \boldsymbol{V}) = \boldsymbol{P}_0\boldsymbol{N}_{1,1}(\mu, \boldsymbol{V}) \tag{3.9}$$

In this system of three equations, notice that $\boldsymbol{\Phi}_{0,2} = \boldsymbol{\Phi}_{0,2}^0 + \boldsymbol{\Phi}^1$ has a component in $\boldsymbol{X}_1$ only containing the quadratic terms A^2 and $\bar{A}^2$, since

$$\boldsymbol{P}_1\boldsymbol{N}_{0,2}(\boldsymbol{V}, \boldsymbol{V}) - \boldsymbol{H}(\boldsymbol{V}) = \mathbf{h}_1 A^2 + \overline{\mathbf{h}}_1 \overline{A}^2,$$

and

$$\boldsymbol{H}(\boldsymbol{V}) = \mathbf{h}_2 A\overline{B} + \overline{\mathbf{h}}_2 \overline{A} B + \mathbf{h}_3 B^2 + \overline{\mathbf{h}}_3 \overline{B}^2$$

where $\mathbf{h}_j$, $j = 1, 2, 3$ lie in $\boldsymbol{X}_1$.

with real polynomials P and Q. In the finite depth problem, the full b.v.p. is – by center manifold theory – locally equivalent to a system of ordinary differential equations which consists of the normal form system (3.1) plus higher order perturbation terms in $\left(\mu, A, B, \overline{A}, \overline{B}\right)$.

In the present situation of a fluid of infinite depth, a center manifold reduction is not available because of lack of gap in the spectrum of $\boldsymbol{L}_1$ near the imaginary axis. However, normal form theory is applicable to (2.12) with some modifications and restrictions which are due to the non–invertibility of the operator $\boldsymbol{L}_1$. System (2.12) will be transformed into normal form by introducing new coordinates $\boldsymbol{V}$ and $\boldsymbol{W}$

$$\boldsymbol{V} = \left(A, B, \overline{A}, \overline{B}\right)^t \in \mathbb{C}^4, \quad \boldsymbol{W} = (\xi_0, \xi_1, \theta_0, \theta_1, \tau_0, \tau_1) \in D(\boldsymbol{L}_1). \tag{3.2}$$

The central equation of (2.12) can be put into normal form with a remainder term depending on $(\mu, \boldsymbol{V}, \boldsymbol{W})$. As regards the hyperbolic equation of (2.12), it turns out to be useful to remove certain second order terms only.

The result reads as follows.

Theorem 1 *There exist polynomials*

$$\boldsymbol{\Phi}^0 : \mathbb{R} \times \mathbb{C}^4 \to \mathbb{C}^4, \quad \boldsymbol{\Phi}^1 : \mathbb{C}^4 \to D(\boldsymbol{L}_1)$$

of degree three and two with respect to $\boldsymbol{V}$, respectively, such that, by the change of variables

$$\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{\Phi}^0(\mu; \boldsymbol{V}), \ \boldsymbol{w} = \boldsymbol{W} + \boldsymbol{\Phi}^1(\boldsymbol{V}). \tag{3.3}$$

system (2.12) *is transformed into*

$$\begin{cases} \dfrac{d}{dx}\boldsymbol{V}(x) = \boldsymbol{L}_0\boldsymbol{V}(x) + \boldsymbol{G}(\mu; \boldsymbol{V}(x)) + \boldsymbol{R}^0(\mu; \boldsymbol{V}(x), \boldsymbol{W}(x)), \\ \dfrac{d}{dx}\boldsymbol{W}(x) = \boldsymbol{L}_1\boldsymbol{W}(x) + \boldsymbol{H}(\boldsymbol{V}(x)) + \boldsymbol{R}^1(\mu; \boldsymbol{V}(x), \boldsymbol{W}(x)) \end{cases} \tag{3.4}$$

