Asymptotic expansions of vibrations with small unilateral contact

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Abstract We study some spring mass models for a structure having a small unilateral contact with a small parameter ε. We valid an asymptotic expansion with the method of strained coordinates with new tools to handle such defects, including a non negligible cumulative effect over a long time: $T_\varepsilon \sim \varepsilon^{-1}$ as usual; or, for a new critical case, we can only expect: $T_\varepsilon \sim \varepsilon^{-1/2}$.

1 Introduction

For spring mass models, the presence of a small piecewise linear rigidity can model a small defect which implies unilateral reactions of the structure. For nondestructive testing we study a such singular nonlinear effect for large time by asymptotic expansion of the vibrations. New features and comparisons with classical cases of smooth perturbations are given, for instance for the Duffing equations: $\ddot{u} + u + \varepsilon u^3 = 0$. Indeed, piecewise non linearity is singular, lipschitz but not differentiable. We give some new results to validate such asymptotic expansions. Furthermore, these tools are also valid for a more general piecewise non linearity.
For short time, a linearization procedure is enough to compute a good approximation. But for large time, nonlinear cumulative effects drastically alter the nature of the solution. We will consider the classical method of strained coordinates to compute asymptotic expansions. The idea goes further back to Stokes, who in 1847 calculated periodic solutions for a weakly nonlinear wave

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propagation problem. Subsequent authors have generally referred to this as the method of Poincaré or the Lindstedt's method. It is a simple and efficient method which gives us approximate nonlinear normal modes with 1 or more degrees of freedom.
In section 2 we present the method on an explicit case with lipschitz force. We focus on an equation with one degree of freedom with expansions valid for time of order ε^{-1} or, more surprisingly, $\varepsilon^{-1/2}$. Section 3 contains a tool to expand $(u+\varepsilon v)_+$ and some accurate estimate for the remainder. This is a new key point to validate the method of strained coordinates with unilateral contact. In Section 4, we extend previous results for systems with N degrees of freedom, first, with the same accuracy for approximate nonlinear normal modes, second, with less accuracy with all modes. Section 5 is an appendix containing some technical proofs and results.

2 One degree of freedom

2.1 Explicit pulsation

We replace in the Duffing equation u^3 by the piecewise linear term $u_+ = \max(0, u)$.

$$\ddot{u} + \omega_0^2 u + \varepsilon u_+ = 0, \tag{1}$$

where ω_0 a positive constant. This case has got a conserved energy E: $\dot{E} = 0$,

Fig. 1 Two springs, one on the right with an unilateral contact.

where $2E = \dot{u}^2 + \omega_0^2 u^2 + \varepsilon (u_+)^2$. Therefore, the level sets of $E(u, \dot{u})$ will be made of two half ellipses. Indeed, for $u < 0$ the level set is an half ellipse, and for $u > 0$ is another half ellipse. Any solution $u(t)$ is confined to a closed level curve of $E(u, \dot{u})$ and is necessarily a periodic functions of t.
More precisely, a non trivial solution ($E > 0$) is on the half ellipse: $\dot{u}^2 + \omega_0^2 u = 2E$, in the phase plane during the time $T_C = \pi/\omega_0$, and on the half ellipse $\dot{u}^2 + (\omega_0^2 + \varepsilon)u = 2E$ during the time $T_E = \pi/\sqrt{\omega_0^2 + \varepsilon}$. The period $P(\varepsilon)$ is then
$P(\varepsilon) = (1 + \left(1 + \varepsilon/\omega_0^2\right)^{-1/2})\pi/\omega_0$, and the exact pulsation is:

$$\omega(\varepsilon) = 2\omega_0(1 + \left(1 + \varepsilon/\omega_0^2\right)^{-1/2})^{-1} = \omega_0 + \frac{\varepsilon}{(4\omega_0)} - \frac{\varepsilon^2}{(8\omega_0^3)} + \mathcal{O}(\varepsilon^3). \tag{2}$$

Let us compare with the pulsation for Duffing equation which depends on the amplitude a_0 of the solution: $\omega_D(\varepsilon) = \omega_0 + \frac{3}{8\omega_0^2} a_0^2 \varepsilon - \frac{15}{256\omega_0^4} a_0^4 \varepsilon^2 + \mathcal{O}(\varepsilon^3)$.

