Fourier-based distances and Berry-Esseen like inequalities

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Abstract

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with values in Euclidean space $\mathbb{R}^N$, $N \geq 1$. Assuming some regularity on the common density, we prove a Berry-Esseen like theorem: the boundedness of moments higher than two improves the rate of convergence to the related Gaussian in Sobolev norms. Our method is noticeable simple and it applies also when studying convergence to stable laws.

1 Introduction

Consider a sequence of independent identically distributed random variables $(X_n)_{n\in\mathbb{N}}$ with values in Euclidean space $\mathbb{R}^N$, $N \geq 1$. We assume that the probability density f of X_1 satisfies

$$\int_{\mathbb{R}^N} f\,dx = 1, \qquad \int_{\mathbb{R}^N} xf\,dx = 0, \qquad \int_{\mathbb{R}^N} |x|^2 f\,dx < \infty. \tag{1}$$

Denote by f_n the probability density of the normalized partial sum

$$S_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}.$$

Let ω_K be the normal law with zero mean and covariance operator K, whose coefficients are

$$K_{ij} = \int_{\mathbb{R}^N} x_i x_j f(x)\,dx \qquad \text{for } 1 \leq i, j \leq N.$$

The classical central limit theorem asserts that f_n converges to the Gaussian density

$$\omega_K(x) = \frac{1}{(2\pi)^{N/2}\sqrt{\det(K)}} \exp\left(-\frac{K^{-1}x \cdot x}{2}\right)$$

as $n \to \infty$, see [8] for instance. In a recent paper, [7], Lions and Toscani proved that, under some additional regularity assumption on f (see condition (2) below), the convergence $f_n \to \omega_K$ holds in any Sobolev space

$$H^k(\mathbb{R}^N) = \{\phi \in \mathcal{S}'(\mathbb{R}^N),\ \int_{\mathbb{R}^N} (1+|\xi|^2)^k\, |\widehat{\phi}(\xi)|^2\, d\xi < \infty\}$$

where $\widehat{\phi}(\xi) = \displaystyle\int_{\mathbb{R}^N} e^{-ix\cdot\xi}\phi(x)\, dx$ stands for the Fourier transform of ϕ. However, in [7] the rate of convergence has not been discussed. In view of the usual Berry-Esseen theorem [8], such kind of information may be expected from extra moment assumptions on f. The aim of this paper is to prove this quite intuitive result. Namely, we shall show the following

Theorem 1 *Let f satisfy (1). Moreover, assume there exists $M > 0$ and $k \geq 0$ such that*

$$\int_{\mathbb{R}^N} (1+|\xi|^2)^k\, |\widehat{f}(\xi)|^M\, d\xi = C_{M,k} < \infty \tag{2}$$

and

$$\int_{\mathbb{R}^N} |x|^s f(x)\, dx = \rho_s < \infty \text{ for some } 2 < s \leq 3. \tag{3}$$

Then

$$\|f_n - \omega_K\|_{H^k} \leq C \frac{1}{n^{(s-2)/2}},$$

the constant C depending on K, ρ_s, M and k.

Remark 1 *Clearly, (2) is fulfilled with $M = 2$ as f belongs to H^k. Furthermore, according to [7], it is worth pointing out that Theorem 1 holds in any H^k if, instead of satisfying hypothesis (2) the probability density f is such that, for some $\varepsilon > 0$*

$$|\xi|^\varepsilon\, |\widehat{f}(\xi)| \leq C \text{ for } |\xi| \geq 1. \tag{4}$$

Indeed, (4) yields (2) with $M > (2k+N)/\varepsilon$. Notice also that $\sqrt{f} \in H^1$ ensures (4) with $\varepsilon = 1$.

Remark 2 *The usual Berry-Esseen theorem gives the same decay for the uniform metric between the associated distribution functions* $\rho(F_n, \Omega_K) = \sup\left\{|F_n(x) - \Omega_K(x)|\,;\, x \in \mathbb{R}^N\right\}$. *This has also to be compared to results in [9] describing the rate of convergence evaluated in terms of a certain minimal metric, implying weak convergence of the probabillity density.*

Berry-Esseen type estimates are of vital interest in statistics, where many results about the asymptotic distribution of estimators and testors depend on the central limit theorem, and the validity of the statistical method depends on how good the asymptotic distribution approximates the real distribution (see [6] for further details). Classical estimates refer to the uniform metric (see Remark 2) but the novelty here is the study of the decay in Sobolev spaces H^k, in the spirit of [7].