where $\boldsymbol{R}^0 \in \mathbb{C}^4, \boldsymbol{R}^1 \in \boldsymbol{X}_1$

$$\boldsymbol{R}^j(\mu; \boldsymbol{V}, \boldsymbol{W}) = \mathcal{O}\left(\left(|\mu||\boldsymbol{V}|^{1-j} + |\mu|^2 + |\boldsymbol{V}|^{3-j}\right)|\boldsymbol{V}| + \left(|\boldsymbol{V}| + |\mu| + \|\boldsymbol{W}\|_{D(L)}\right)\|\boldsymbol{W}\|_{D(L)}\right)$$

as $|\mu| + |\boldsymbol{V}| + \|\boldsymbol{W}\|_{D(L)} \to 0$. Moreover, $\boldsymbol{H}$ is a quadratic polynomial with

$$\boldsymbol{H}\left(A, 0, \overline{A}, 0\right) = \boldsymbol{0} \tag{3.5}$$

The truncated equation for the central part

$$\frac{d}{dx}\boldsymbol{V}(x) = \boldsymbol{L}_0\boldsymbol{V}(x) + \boldsymbol{G}(\mu; \boldsymbol{V}(x))$$

takes the normal form (3.1) *with real polynomials P and Q of degree three.*

Remark. Since $\boldsymbol{L}_1$ is not invertible, we cannot eliminate the polynomial $\boldsymbol{H}$ completely. However, each monomial of $\boldsymbol{H}$ contains B or $\overline{B}$ as factor. This observation will be crucial in the proof of the existence of solitary waves.

$\boldsymbol{u}_1$ lying in the complementary subspace $\boldsymbol{X}_1$. The components of $\boldsymbol{u}$ can be calculated using the generalized eigenvectors of the adjoint operator as follows.

$$\begin{aligned}
&\boldsymbol{u}_0 = \boldsymbol{P}_0\boldsymbol{u} = a\boldsymbol{\varphi}_+^0 + b\boldsymbol{\varphi}_+^1 + \bar{a}\,\boldsymbol{\varphi}_-^0 + \bar{b}\,\boldsymbol{\varphi}_-^1,\\
&a = \left(\boldsymbol{u}, \boldsymbol{\psi}_0^-\right),\ b = \left(\boldsymbol{u}, \boldsymbol{\psi}_1^-.\right)\\
&\boldsymbol{u}_1 = (\boldsymbol{I} - \boldsymbol{P}_0)\,\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}_0.
\end{aligned}$$

The central component $\boldsymbol{u}_0$ is identified with the complex vector $\boldsymbol{v} = \left(a, b, \bar{a}, \bar{b}\right)^t$ as given above. We write $\boldsymbol{w} = \boldsymbol{P}_1\boldsymbol{u} = (\boldsymbol{I} - \boldsymbol{P}_0)\,\boldsymbol{u}$ for the hyperbolic component.

According to the decomposition of $\boldsymbol{u}$, (2.1) splits into a system of coupled differential equations in $\mathbb{C}^2$ and $\boldsymbol{X}_1$. The projected system is given by

$$\left\{\begin{aligned}
&\frac{d}{dx}\boldsymbol{v}(x) = \boldsymbol{L}_0\boldsymbol{v}(x) + \boldsymbol{N}^0\left(\mu; \boldsymbol{v}(x), \boldsymbol{w}(x)\right),\\
&\frac{d}{dx}\boldsymbol{w}(x) = \boldsymbol{L}_1\boldsymbol{w}(x) + \boldsymbol{N}^1\left(\mu; \boldsymbol{v}(x), \boldsymbol{w}(x)\right)
\end{aligned}\right. \tag{2.12}$$

with

$$\boldsymbol{L}_0 = \begin{pmatrix} 2i & 1 & 0 & 0\\ 0 & 2i & 0 & 0\\ 0 & 0 & -2i & 1\\ 0 & 0 & 0 & -2i \end{pmatrix},\ \boldsymbol{N}^0\left(\mu; \boldsymbol{v}, \boldsymbol{w}\right) = \begin{pmatrix} \left(\boldsymbol{N}\left(\mu; \boldsymbol{u}_0(\boldsymbol{v}) + \boldsymbol{w}\right), \boldsymbol{\psi}_0^-\right)\\ \left(\boldsymbol{N}\left(\mu; \boldsymbol{u}_0(\boldsymbol{v}) + \boldsymbol{w}\right), \boldsymbol{\psi}_1^-\right) \end{pmatrix},$$

$$\boldsymbol{N}^1\left(\mu; \boldsymbol{v}, \boldsymbol{w}\right) = \boldsymbol{P}_1\boldsymbol{N}\left(\mu; \boldsymbol{u}_0(\boldsymbol{v}) + \boldsymbol{w}\right).$$