2.2 The method of strained coordinates

Now, we compute, with the method of strained coordinates, ω_ε, *an approximation* of the exact pulsation $\omega(\varepsilon)$. We expose completely this case to use the same method further when we will not have such explicit pulsation.
Let us define the new time $s = \omega_\varepsilon t$ and the following notations:

$$\omega_\varepsilon = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2, \quad \omega_\varepsilon^2 = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \mathcal{O}(\varepsilon^3) \tag{3}$$

$$\alpha_0 = \omega_0^2, \ \alpha_1 = 2\omega_0\omega_1, \quad \alpha_2 = \omega_1^2 + 2\omega_0\omega_2. \tag{4}$$

The unknowns are ω_1, ω_2 or α_1, α_2. Replacing the solution of (1) by the following anzatz with the following initial data to simplify the exposition:

$$\begin{aligned} u_\varepsilon(t) &= v_\varepsilon(\omega_\varepsilon t) + \varepsilon^2 r_\varepsilon(\omega_\varepsilon t), \\ v_\varepsilon(s) &= v_0(s) + \varepsilon v_1(s), \qquad \text{where } s = \omega_\varepsilon t, \\ u_\varepsilon(0) &= a_0 > 0, \qquad \dot{u}_\varepsilon(0) = 0, \end{aligned}$$

then, we obtain initial data and next differential equations for v_0, v_1, r_ε:

$$v_0(0) = a_0 \, \dot{v}_0(0) = 0, \quad 0 = v_1(0) = \dot{v}_1'(0), \quad 0 = r_\varepsilon(0) = \dot{r}_\varepsilon(0).$$

We use the natural expansion: $(u + \varepsilon v)_+ = u_+ + \varepsilon H(u) v + \cdots$, where H is the Heaviside function, equal to 1 if $u > 0$ and else 0, (see Lemma 3.1 below).

$$\ddot{v}_0 + v_0 = 0, \tag{5}$$

$$-\alpha_0(\ddot{v}_1 + v_1) = (v_0)_+ + \alpha_1 \ddot{v}_0, \tag{6}$$

$$-\alpha_0(\ddot{r}_\varepsilon + r_\varepsilon) = H(v_0)v_1 + \alpha_2\ddot{v}_0 + \alpha_1\ddot{v}_1 + R_\varepsilon(s). \tag{7}$$

We now compute, α_1, v_1 and then α_2. We have $v_0(s) = a_0 \cos(s)$. A key point in the method of strained coordinates is to keep bounded v_1 and r_ε for large time by a choice of α_1 for u_1 and α_2 for r_ε. For this purpose, we avoid resonant or **secular** term in the right-hand-side of equations (6), (7). Let us first focus on α_1. Notice that, $u_+ = \dfrac{u}{2} + \dfrac{|u|}{2}$. $|\cos(s)|$ has no term with frequencies ± 1, since there are only even frequencies. Thus $-\alpha_0((v_0)_+ - \alpha_1 v_0) = a_0 \cos(s)(1/2 - \alpha_1) + a_0|\cos(s)|/2$ has no secular term if and only if $\alpha_1 = 1/2$, $\omega_1 = 1/(4\omega_0)$. Now, v_1 satisfies:

$$-\omega_0^2(\ddot{v}_1 + v_1) = |v_0|/2, \qquad v_1(0) = 0, \quad \dot{v}_1(0) = 0.$$

To remove secular term in the equation (7) we have to obtain the Fourier expansion for $H(v_0)$ and v_1. Some computations give us:

$$|\cos(s)| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{+\infty}\frac{(-1)^k}{4k^2-1}\cos(2ks),$$
$$v_1(s) = \frac{-a_0}{\omega_0^2}\left(\frac{1}{\pi} - \frac{2}{\pi}\sum_{k=1}^{+\infty}\frac{(-1)^k}{(4k^2-1)^2}\cos(2ks)\right),$$
$$H(v_0) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{+\infty}\frac{(-1)^j}{2j+1}\cos((2j+1)s)$$