On the other hand, it is worth emphasizing the simplicity of our arguments which take advantage of the introduction of a suitable Fourier-based metric for probability distributions (see definition (5)). We shall describe this metric in Section 2. Interest and easiness of use of this metric is illustrated with an amazing application to the heat equation. Next, coming back to the behaviour of f_n, a slow rate of decay in Sobolev spaces can be found by means of the method developped by Carrillo and Toscani [2]. The simple proof of this fact is also given in Section 2; combined to the results of Lions-Toscani [7], we obtain convergence in any Sobolev space at a rate which is not sharp. Section 3 is devoted to the proof of Theorem 1, with convergence in H^k norm at the Berry-Esseen rate. Finally, in Section 4, we extend the discussion to stable laws.

2 Fourier-based metrics

2.1 Definition

Denote by $P_s(\mathbb{R}^N)$, $s > 0$, the class of all probability distributions F on $\mathbb{R}^N$, $N \geq 1$, such that

$$\int_{\mathbb{R}^N} |x|^s \, dF(x) < \infty.$$

We introduce a metric on $P_s(\mathbb{R}^N)$ by

$$d_s(F, G) = \sup_{\xi \in \mathbb{R}^N \setminus \{0\}} \left\{ \frac{|\widehat{f}(\xi) - \widehat{g}(\xi)|}{|\xi|^s} \right\}, \tag{5}$$

where f, g are the densities associated to F, G respectively. In the following, if X and Y have distributions F and G, or f and g are probability densities we will write $d_s(X, Y)$ or $d_s(f, g)$ instead of $d_s(F, G)$.

Connection to moment properties appear clearly with the following remark. Let us write $s = m + \delta$, where m is an integer and $0 \leq \delta < 1$. Then, an easy computation shows that in order that $d_s(F, G)$ be finite, it suffices that F and G have the same moments up to order m.

The metric (5) has been introduced in [4] to investigate the trend to equilibrium of the solutions to the Boltzmann equation for Maxwell molecules. There, the case $s = 2 + \delta$, $\delta > 0$, was considered. Further applications of d_s, with $s = 4$, were studied in [1], while the case $s = 2$ has been considered in [10] in connection with the uniqueness of the solutions to the Boltzmann equation for Maxwell molecules. There, the main properties and the relationships with other classical metrics have been investigated. Here, we remark that, for all $s > 0$, d_s is a regular metric, i.e. for any distributions F, G and H

$$d_s(F \star H, G \star H) \leq d_s(F, G) \tag{6}$$

where $\star$ denotes convolution of distributions. Moreover, for any constant $a > 0$ and any random variables X and Y,

$$d_s(aX, aY) \leq a^s d_s(X, Y) \tag{7}$$

Properties (6) and (7) characterize ideal metrics [11].

This type of metrics proved to be a very useful instrument in the estimate of the convergence rate in the central limit theorem. It seems to us worth illustrating how this tool can be used with the following very simple examples. The former deals with the heat equation: by means of the metric (5) we shall find the rate of decay of the solution of the heat equation to the fundamental one, showing that this rate is related to the number of moments which are initially finite. The latter explains how we can use this metric to prove Berry-Esseen like inequalities with an easy proof of convergence in Sobolev spaces at a suboptimal rate.

2.2 A remark on the heat equation

It is well known that a solution of the heat equation

$$\begin{cases} \partial_t u - k\Delta_x u = 0 \ \ in\ \mathbb{R}^N \\ u(0, x) = f(x), \end{cases} \tag{8}$$

where $f(x)$ is a probability density function satisfying (1), behaves asymptotically in time as the fundamental solution of (8), which is given by ω_{2kt}, and the rate of convergence in L^2 norm is governed by $t^{-N/2}$. The purpose of this section is to show that we can improve this rate of convergence as soon as we have more information on the moments of the initial data.

THEOREM 2 *Let u_1 and u_2 be solutions of (8) with initial data f_1 and f_2 respectively, and let s be such that $d_s(f_1, f_2)$ is finite . Then, we have*

$$\|(u_1 - u_2)(t, \cdot)\|^2_{L^2(\mathbb{R}^N)} \leq (d_s(f_1, f_2))^2 \ C \, t^{-(s+N/2)},$$

with $C = \dfrac{\Gamma(s + N/2)}{\Gamma(N/2)} (4\pi)^{-N/2}(2k)^{-(s+N/2)}$.

In particular, as $s \to 0$, we recover the usual $t^{-N/2}$ rate. Let $s = m + \delta$, where m is an integer and $0 \leq \delta < 1$. Looking at the definition of the metric d_s, practically Theorem 2 means that if the data f_1 and f_2 have the same moments up to m, and

$$\int_{\mathbb{R}^N} |x|^s f_i(x) < \infty, \qquad i = 1, 2$$

then the rate of convergence becomes $t^{-(s+N/2)}$.