3 Application of a normal form algorithm

We are adapting a normal form algorithm which was developed by Elphick et al. [6].

For the problem of capillary gravity waves on the free surface of a fluid of *finite* depth h, a normal form in the 1:1 resonance case was calculated by Dias & Iooss [5] and the limit $h \to \infty$ was considered. As regards the mathematics, there are substantial differences between the "finite depth" and the "infinite depth" problems. In the forementioned case, a centre manifold reduction theorem is available which guarantees that all small bounded solutions lie on a center manifold over the linear subspace spanned by the eigenelements and generalized eigenelements corresponding to the eigenvalues with zero real part. In the case of a 1:1 resonance bifurcation, center manifold theory assures that all small bounded solutions can be written in the form

$$\begin{aligned}
\boldsymbol{u}(x) \;=\;& A(x)\boldsymbol{\varphi}_0^+ + B(x)\boldsymbol{\varphi}_1^+ + \overline{A(x)}\boldsymbol{\varphi}_0^- + \overline{B(x)}\boldsymbol{\varphi}_1^-\\
&+ \boldsymbol{\Phi}(\mu; A(x), B(x), \overline{A(x)}, \overline{B(x)}).
\end{aligned}$$

A normal form corresponding to a 1:1 resonance bifurcation problem with reversibility is given in the book of Iooss & Adelmeyer [7], pp.58–59. The normal form reads as follows

$$\left\{\begin{aligned}
A_x &= 2\mathrm{i}A + B + \mathrm{i}AP\left(\mu; A\overline{A}, \tfrac{\mathrm{i}}{2}\left(A\overline{B} - \overline{A}B\right)\right)\\
B_x &= 2\mathrm{i}B + \mathrm{i}BP\left(\mu; A\overline{A}, \tfrac{\mathrm{i}}{2}\left(A\overline{B} - \overline{A}B\right)\right) + AQ\left(\mu; A\overline{A}, \tfrac{\mathrm{i}}{2}\left(A\overline{B} - \overline{A}B\right)\right)
\end{aligned}\right. \tag{3.1}$$

$$\boldsymbol{\varphi}_{\pm}^{0} = y \mapsto \begin{pmatrix} \mp\mathrm{i} \\ 2 \\ \mp\mathrm{i}\mathrm{e}^{2y} \\ 2\mathrm{e}^{2y} \\ \mathrm{e}^{2y} \\ \pm 2\mathrm{i}\mathrm{e}^{2y} \end{pmatrix}, \quad \boldsymbol{\varphi}_{\pm}^{1} = y \mapsto \begin{pmatrix} 0 \\ \mp\mathrm{i} \\ -y\mathrm{e}^{2y} \\ \mp\mathrm{i}\,(1+2y)\,\mathrm{e}^{2y} \\ \mp\mathrm{i}y\mathrm{e}^{2y} \\ (1+2y)\,\mathrm{e}^{2y} \end{pmatrix}. \tag{2.6}$$

The eigenvectors and generalized eigenvectors of the formal adjoint operator $\boldsymbol{L}^*$ of $\boldsymbol{L}$ will be shown to exist. They are needed in the normal form algorithm.