To remove secular term of order one in (7), it suffices to take α_2 such that:

$$0 = \int_0^{2\pi}[H(v_0(s))v_1(s) + \alpha_2\ddot{v}_0(s) + \alpha_1\ddot{v}_1(s)]\cdot v_0 ds. \tag{8}$$

For Duffing equation, see [6, 7, 8], the source term involve only few complex exponentials and the calculus of α_2 is explicit. For general smooth source term, Fourier coefficients decay very fast. Here, we have an infinite set of frequencies for v_1 and $H(v_0)$, with only a small algebraic rate of decay for Fourier coefficients. So, numerical computations need to compute more Fourier coefficients. For our first simple example, we can compute explicitly α_2. After lengthy and tedious computations involving numerical series, we obtain $\alpha_2 = -3(4\omega_0)^{-2}$, thus $\omega_2 = -(2\omega_0)^{-3}$ as we have yet obtained in (2). More generally, we have:

Proposition 2.1. *Let u_ε be the solution of (1) with $u_\varepsilon(0) = a_0 + \varepsilon a_1$, $\dot{u}_\varepsilon(0) = 0$, then, there exists $\gamma > 0$, such that, for all $t < T_\varepsilon = \gamma\varepsilon^{-1}$:*

$$u_\varepsilon(t) = v_0(\omega_\varepsilon t) + \varepsilon v_1(\omega_\varepsilon t) + \mathcal{O}(\varepsilon^2), \qquad \omega_\varepsilon = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2,$$

where $v_0(s) = a_0\cos(s)$, $\ddot{v}_1 + v_1 = -\frac{|v_0|}{2\omega_0^2}$, $v_1(0) = a_1$, $\dot{v}_1(0) = 0$.
$\omega_1 = 1/4\omega_0$ and ω_2 is given by α_2 thanks equations (8), (3).

Remarks:
a new critical case: we give another simple example, with an asymptotic expansion only valid for time of order $\frac{1}{\sqrt{\varepsilon}}$. Consider, the solution u_ε of:

$$\ddot{u} + u + \varepsilon(u-1)_+ = 0,\ u_\varepsilon(0) = 1+\varepsilon,\ \dot{u}_\varepsilon(0) = 0. \tag{9}$$

The method of strained coordinates gives us the following approximation for $u_\varepsilon(t)$: $v_\varepsilon(t) = (1+\varepsilon)\cos(t)$ for $t \le T_\varepsilon$. This system has got an energy: $2E_\varepsilon = \dot{u}^2 + u^2 + \varepsilon[(u-1)_+]^2$. Since, 1 is the maximum of $v_0(t) = \cos(t)$, a new phenomenon appears, during each period, $u_\varepsilon > 1$ for interval of time of order $\sqrt{\varepsilon}$ instead of ε, and then T_ε is smaller and of order $\frac{1}{\sqrt{\varepsilon}}$. To explain this new phenomenon, we give precise estimates of the remainder when we expand $(v_0 + \varepsilon v_1 + \varepsilon^2 r_\varepsilon)_+$ in the next section.

Nonlinear dependence of pulsation with respect to the amplitude : Previous examples have pulsation independent of the amplitude. It is not always the case, as we can see on following case. Let b be a real number and consider, the solution u_ε of:

$$\ddot{u} + u + \varepsilon(u-b)_+ = 0, \;\; u_\varepsilon(0) = a_0 > |b|, \;\; \dot{u}_\varepsilon(0) = 0. \tag{10}$$

At the first order, the method of strained coordinates gives us following equations:

$$\ddot{v}_0 + v_0 = 0, \qquad -\alpha_0(\ddot{v}_1 + v_1) = (v_0 - b)_+ + \alpha_1 \ddot{v}_0 + O(\varepsilon).$$

