VANISHING PRESSURE
IN GAS DYNAMICS EQUATIONS

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Abstract. This work is devoted to the analysis of the behaviour of solutions of gas dynamics equations as the pressure goes to 0 in the context of regular solutions. We obtain in this way a first justification of the connection to pressureless gases model.

Key Words. One-dimensional degenerate hyperbolic system, Zero pressure gas dynamics

1. INTRODUCTION

This work is a first attempt to deal with vanishing pressure in the following system of gas dynamics equations

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0 \ \ in \ \mathbb{R}_t^+ \times \mathbb{R}_x, \\ \partial_t(\rho u) + \partial_x(\rho u^2/2 + \varepsilon^2 p(\rho)) = 0 \ \ in \ \mathbb{R}_t^+ \times \mathbb{R}_x. \end{array} \right. \tag{1}$$

In (1), the pressure law $p : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is required to satisfy

$$p'(\rho) > 0 \ for \ \rho > 0, \tag{2}$$

$$p"(\rho) \geq 0, \tag{3}$$

$$2p'(\rho) \geq \rho p"(\rho). \tag{4}$$

In the sequel, we will denote by $P(\rho)$ a function verifying

$$P'(\rho) = \sqrt{p'(\rho)}/\rho.$$

Notice in particular that the power laws $p(\rho) = \rho^\gamma$, with $\gamma > 1$ as well as the isothermal law $p(\rho) = \rho$ fulfill (2-3), but (4) restricts to $1 \leq \gamma \leq 3$. We obtain in these cases $P(\rho) = 2\sqrt{\gamma}/(\gamma - 1)\,\rho^{(\gamma-1)/2}$ for

$\gamma > 1$ and $P(\rho) = \ln(\rho)$ for $\gamma = 1$. The problem is supplemented by prescribying Cauchy data for the density $\rho(0,x) = \rho_0(x) \geq 0$ and the velocity $u(0,x) = u_0(x)$. Formally, as ε vanishes, we are led to the following pressureless gases equations

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \;\; in \; I\!R_t^+ \times I\!R_x, \\ \partial_t(\rho u) + \partial_x(\rho u^2/2) = 0 \;\; in \; I\!R_t^+ \times I\!R_x. \end{cases} \tag{5}$$

The system (5) was introduced as a simplified model of the dynamics of galaxies by Zeldovich, see [9] and a first mathematical analysis is due to Bouchut in [1]. Existence of solutions and connection to scalar conservation laws are studied by Grenier [5], Brenier-Grenier [4] and, also by Bouchut-James who used in [3] the notion of duality solutions of [2] (see also Poupaud-Rascle [8] where an equivalent formulation is proposed). Regular solutions, even in the multidimensional context, are investigated by Poupaud [7].

In this work, we restrict to the case of regular solutions: crucial bounds are obtained by considering the Riemann invariants associated to (1), and, then we pass to the limit as ε goes to 0. In this way, we obtain the following

Theorem 1 *Let $\rho_0^\varepsilon, u_0^\varepsilon$ be a sequence of initial data for (1) satisfying*

$$\begin{cases} \rho_0^\varepsilon \geq d_\varepsilon > 0, \quad \partial_x(u_0^\varepsilon \pm \varepsilon P(\rho_0^\varepsilon)) \geq 0, \\ \rho_0^\varepsilon \rightharpoonup \rho_0 \, weakly * in \, L^\infty_{loc}(I\!R), \\ u_0^\varepsilon \rightharpoonup u_0 \, in \, W^{1,\infty}_{loc}(I\!R). \end{cases} \tag{6}$$

where d_ε is a sequence of positive reals, possibly tending to 0. Then the associated solutions of (1) satisfy

$$\begin{aligned} &\rho^\varepsilon \rightharpoonup \rho \, weakly * in \, L^\infty_{loc}(I\!R^+ \times I\!R), \\ &u^\varepsilon \to u \, in \, C^0_{loc}(I\!R^+ \times I\!R), \end{aligned}$$

where (ρ, u) is the unique solution to (5) with initial data (ρ_0, u_0).

Remark 1 *Note that (6) implies that* $\partial_x u_0^\varepsilon \geq \varepsilon|\partial_x(P(\rho_0^\varepsilon))|$*, thus* $\partial_x u_0^\varepsilon$ *is non-negative, and, passing to the limit,* $\partial_x u_0$ *is too. Moreover, in terms of* ρ_0^ε*, this yields* $|\partial_x(\rho_0^\varepsilon)| \leq 1/\varepsilon\, \rho_0^\varepsilon/\sqrt{p'(\rho_0^\varepsilon)}\, \partial_x u_0^\varepsilon$ *which essentially means that* $\partial_x \rho_0^\varepsilon$ *may blow up as* $(\varepsilon\sqrt{p'(d_\varepsilon)})^{-1}$.