$$D(\boldsymbol{L}^*) := \left\{ \begin{array}{r} \boldsymbol{u} = (\xi_0, \xi_1, \alpha_0, \alpha_1, \beta_0, \beta_1)^t \in \mathbb{C}^2 \times \left[W^{1,1}(-\infty, 0)\right]^4 : \\ \alpha_0|_{y=0} = 0,\ \alpha_1|_{y=0} = -4\xi_1 \end{array} \right\}, \tag{2.7}$$

$$\boldsymbol{L}^* \begin{pmatrix} \xi_0 \\ \xi_1 \\ \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \end{pmatrix} := \begin{pmatrix} 4\xi_1 - \beta_0^0 \\ \xi_0 - \beta_1^0 \\ \beta_0' \\ \beta_1' \\ -\alpha_0' \\ -\alpha_1' \end{pmatrix}. \tag{2.8}$$

The normalized eigenfunctions of $\boldsymbol{L}^*$ corresponding to the eigenvalues $\pm 2\mathrm{i}$ are

$$(\boldsymbol{L}^* \mp 2\mathrm{i})\,\boldsymbol{\psi}_1^{\pm} = \mathbf{o}, \quad (\boldsymbol{L}^* \mp 2\mathrm{i})\,\boldsymbol{\psi}_0^{\pm} = \boldsymbol{\psi}_1^{\pm}, \tag{2.9}$$

$$\boldsymbol{\psi}_1^{\pm} = y \mapsto \begin{pmatrix} 2 \\ \pm\mathrm{i} \\ 0 \\ \mp 4\mathrm{i}\mathrm{e}^{2y} \\ 0 \\ 4\mathrm{e}^{2y} \end{pmatrix}, \quad \boldsymbol{\psi}_0^{\pm} = y \mapsto \begin{pmatrix} \pm\mathrm{i} \\ 0 \\ 0 \\ -4y\mathrm{e}^{2y} \\ 0 \\ \mp 4\mathrm{i}y\mathrm{e}^{2y} \end{pmatrix}. \tag{2.10}$$

The normalization implies the biorthogonality conditions

$$\left(\boldsymbol{\varphi}_{\pm}^{j}, \boldsymbol{\psi}_{k}^{\mp}\right) = \delta_{k}^{j}, \quad \left(\boldsymbol{\varphi}_{\pm}^{j}, \boldsymbol{\psi}_{k}^{\pm}\right) = 0, \tag{2.11}$$

where the scalar product is the one of $\mathbb{C}^2 \times \left[L^2(-\infty, 0)\right]^4$, and where these quantities are well defined due to the exponential decay of eigenfunctions as $y \to -\infty$.

An element $\boldsymbol{u} \in \boldsymbol{X}$ can be decomposed into a "central" component $\boldsymbol{u}_0$ lying in the subspace $\boldsymbol{X}_0$ spanned by the generalized eigenvectors $\boldsymbol{\varphi}_{\pm}^{0}$ and $\boldsymbol{\varphi}_{\pm}^{1}$ and a "hyperbolic" component and similarly for β. Then we can write (1.4) as a differential equation in $\boldsymbol{X} = \mathbb{C} \times \mathbb{C} \times [L_1(-\infty,0)]^4$ of the form

$$\frac{d}{dx}\boldsymbol{u}(x) = \boldsymbol{L}\boldsymbol{u}(x) + \boldsymbol{N}(\mu;\boldsymbol{u}(x)), \quad x \in \mathbb{R}. \tag{2.1}$$

A solution $\boldsymbol{u} : \mathbb{R} \mapsto D(\boldsymbol{L})$ is a mapping from $\mathbb{R}$ into the domain of the linear operator $\boldsymbol{L}$, more precisely we require $\boldsymbol{u} \in C^1(\mathbb{R},\boldsymbol{X}) \cap C^0(\mathbb{R},D(\boldsymbol{L}))$. The derivative in (2.1) is the Fréchet derivative. The linear operator $\boldsymbol{L} : D(\boldsymbol{L}) \to \boldsymbol{X}$ and the nonlinear mapping $\boldsymbol{F} : \mathbb{R} \times D(\boldsymbol{L}) \to \boldsymbol{X}$ are defined as follows