Then $v_0(s) = a_0 \cos(s)$ and α_1 satisfies following equation:

$$\alpha_1 = \frac{1}{\pi}\int_0^{2\pi} (a_0\cos(s) - b)_+ \cos(s) ds = \frac{a_0}{2\pi}\left(2\beta + \sin(2\beta) - 4b\sin(\beta)\right),$$
$$\beta = \beta(b, a_0) = \arccos\left(\frac{b}{a_0}\right) \in [0, \pi].$$

Notice the nonlinear dependence of $\omega_1 = \alpha_1/2$ with respect to b and a_0. Furthermore, at the first order, and for time of the order ε^{-1}, we have: $u_\varepsilon(t) = a_0\cos((1 + \varepsilon\alpha_1/2)t) + \mathcal{O}(\varepsilon)$.

3 Expansion of $(u + \varepsilon v)_+$

We give some useful lemmas to make asymptotic expansions and to estimate precisely the remainder for the basic piecewise linear map $u \to u_+ = \max(0, u)$.

Lemma 3.1. [Asymptotic expansion for $(u+\varepsilon v)_+$ **]** *Let be $T > 0$, $M > 0$, u, v two real valued functions defined on $I = [0, T]$, $J_\varepsilon = \{t \in I, |u(t)| \leq \varepsilon M\}$, $\mu_\varepsilon(T)$ the measure of the set J_ε and H is the Heaviside step function, then*

$$(u + \varepsilon v)_+ = (u)_+ + \varepsilon H(u) v + \varepsilon \chi_\varepsilon(u, v), \;\; \textit{with} \;\; H(u) = \begin{cases} 1 & \textit{if } u > 0 \\ 0 & \textit{else} \end{cases},$$

where $\chi_\varepsilon(u, v)$ is a non negative piecewise linear function and 1-Lipschitz with respect to v, which satisfies for all ε, If $|v(t)| \leq M$ for any $t \in I$,:

$$|\chi_\varepsilon(u, v)| \leq |v| \leq M, \;\; \int_0^T |\chi_\varepsilon(u(t), v(t))|\, dt \leq M\mu_\varepsilon(T). \tag{11}$$

The point in inequality (11) is the remainder $\varepsilon\chi_\varepsilon$ is only of order ε in L^∞ but of order $\varepsilon\mu_\varepsilon$ in L^1. In general, μ_ε is not better than a constant, take for instance $u \equiv 0$. Fortunately, it is proved below that μ_ε is often of order ε, and for some critical cases of order $\sqrt{\varepsilon}$.

Proof : Equality (11) defines χ_ε and can be rewritten as follow:

$$\chi_\varepsilon(u,v) = \frac{(u+\varepsilon v)_+ - u_+ - \varepsilon H(u)v}{\varepsilon}. \tag{12}$$

So, χ_ε is non negative since $u \to u_+$ is a convex function. We also easily see that the map $(u,v) \to \chi_\varepsilon(u,v)$ is piecewise linear, continuous except on the line $u = 0$ where χ_ε has a jump $-v$. This jump comes from the Heaviside step function. An explicit computations gives us the simple and useful formula: $0 \leq \varepsilon\chi_\varepsilon(u,v) = \begin{cases} |u+\varepsilon v| & \text{if } |u+\varepsilon v| < |\varepsilon v| \\ 0 & \text{else} \end{cases}$ .We then have immediately $0 \leq \chi_\varepsilon(u,v) \leq |v|$. Let u be fixed, then $v \to \chi_\varepsilon(u,v)$ is one Lipschitz with respect to v. Furthermore, the support of χ_ε is included in J_ε, which concludes the proof. □

Now, we investigate the size of $\mu_\varepsilon(T)$ with notations of Lemma 3.1.

Lemma 3.2 (Order of $\mu_\varepsilon(T)$). *Let u be a smooth periodic function. If u has only simple roots on $I = [0,T]$, then , for some positive C: $\mu_\varepsilon(T) \leq C\varepsilon T$. More generally, if u has also double roots then $\mu_\varepsilon(T) \leq C\sqrt{\varepsilon}T$.*

Notice that any non zero solution of any linear homogeneous second order ordinary differential equation has always simple zeros.