In Section 2, we perform some computations on the Riemann invariants $u \pm \varepsilon P(\rho)$ associated to solutions of (1), following essentially ideas of P. D. Lax [6]. This allows us to discuss crucial bounds on the solutions in Section 3 and we conclude in Section 4.

2. Riemann Invariants

Let $c = \sqrt{p'(\rho)}$. We set $w = u + \varepsilon P(\rho)$ and $z = u - \varepsilon P(\rho)$, where (ρ, u) satisfies (1). Then, assuming that ρ are u are regular enough, some elementary computations leads to

$$\begin{cases} \partial_t w + (u + \varepsilon c)\partial_x w = 0, \\ \partial_t z + (u - \varepsilon c)\partial_x z = 0. \end{cases} \tag{7}$$

Following P. D. Lax, [6], possible loss of regularity of the solutions can be observed by considering the equations satisfied by the first derivative of w and z. Let $W = \partial_x w$, $Z = \partial_x z$ and set $\alpha = \dfrac{\rho p''(\rho)}{2p'(\rho)} = c'(\rho)/P'(\rho)$. By deriving (7) we get

$$\begin{cases} \partial_t W + (u + \varepsilon c)\partial_x W + (1+\alpha)/2\, W^2 + (1-\alpha)/2\, ZW = 0, \\ \partial_t Z + (u - \varepsilon c)\partial_x Z + (1+\alpha)/2\, Z^2 + (1-\alpha)/2\, ZW = 0, \end{cases} \tag{8}$$

since $\partial_x u = \partial_x(\dfrac{w+z}{2}) = 1/2(W+Z)$ and $\partial_x(\varepsilon P(\rho)) = 1/2(W-Z)$. We introduce the characteristics associated to the velocities $u \pm \varepsilon c$, namely

$$\begin{cases} \dfrac{d}{dt}X(t,x) = u(t, X(t,x)) + \varepsilon c(t, X(t,x)), \\ \dfrac{d}{dt}Y(t,x) = u(t, Y(t,x)) - \varepsilon c(t, Y(t,x)), \end{cases}$$

and $X(0,x) = x = Y(0,x)$. For ϕ a real-valued function depending on time and position, we set $\phi^\sharp(t,x) = \phi(t, X(t,x))$ and $\phi^\flat(t,x) =$

$\phi(t, Y(t,x))$. Now, we can rewrite (8) as the following ode

$$\frac{d}{dt}S + AS^2 + BS = 0, \tag{9}$$

where S stands for $W^\sharp$ and $A = (1+\alpha^\sharp)/2$, $B = (1-\alpha^\sharp)/2\, Z^\sharp$ or $S = Z^\flat$, $A = (1+\alpha^\flat)/2$, $B = (1-\alpha^\flat)/2\, W^\flat$, respectively. In (9), the function B can be viewed as the time derivative of another function. We introduce h and k as functions of the two variables (w, z) verifying $\partial_z h(w,z) = (1-\alpha(\rho))/(2\varepsilon c(\rho)) = \partial_w k(w,z)$ with $\rho = P^{-1}((w-z)/2\varepsilon)$, P^{-1} being the inverse of P. Indeed, we remark that

$$\begin{aligned} \frac{d}{dt}(h^\sharp(t,x)) &= \frac{d}{dt}(h(w^\sharp(t,x), z^\sharp(t,x)) \\ &= (\partial_z h)^\sharp(t,x)\,(2\varepsilon c\partial_x z)(t, X(t,x)) \\ &= ((1-\alpha)Z)^\sharp = +2B \end{aligned} \tag{10}$$

by (7). Similarly, we get

$$\frac{d}{dt}(k^\flat) = (\partial_w k)^\flat(-2\varepsilon c(\partial_x w)(t, Y(t,x)) = -((1-\alpha)W)^\flat = -2B. \tag{11}$$

Combining (10) and (11) to (9) leads to

$$\frac{d}{dt}S + AS^2 + (\frac{d}{dt}C)S = 0 \tag{12}$$

with $C = h^\sharp/2$ or $-k^\flat/2$. This Ricatti-like equations can be integrated easily; we obtain

$$\begin{aligned} S(t,x) = \Big(&1/S(0,x)\,\exp((C(t,x) - C(0,x)) \\ &+ \int_0^t A(\tau,x)\exp(C(t,x) - C(\tau,x))\,d\tau\Big)^{-1}. \end{aligned} \tag{13}$$

Since $A = (1+\alpha)/2 \geq 1/2 > 0$ by (2-3), for $S(0,x) \geq 0$ the solution of (13) exists globally in time which in turn leads to global existence for (1).