$$D(\boldsymbol{L}) = \left\{ \begin{array}{r} \boldsymbol{u} = (\xi_0,\xi_1,\alpha_0,\alpha_1,\beta_0,\beta_1)^t \in \mathbb{C}^2 \times \left[W^{1,1}(-\infty,0)\right]^4 \; : \\ \xi_0 = \alpha_0|_{y=0}, \; \xi_1 = \alpha_1|_{y=0} \end{array} \right\}, \tag{2.2}$$

$$\boldsymbol{L}\begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \zeta_1 \\ 4\beta_1^0 + 4\zeta_0 \\ \beta_0' \\ \beta_1' \\ -\alpha_0' \\ -\alpha_1' \end{pmatrix}, \qquad \boldsymbol{N}(\mu;\boldsymbol{u}) = \begin{pmatrix} 0 \\ F(\mu;\alpha_0^0,\alpha_1^0,\beta_0^0,\beta_1^0) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.3}$$

The prime $'$ stands for the derivative in y, the superscript ^{0} means evaluation on the upper boundary $y = 0$. The nonlinear function $F(\mu;\alpha_0^0,\alpha_1^0,\beta_0^0,\beta_1^0)$ is given by

$$\begin{aligned} F(\mu;\alpha_0,\alpha_1,\beta_0,\beta_1) &= -\alpha_1\beta_1 - 4\beta_1 + 4\beta_1 \mathrm{e}^{\beta_0} - 4\alpha_0 + 4\mathrm{e}^{-2\beta_0}\sin\alpha_0 + 4\mu\mathrm{e}^{-2\beta_0}\sin\alpha_0 \\ &= \mathcal{O}\left(|(\mu;\alpha_0,\alpha_1,\beta_0,\beta_1)|^2\right), \text{ as } \quad (\mu;\alpha_0,\alpha_1,\beta_0,\beta_1) \to 0. \end{aligned}$$

We shall construct a "normal form" of the system (2.1), which is determined by the spectrum of the linear operator $\boldsymbol{L}$ and symmetry properties. One of the notable features of equation (2.1) is the reversibility w.r.t. the isometry

$$R : (\alpha_0^0,\alpha_1^0,\alpha_0,\alpha_1,\beta_0,\beta_1)^t \mapsto (-\alpha_0^0,\alpha_1^0,-\alpha_0,\alpha_1,\beta_0,-\beta_1)^t. \tag{2.4}$$

Let $\boldsymbol{u}(x)$ be a solution of (2.1) then $\boldsymbol{u}_{\mathrm{R}}(x) = R\boldsymbol{u}(-x)$ is also a solution, since both $\boldsymbol{L}$ and $\boldsymbol{N}$ anticommute with R.

Now we are going to study the spectrum of the linear operator $\boldsymbol{L}$. First set $\boldsymbol{u}(x) = \boldsymbol{u}_0 \mathrm{e}^{\mathrm{i}kx}$ and introduce this into the linearized equation, one obtains the relation $-k^2 + 4|k| - 4 = 0$, i.e. there exist two eigenvalues $\pm 2\mathrm{i}$ of $\boldsymbol{L}$, which can be shown to be double. The full real line constitutes the essential spectrum of $\boldsymbol{L}$. In particular, the closure of the range of $(\boldsymbol{L} - \lambda)$, $\lambda \in \mathbb{R}$, has codimension two, while the kernel is zero. The spectrum at $\lambda = 0$ is reflected by the nonsmoothness of the symbol which contains a term $|k|$. The geometric multiplicity of the two eigenvalues is one and there exists a Jordan chain of length two.

$$(\boldsymbol{L} \mp 2\mathrm{i})\,\boldsymbol{\varphi}_\pm^0 = \mathbf{o}, \quad (\boldsymbol{L} \mp 2\mathrm{i})\,\boldsymbol{\varphi}_\pm^1 = \boldsymbol{\varphi}_\pm^0, \tag{2.5}$$

in the lower complex halfplane, the trace of β on $y=0$ is the Hilbert transform of α evaluated on $y=0$. For the proof of the last statement, we consider the linear boundary value problem

$$\begin{cases} \alpha_x = \beta_y, \quad \beta_x = -\alpha_y, \qquad (x,y) \in I\!R \times (-\infty, 0), \\ \lim_{y\uparrow 0} \alpha(x,y) = \alpha^0(x), \quad \alpha, \beta \to 0, \quad y \to -\infty, \end{cases} \tag{1.5}$$

where the trace α^0 of α is prescribed. If α^0 belongs to $L_2(I\!R)$ for example, then the solution to (1.5) can be found by applying the Fourier transform, solving the transformed system and applying the inverse Fourier transform. One finds that α and β are given by convolution integrals with fundamental solutions as kernels.