Proof : First assume u only has simple roots on a period $[0,P]$, and let $Z = \{t_0 \in [0,P], u(t_0) = 0\}$. A well known result state that Z is a discret set since u has only simple roots. Thus Z is a finite subset of $[0,P]$: $Z = \{t_1, t_2, \cdots, t_N\}$. We can choose an open neighborhood V_j of each t_j such that u is a diffeomorphism on V_j with derivative $|\dot{u}| > |\dot{u}(t_j)|/2$. On the compact set $K = [0,P] - \cup V_j$, u never vanishes, then $\min_{t\in K} |u(t)| = \varepsilon_0 > 0$. Thus, we have for all $\varepsilon M < \varepsilon_0$, the length of J_ε in V_j is $|V_j \cap J_\varepsilon| \leq \dfrac{4\varepsilon M}{|\dot{u}(t_j)|}$. μ_ε is additive: $\mu_\varepsilon(P+t) = \mu_\varepsilon(P) + \mu_\varepsilon(t)$ which give the linear growth of $\mu_\varepsilon(T) = \mathcal{O}(\varepsilon T)$ for the case with simple roots.

For the general case, on each small neighborhood of t_j: V_j, we have with a Taylor expansion, $|u(t_j + s)| \geq d_j|s|^l$, with $1 \leq l \leq 2$, $d_j > 0$, so, $|V_j \cap J_\varepsilon| \leq 2(\varepsilon M/d_j)^{1/l}$, then $\mu_\varepsilon(P) = \mathcal{O}(\sqrt{\varepsilon})$,which is enough to conclude the proof. □

4 N degrees of freedom

$$M\ddot{U} + KU + \varepsilon(AU - B)_+ = 0, \text{ where } [(AU - B)_k]_+ = \left(\sum_{j=1}^{N} a_{kj}u_j - b_k\right)_+,$$

M is the diagonal mass matrix with positive term on the diagonal, K is the stiffness matrix which is symmetric definite positive. For the term $\varepsilon(AU - B)_+$, modeling small defect, it is possible to add many of such terms. For a such system, endowed with a natural energy for the linearized part, we control the ε-Lipschitz last term, and the solutions remain bounded for all time. Without loosing generality, with a change of variables, we deal with following diagonalized system for the linear part, keeping the same notation, except for the positive diagonal matrix Λ:

$$\ddot{U} + \Lambda^2 U + \varepsilon(AU - B)_+ = 0, \tag{13}$$

4.1 Nonlinear normal mode, second order approximation

For the system (13) with an initial condition on an eigenmode of the linearized system: $u_1^\varepsilon(0) = a_0 + \varepsilon a_1$, $\dot{u}_1^\varepsilon(0) = 0$ and, for $k \neq 1$: $u_k^\varepsilon(0) = 0$, $\dot{u}_k^\varepsilon(0) = 0$. Using the same time $s = \omega_\varepsilon t$ for each component and following notations:

$$\omega_\varepsilon = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2, \qquad \omega_0 = \lambda_1, \qquad (\omega_\varepsilon)^2 = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \mathrm{O}(\varepsilon^3),$$
$$u_j^\varepsilon(t) = v_j^\varepsilon(s), \qquad v_j^\varepsilon(s) = v_j^0 + \varepsilon v_j^1 + \varepsilon^2 r_j^\varepsilon, \qquad j = 1, \cdots, N.$$

Replacing, this anzatz in the System (13) we have in variable s:

$$(\omega_\varepsilon)^2 \ddot{v}_k^\varepsilon + \lambda_k^2 v_k^\varepsilon = -\varepsilon \left(\sum_{j=1}^{N} a_{kj} v_j^\varepsilon(s) - b_k\right)_+,$$

$$L_k v_k^0 = \alpha_0 \ddot{v}_k^0 + \lambda_k^2 v_k^0 = 0, \quad -L_k v_k^1 = \left(\sum_{j=1}^{N} a_{kj} v_j^0 - b_k\right)_+ + \alpha_1 \ddot{v}_k^0 = S_k^1,$$

$$-L_k r_k^\varepsilon = H\left(\sum_{j=1}^{N} a_{kj} v_j^0 - b_k\right)\left(\sum_{j=1}^{N} a_{kj} v_j^1\right) + \alpha_2 \ddot{v}_k^0 + \alpha_1 \ddot{v}_k^1 + \varepsilon \cdots = S_k^2 + \varepsilon \cdots$$