3. Estimates

The computations made in the previous Section allow us to derive some estimates on the solution of (1). We have

Proposition 1 *Let $\rho_0^\varepsilon, u_0^\varepsilon$ be the initial data of (1) satisfying the requirements of Theorem 1. Then there exists a global regular solution $(\rho^\varepsilon, u^\varepsilon)$ of (1) and the sequences $(u_\varepsilon, \varepsilon P(\rho^\varepsilon))_{\varepsilon>0}$ are bounded in $L^\infty(\mathbb{R}^+; W^{1,\infty}_{loc}(\mathbb{R}))$ while $(\rho_\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R})$.*

We turn to the Riemann invariant. Assumption (6) means that $W(0,x)$ and $Z(0,x)$ are non-negative. By (13), one deduces that W and Z remain non-negative and are defined for all time. In turn, this implies that $t \longmapsto h^\sharp(t,x)$ is non-decreasing by (10) and $t \longmapsto k^\flat(t,x)$ is non-increasing by (11), since (4) means that $1-\alpha \geq 0$. Thus $t \longmapsto C(t,x)$ is non-decreasing and it follows that

$$0 \leq S(t,x) \leq \Big(1/S(0,x) + t/2\Big)^{-1}. \tag{14}$$

Therefore $W = \partial_x w^\varepsilon$ and $Z = \partial_x z^\varepsilon$ are bounded in $L^\infty(\mathbb{R}^+ \times \mathbb{R}_{loc})$ while integrating (7) along the characteristics provides a $L^\infty(\mathbb{R}^+ \times \mathbb{R}_{loc})$ bound on $w^\varepsilon, z^\varepsilon$. Thus, $u^\varepsilon = 1/2(w^\varepsilon + z^\varepsilon)$ and $\varepsilon P(\rho^\varepsilon) = 1/2(w^\varepsilon - z^\varepsilon)$ are bounded in $L^\infty(\mathbb{R}; W^{1,\infty}(-R,+R))$ for all $0 < R < \infty$. Finally, coming back to (1) we get

$$\partial_t \rho^\varepsilon + u^\varepsilon \partial_x \rho^\varepsilon + \rho^\varepsilon \partial_x u^\varepsilon = 0$$

which can be integrated along the characteristic curves associated to the velocity u^ε and provides the estimate on ρ^ε in $L^\infty((0,T)\times(-R,+R))$ for all $0 < T, R < \infty$.

Remark 2 *One needs assumption (4) to obtain a uniform bound on $\partial_x u^\varepsilon$ and $\partial_x(\varepsilon P(\rho^\varepsilon))$. Indeed, for $\varepsilon > 0$ given, by (13) W, Z belongs to*

$L^\infty(\mathbb{R} \times \mathbb{R}_{loc})$, but this estimate is not uniform wrt. ε. For instance, considering power law with $\gamma > 3$, W, Z blow up as $e^{1/\varepsilon}$.

4. Passage to the limit

We are ready to end the proof of Theorem 1. Since u^ε satisfies

$$\begin{aligned} &\partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + \varepsilon^2/\rho^\varepsilon \partial_x(p(\rho^\varepsilon)) = 0 \\ &= \partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + \varepsilon\sqrt{p'(\rho^\varepsilon)}\partial_x(\varepsilon P(\rho^\varepsilon)) = 0 \end{aligned} \tag{15}$$

by (1), we deduce from Proposition 1 that $\partial_t u^\varepsilon$ lies in a bounded set of $L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R})$. Indeed, ρ^ε is bounded in $L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R})$ and by (2), p' is non decreasing, thus $p'(\rho^\varepsilon)$ is bounded in $L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R})$. Therefore, Proposition 1 implies that the last term in (15) is bounded (and actually tends to 0!). One deduces that u^ε belongs to a bounded set in $W^{1,\infty}_{loc}(\mathbb{R}^+ \times \mathbb{R})$ and by Ascoli's theorem, we can assume

$$u^\varepsilon \to u \ in \ C^0((0,T) \times (-R, +R)), \ 0 < T, R < \infty, \tag{16}$$

at least for a subsequence. Furthermore, we can also suppose, say, $\rho^\varepsilon \rightharpoonup_* \rho$ in $L^\infty((0,T) \times (-R,+R))$. These convergence properties permit us to pass to the limit in the $\mathcal{D}'$ sense in (1) and we are led to (5) as ε goes to 0. We check easily that the initial data for (ρ, u) are the limits (ρ_0, u_0) of $(\rho_0^\varepsilon, u_0^\varepsilon)$. Since ρ_0 has no atomic part, uniqueness for (5) follows, see [3], which ensures that the whole sequence converges and achieves the proof of Theorem 1. Note that we can also pass to the limit in (15) which shows that u is a regular solution of the Burgers equation.

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