$$\begin{cases} \alpha(x,y) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{y}{y^2+(x-\xi)^2}\alpha^0(\xi)\,\mathrm{d}\xi, \\ \beta(x,y) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{x-\xi}{y^2+(x-\xi)^2}\alpha^0(\xi)\,\mathrm{d}\xi. \end{cases} \tag{1.6}$$

The first equation represents Poisson's formula in the halfplane. Taking the limit $y\uparrow 0$ for β we obtain

$$\beta(x,0) = \left(\mathcal{H}\alpha(\cdot,0)\right)(x) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{\alpha^0(\xi)}{x-\xi}\,\mathrm{d}\xi, \tag{1.7}$$

where $\mathcal{H}$ denotes the Hilbert transform. The Fourier transform of $\beta^0=\beta(\cdot,0)$ is given by

$$\hat{\beta^0}(k) = \mathrm{i}\,\mathrm{sign}(k)\hat{\alpha^0}(k), \tag{1.8}$$

which shows that the linearized equation of (1.4) is an integro differential equation with symbol $-k^2+4|k|-4\,(1+\mu)$. Replacing $\beta(x,0)$ by $\mathcal{H}\alpha(x,0)$ we obtain the integro differential equation

$$\alpha_{xx} + \alpha_x\beta_x - 4\beta_x \mathrm{e}^{\beta} - 4\,(1+\mu)\,\mathrm{e}^{-2\beta}\sin\alpha = 0, \quad \beta = \mathcal{H}\alpha. \tag{1.9}$$

Since the zeros of the symbol are $2 \pm 2\sqrt{-\mu}$ and $-2 \pm 2\sqrt{-\mu}$, a 1:1 resonance bifurcation takes place at $\mu = 0$.

In the next section the boundary value problem (1.4) will be put into a form which makes it amenable to the normal form algorithm.

2 Preparatory analysis

We treat the problem in the formulation (1.4). In view of the normal form algorithm to be applied, it is convenient to formulate the boundary value problem as a dynamical system in a Banach space. The horizontal variable x plays the role of the evolutionary variable. We define the vector of unknowns

$$\boldsymbol{u} = \boldsymbol{u}(x) = \left(\alpha_0^0(x), \alpha_1^0(x), \alpha_0(x,\cdot), \alpha_1(x,\cdot), \beta_0(x,\cdot), \beta_1(x,\cdot)\right)^t,$$

where the following identifications are made

$$\alpha_0^0(x) = \alpha(x,0),\ \alpha_1^0 = \alpha_x(x,0), \quad \alpha_0(x,\cdot) = \alpha(x,\cdot),\ \alpha_1(x,\cdot) = \alpha_x(x,\cdot)$$

into the lower halfplane. We set $\zeta = \xi + \mathrm{i}\eta$ and introduce the analytic complex function $w(\zeta) = \phi(\zeta) + \mathrm{i}\psi(\zeta)$ and define two real functions α and β of the complex variable w by

$$w'(\zeta) = U(\zeta) - \mathrm{i}V(\zeta) = \mathrm{e}^{\beta(w(\zeta)) - \mathrm{i}\alpha(w(\zeta))}.$$