Equations for v_k^0, for all $k \neq 1$, with zero initial data give us $v_k^0 = 0$. In equation for v_1^1, we remove the secular term for the right hand side. If $b_1 = 0$ we have $\omega_1 = \dfrac{a_{11}}{4\lambda_1}$. Then, for $k \neq 1$, we can compute v_k^1 since: $\alpha_0 \ddot{v}_k^1 + \lambda_k^2 v_k^1 = -\left(a_{k1} v_1^0 - b_k\right)_+$. v_k^1 is a 2π-periodic bounded function since $\lambda_k \neq \lambda_1$.

Simplifying equation for r_1^ε we can compute numerically α_2 and then ω_2 as in the Propoposition 2.1. Then we check, for all $k \neq 1$ that r_k^ε stay bounded for large time since there is no resonance of order one. We have obtained following results with previous notations:

Theorem 4.1. *The following expansion of the nonlinear normal mode is valid on $(0, T_\varepsilon)$, under assumption $\lambda_1 \notin \lambda_k \mathbb{Z}$ for all $k \neq 1$:*

$$u_1^\varepsilon(t) = v_1^0(\omega_\varepsilon t) + \varepsilon v_1^1(\omega_\varepsilon t) + \mathcal{O}(\varepsilon^2), \;\; u_k^\varepsilon(t) = 0 + \varepsilon v_k^1(\omega_\varepsilon t) + \mathcal{O}(\varepsilon^2),$$

where $v_1^0(s) = a_0 \cos(s)$, and ω_1, v_1^1, v_k^1, ω_2 are given by following equations, in the sense that we compute successively α_1, ω_1, v_1^1, v_k^1, α_2, ω_2:

$$\begin{aligned}
&0 = \int_0^{2\pi} S_1^1 \cdot v_1^0 ds, \quad \textit{where } S_1^1 = (a_{11} v_1^0 - b_1)_+ + \alpha_{\mathbf{1}} \ddot{v}_1^0, \\
&-L_1 v_1^1 = \left(a_{11} v_1^0 - b_1\right)_+ + \alpha_1 \ddot{v}_1^0 = S_1^1, \quad v_1^1(0) = a_1, \; \dot{v}_1^1(0) = 0, \\
&-L_k v_k^1 = \left(a_{k1} v_1^0 - b_k\right)_+ = S_k^1, \quad v_k^1(0) = 0, \; \dot{v}_k^1(0) = 0, \; \textit{for } k \neq 1, \\
&0 = \int_0^{2\pi} S_1^2 \cdot v_1^0 ds \quad \textit{where } S_1^2 = H(a_{11} v_1^0 - b_1) \left(\sum_{j=1}^N a_{1j} v_j^1 \right) + \alpha_{\mathbf{2}} \ddot{v}_1^0 + \alpha_1 \ddot{v}_1^0.
\end{aligned}$$

Furthermore, if $(a_{j1} v_1^0 - b_j)$ has got only simple roots for all $j = 1, \cdots, N$, then T^ε is of order ε^{-1}, else T^ε is of order $\varepsilon^{-1/2}$.

4.2 First order asymptotic expansion

The method of strained coordinates is used for each normal component, with general initial data $u_k^\varepsilon(0) = a_k$, $\dot{u}_k^\varepsilon(0) = 0$ and, with following anzatz:

$$\lambda_k^\varepsilon = \lambda_k^0 + \varepsilon \lambda_k^1, \quad \lambda_k^0 = \lambda_k, \quad u_k^\varepsilon(t) = v_k^\varepsilon(s_k) \text{ where } s_k = \lambda_k^\varepsilon t, v_k^\varepsilon(s) = v_k^0 + \varepsilon r_k^\varepsilon.$$

Replacing, this anzatz in the system (13) we have:

$$\begin{aligned}
(\lambda_k^\varepsilon)^2 \ddot{v}_k^\varepsilon(s_k) + \lambda_k^2 v_k(s_k) &= -\varepsilon \left(\sum_{j=1}^N a_{kj} v_j^\varepsilon \left(\frac{\lambda_j^\varepsilon}{\lambda_k^\varepsilon} s_k \right) - b_k \right)_+, \\
L_k v_k^0 &= (\lambda_k^0)^2 \ddot{v}_k^0(s_k) + \lambda_k^2 v_k^0(s_k) = 0, \\
-L_k r_k^\varepsilon(s_k) &= \left(\sum_{j=1}^N a_{kj} v_j^0 \left(\frac{\lambda_j^0}{\lambda_k^0} s_k \right) - b_k \right)_+ + 2\lambda_k \lambda_k^1 \ddot{v}_k^0 + \varepsilon(\cdots) \equiv S_k^1 + \varepsilon(\cdots).
\end{aligned}$$

If $b_k = 0$, we identify the secular term with the Lemma 5.4 since $S_+ = S/2 + |S|/2$. Then, we remove the resonant term in the source term for the remainder r_k^ε, which gives us $\lambda_k^1 = \dfrac{a_{kk}}{4\lambda_k}$. If $b_k \neq 0$, we compute λ_k^1 numerically.

Noting that, replacing $v_j^\varepsilon(s_j)$ by $v_j^0\left(\frac{\lambda_j^0}{\lambda_k^0}s_k\right)$ implies a secular term of order εt. Since the map $S \to S_+$ is one-Lipschitz, the error goes to the right-hand side of equation (14). Furthermore, $r_k^\varepsilon = r_k^\varepsilon(s_1, \cdots, s_N)$, and the method of strained coordinates is not valid to get u_k^1 and λ_k^2. Nevertheless, we obtain:

Theorem 4.2. *If $\lambda_1, \cdots, \lambda_N$ are $\mathbb{Z}$ independent, then, for all k, for $t < T_\varepsilon \sim \varepsilon^{-1}$:*

$$u_k^\varepsilon(t) = v_k^0(\lambda_k^\varepsilon\, t) + \mathcal{O}(\varepsilon) \qquad \text{where } \lambda_k^\varepsilon = \lambda_k + \varepsilon\lambda_k^1,$$

where $v_k^0(s) = a_k \cos(s)$, and λ_k^1 is defined by the equation:

$$0 = \int_0^{2\pi} \left[\left(\sum_{j=1}^N a_{kj} v_j^0 \left(\frac{\lambda_j^0}{\lambda_k^0} s_k \right) - b_k \right)_+ + 2\lambda_k \lambda_{\mathbf{k}}^{\mathbf{1}} \ddot{v}_k^0 \right] \cdot v_k^0 ds$$

Furthermore, if $b_k = 0$ we have: $\lambda_k^1 = \dfrac{a_{kk}}{4\lambda_k}$.

5 Appendix: technical proofs

We briefly give some results used before. Complete proofs are avaible in [4]. The following Lemma is useful to prove an expansion for large time. There is a similar version for system.

Lemma 5.3. [Bounds for large time] *Let w_ε be a solution of*

$$w_\varepsilon" + w_\varepsilon = S_\varepsilon(s) + f_\varepsilon(s) + \varepsilon g_\varepsilon(s, w_\varepsilon),\ w_\varepsilon'(0) = 0, w_\varepsilon'(0) = 0. \qquad (14)$$

If source terms satisfy the following conditions with $M > 0$:

1. *S_ε are periodic functions orthogonal to $e^{\pm it}$, and $|S_\varepsilon(t)| \le M$*
2. *$|f_\varepsilon| \le M$ and for all T, $\int_0^T |f_\varepsilon(s)| ds \le C\varepsilon T$ or $C\sqrt{\varepsilon}T$,*
3. *there exists $R > 0$ such that: $M_R = \sup_{\varepsilon \in (0,1), s>0, R>u^2} |g_\varepsilon(s, u)| < \infty$,*

then, w_ε is uniformly bounded in $L^\infty(0, T_\varepsilon)$, where $T_\varepsilon = \dfrac{\gamma}{\varepsilon}$ or $\dfrac{\gamma}{\sqrt{\varepsilon}}$ and $\gamma > 0$.