Thus, the magnitude of the velocity is given by e^{β} and α is the angle between the velocity and the horizontal line measured in anticlockwise direction. The independent variable ζ is replaced by w, so that the new domain for w is the lower halfplane. As it is shown in the forementioned paper [2], the equations in the transformed variables become

$$\begin{cases} \alpha_{\phi\phi} + \alpha_\phi\beta_\phi - \frac{c^2\rho_0}{T}\beta_\phi \mathrm{e}^{\beta} - \frac{g\rho_0}{T}\mathrm{e}^{-2\beta}\sin\alpha = 0, & \psi = 0, \\ \alpha_\phi = \beta_\psi, \quad \beta_\phi = -\alpha_\psi, & \psi < 0. \end{cases}$$

In order to make all quantities dimensionless, we are scaling the independent variables by introducing a new unit of length ℓ.

$$(\phi, \psi) = (\ell x, \ell y), \quad \ell = \frac{4T}{c^2\rho_0}.$$

A further condition at $\eta = -\infty$ has to be added to the boundary value problem. In order to obtain solutions with physical relevance, we require that both α and β vanish at infinite depth. Equations governing the problem then read as follows

$$\begin{cases} \alpha_{xx} + \alpha_x\beta_x - 4\beta_x\mathrm{e}^{\beta} - 4\,(1+\mu)\,\mathrm{e}^{-2\beta}\sin\alpha = 0, & y = 0, \\ \alpha_x = \beta_y, \quad \beta_x = -\alpha_y, & y < 0, \\ \alpha, \beta \to 0, \quad y \to -\infty \end{cases} \tag{1.4}$$

where the dimensionless parameter μ is given by $\mu = 4\dfrac{gT}{\rho_0 c^4} - 1 \approx 0$. Formally, the fraction coincides with the product $b\lambda$ of the Bond number and the inverse of the square of the Froude number,

$$b = \frac{T}{\rho_0 h c^2}, \quad \lambda = \frac{gh}{c^2}.$$

These two parameters are defined for a fluid of finite depth h. It was shown in [8] that, in the limit $h \to \infty$, the product $b\lambda$ approaches the constant value $\frac{1}{4}$ along a certain branch in the (b, λ) bifurcation diagram where a 1:1 resonance bifurcation occurs. The assumption $|\mu| \ll 1$ agrees with the range of parameter values one obtains in that limit.

We remark that the first equation in (1.4) represents an equation in differentiated form

$$\alpha_x\mathrm{e}^{\beta} - 2\mathrm{e}^{2\beta} - 4(1+\mu)\int_{-\infty}^{0}\left(\mathrm{e}^{-\beta}\cos\alpha - 1\right)\,\mathrm{d}y = \mathrm{const.}$$

Differentiating the above equation w.r.t. x, one obtains the equation at $y = 0$ of (1.4). The trivial solution $(\alpha, \beta) = (0, \mathrm{const})$ is ruled out by the decay condition at $y = -\infty$.

The boundary value problem given above can be viewed as an integro differential equation in x with the trace of α on $y = 0$ as the unknown function. Since $\alpha + \mathrm{i}\beta$ is an analytic function as the Benjamin–Ono equation in the literature, admits an explicit solitary wave solution decaying algebraically.

The full Euler equations for a two fluid system, with one infinite layer, were considered by Amick [1] and Sun [16]. They independently proved, by a fixed point technique, the existence of a solitary wave, near such a solitary solution of the Benjamin-Ono equation. Recently, Benjamin proposed in [4] an approximate model equation for the interface problem of a two fluid system, the lower being of infinite depth, the interface being subject to capillarity. The linear singularity in this problem is the same as the one we are treating below, but a mathematical proof of the existence of solitary waves has still to be given.

The existence of solitary capillary gravity waves on the free surface of a fluid of large but finite depth (i.e. for small Bond number and Froude number less than one) was proved by Iooss & Kirchgässner [8]. Their analysis was extended by Dias & Iooss [5] who also considered the limit from finite depth to infinite depth.

The subject of the present work is to provide a rigorous existence proof for solitary capillary gravity waves on deep water.

The investigation is confined to waves of permanent form moving with constant velocity c from the right to the left on the free surface of an inviscid, incompressible fluid of uniform density $\rho = \rho_0$.