For system we have to work with linear combination of periodic functions with different periods and nonlinear function of such sum. So we work with the adherence in $L^\infty(\mathbb{R}, \mathbb{R})$ of span$\{e^{i\lambda t},\ \lambda \in \mathbb{R}\}$, namely the set of almost periodic functions $C_{ap}^0(\mathbb{R}, \mathbb{R})$, see [1]. We first give an useful Lemma about the spectrum of $|w|$ for $u \in C_{ap}^0(\mathbb{R}, \mathbb{R})$. Let us recall definitions for the Fourier coefficient of u associated to frequency λ: $c_\lambda[u]$ and its spectrum: $Sp[u]$,

$$c_\lambda[u] = \lim_{T\to+\infty} \frac{1}{T}\int_0^T u(t)e^{-i\lambda t}dt, \qquad Sp[u] = \{\lambda \in \mathbb{R},\, c_\lambda[u] \neq 0\}.$$

Lemma 5.4. [About spectrum of $|u|$] *If $u \in C^0_{ap}(\mathbb{R},\mathbb{R})$, u has got a finite spectrum: $Sp[u] \subset \{\pm\lambda_1, \cdots, \pm\lambda_N\}$, $(\lambda_1, \cdots, \lambda_N)$ are $\mathbb{Z}$-independent, $0 \notin Sp[u]$, then $\lambda_k \notin Sp[\,|u|\,]$ for all k.*

References

1. Corduneanu, *Almost periodic functions,* Interscience, New York, 1968.
2. H. Hazim, B. Rousselet, *Finite Elements for a Beam System with Nonlinear Contact Under Periodic Excitation,* preprint 2008.
3. D. Jiang, C. Pierre, S.W. Shaw, *Large-amplitude non-linear normal mode of piecewise linear systems.* Journal of Sound and Vibration, 272, p. 869-891, 2004.
4. S. Junca, B. Rousselet, *The Method of Strained Coordinates for Discrete Models with Small Unilateral Contacts,* preprint 2008.
5. J.B. Keller & S. Kogelman, *Asypmtotic solutions of initial value problems for nonlinear partail differential equations*, S.I.A.M. J. Appl. Math. 18, p. 748-758, 1970.
6. Kevorkian, J.; Cole, Julian D., *Perturbation methods in applied mathematics.* Applied Mathematical Sciences, 34. Springer-Verlag, New York-Berlin, 1981.
7. J. Kevorkian and J. Cole, *Multiple Scale and Singular Perturbations Problems,* Applied Mathematical Sciences, volume 114, Springer, Berlin, 1996.
8. Peter D. Miller, *Applied Asymptotic Analysis*, American Mathematical Society, Providence, Rhode Island, volume 75, ch 9-10, 2006.
9. Nayfeh, Ali Hasan, *Introduction to perturbation techniques.* Wiley-Interscience [John Wiley & Sons], New York, 1981.
10. M. Roseau, *Vibrations des systèmes mécaniques. Méthodes analytiques et applications.* Masson, Paris, 489 pp, 1984.
11. B. Rousselet, G. Vanderborck, *Non destructive control of cables: O.D.E. models of non linear vibrations.* Variational Formulations in Mechanics : Theory and Applications - A Workshop dedicated to the 60th Birthday of Professor Raùl A. Feijoo; 3-5/9/2006, 2006.
12. G. Vanderborck, B. Rousselet, *Structural damage detection and localization by non-linear acoustic spectroscopy* , Saviac, 76th Shock and Vibration Symposium, October 31 - November 3, 2005, Destin (Florida / USA), 2005.