In a moving frame of reference with coordinates

$$(\xi, \eta) = (X + ct, Y), \qquad \xi \in I\!R, \quad -\infty < \eta < Z(\xi),$$

where $Z(\xi)$ is the free surface, the flow is steady and the velocity field for the undisturbed fluid is the uniform flow $(c, 0)$. We consider flows which are close to this uniform flow and denote the perturbation by (cU, cV) so that U and V are dimensionless quantities, i.e.

$$u(t, X, Y) = cU(\xi, \eta) - c, \quad v(t, X, Y) = cV(\xi, \eta).$$

From the equation of continuity and the assumption that the flow is irrotational it follows that the flow has a potential ϕ. The potential ϕ and the stream function ψ are given by

$$(U, V) = (\phi_\xi, \phi_\eta) = (\psi_\eta, -\psi_\xi). \tag{1.1}$$

Bernoulli's equation ensures that

$$\frac{1}{2}\rho_0 c^2 \left(U^2 + V^2\right) + p + \rho_0 g \eta = \text{const} \tag{1.2}$$

holds everywhere, where p is the pressure and g stands for the acceleration due to gravity. At large depth $|\eta| \gg 1$ the pressure p is proportional to $\rho_0 g \eta$.

The free surface $\eta = Z(\xi)$ is a streamline corresponding to $\psi = 0$, say; therefore, it satisfies

$$UZ_\xi - V = 0, \quad \eta = Z(\xi), \; -\infty < \xi < \infty. \tag{1.3}$$

On the free surface the jump in the pressure is proportional to the curvature

$$p(\xi) = -T \frac{Z_{\xi\xi}(\xi)}{\left(1 + Z_\xi^2(\xi)\right)^{3/2}}$$

where the factor of proportionality is the constant of surface tension T. The problem is studied in a hodograph form (cf [2]) where the coordinates (ϕ, ψ) are used to map the unknown domain

$$\{\, (\xi, \eta) \in I\!R^2 \; : \; -\infty < \eta < Z(\xi), \, -\infty < \xi < \infty \,\}$$

Capillary Gravity Waves on the Free Surface of an Inviscid Fluid of Infinite Depth –Existence of Solitary Waves–

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Abstract

Permanent capillary gravity waves on the free surface of a two dimensional inviscid fluid of infinite depth are investigated. Applying the hodograph transform, the free boundary value problem becomes a boundary value problem for the Cauchy–Riemann equations in the lower halfplane with nonlinear differential boundary conditions. It can be converted to an integro differential equation with symbol $-k^2+4|k|-4(1+\mu)$, where μ is a bifurcation parameter. A normal form analysis is presented which shows that the boundary value problem can be reduced to an integrable system of ordinary differential equations plus a remainder term containing nonlocal terms of higher order for $|\mu|$ small. This normal form system has been studied thoroughly by several authors (Iooss & Kirchgässner [8], Iooss & Pérouème [10], Dias & Iooss [5]). It admits a pair of solitary wave solutions which are reversible in the sense of Kirchgässner [11]. Applying a method introduced in [11] it is shown that this pair of reversible solitary waves persists for the boundary value problem, and that the decay at infinity of these solitary waves is at least like $1/|x|$.

Key words Capillary gravity waves, solitary waves, normal form.
AMS(MOS) subject classifications 35J65, 76B15, 76B25, 86A05.

1 Statement of the problem

One of the open problems in the area of two dimensional water wave problems is the question of existence of steady capillary gravity waves of solitary type on deep water. It was conjectured by Longuet–Higgins [12] that such waves indeed exist. Steady capillary gravity solitary waves were calculated numerically by the same author in [13]. He observed that these waves do not decay exponentially but only quadratically with the inverse of the distance from the origin in a moving frame of reference. A boundary integral equation technique was used by Vanden–Broeck & Dias [17] to compute both free and forced capillary gravity waves numerically.

There are also numerous papers dealing with solitary waves in a two fluid system where the lower layer is infinitely deep. A model equation for permanent waves in a system of stratified fluids when capillarity is negligible was derived by Benjamin [3]. The corresponding time–dependent equation was derived later by Ono [15]. This model equation, which is known

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