Proof. The proof is entirely explicit. Indeed, by Fourier transform, we get

$$\partial_t \widehat{u_i}(t, \xi) = -k|\xi|^2 \widehat{u_i}(t, \xi).$$

Solving the ordinary differential equation leads to

$$|\widehat{u_1}(t, \xi) - \widehat{u_2}(t, \xi)|^2 = \exp(-2kt|\xi|^2)|\widehat{f_1}(\xi) - \widehat{f_2}(\xi)|^2.$$

It follows that the L^2 norm is estimated by

$$\begin{aligned}
\int_{\mathbb{R}^N} |u_1(t, x) - u_2(t, x)|^2 \, dx &= (2\pi)^{-N} \int_{\mathbb{R}^N} |\widehat{u_1}(t, \xi) - \widehat{u_2}(t, \xi)|^2 \, d\xi \\
&= (2\pi)^{-N} \int_{\mathbb{R}^N} \exp(-2kt|\xi|^2)|\widehat{f_1}(\xi) - \widehat{f_2}(\xi)|^2 \, d\xi \\
&\leq (d_s(f_1, f_2))^2 \ F(t)
\end{aligned}$$

by using the Parseval equality. The function of time $F(t)$ is computed as follows

$$\begin{aligned}
F(t) &= (2\pi)^{-N} \int_{\mathbb{R}^N} \exp(-2kt|\xi|^2)|\xi|^{2s} \, d\xi \\
&= (2\pi)^{-N} \int_{S^{d-1}} \int_0^\infty \exp(-2kt\rho^2)\rho^{2s}\rho^{d-1} \, d\rho d\omega_K \\
&= (2\pi)^{-N} \, 2\pi^{N/2}/\Gamma(N/2) \int_0^\infty \exp(-z) z^{(2s+N-2)/2} \frac{dz}{2.2kt \, (2kt)^{(2s+N-2)/2}} \\
&= \Gamma(s + N/2)/ \left(\Gamma(N/2)(2k)^{s+N/2}(4\pi)^{N/2}\right) \, t^{-(s+N/2)}.
\end{aligned}$$

Optimality of the obtained constant may be seen by considering Dirac mass as initial data. This finishes the proof. □

2.3 A non optimal rate of convergence in H^k norm

The interest of Fourier-based metric for studying convergence properties of S_n is best understood when we realize that the Fourier transform of the density f_n is given by

$$\widehat{f_n}(\xi) = \left(\widehat{f}(\xi/\sqrt{n})\right)^n. \tag{9}$$

This shows that the main contribution comes from small values of ξ, for which $\widehat{f_n}(\xi)$ is close to 1. Furthermore, provided $s > 2$ we obtain

$$d_s(F_n, \Omega_K) \leq \left(\frac{1}{\sqrt{n}}\right)^{s-2} d_s(F, \Omega_K). \tag{10}$$

The rate of convergence in the metric d_s is optimal. By means of (10) we can easily obtain convergence in Sobolev spaces at a suboptimal rate. The proof is immediate. Indeed, we wish to investigate the behaviour of the following (square of) H^k norm

$$I_n = \int_{\mathbb{R}^N} (1+|\xi|^2)^k \left|\widehat{f_n}(\xi) - \widehat{\omega_K}(\xi)\right|^2 d\xi$$

as n goes to ∞. Let $R > 0$ and $p > 0$. Then, we split I_n as follows

$$\begin{aligned} I_n &\leq \int_{|\xi|\leq R} (1+|\xi|^2)^k \left|\widehat{f_n}(\xi) - \widehat{\omega_K}(\xi)\right|^2 d\xi \\ &\quad + \frac{1}{(1+R^2)^p} \int_{|\xi|>R} (1+|\xi|^2)^{k+p} \left|\widehat{f_n}(\xi) - \widehat{\omega_K}(\xi)\right|^2 d\xi \\ &\leq \int_{|\xi|\leq R} (1+|\xi|^2)^k |\xi|^{2s} d_s^2(f_n, \omega_K)\, d\xi + \frac{1}{(1+R^2)^p} \|f_n - \omega_K\|^2_{H^{k+p}} \\ &\leq C_N (1+R^2)^{k+s+N/2} d_s^2(f_n, \omega_K) + \frac{1}{(1+R^2)^p} \|f_n - \omega_K\|^2_{H^{k+p}}. \end{aligned}$$

Optimizing over R we get

$$\|f_n - \omega_K\|_{H^k} \leq c(N,k,p,s) \|f_n - \omega_K\|_{H^{k+p}}^{\frac{2k+2s+N}{2k+2s+N+2p}} d_s(f_n, \omega_K)^{\frac{2p}{2k+2s+N+2p}} \tag{11}$$

where

$$c(N,k,p,s) = C_N^{\frac{p}{2k+2s+N}} \left[\left(\frac{2p}{2k+2s+N}\right)^{\frac{2k+2s+N}{2k+2s+N+2p}} + \left(\frac{2k+2s+N}{2p}\right)^{\frac{2p}{2k+2s+N+2p}} \right]^{\frac{1}{2}}$$

A simple application of (10) and (11) shows the following

Theorem 3 *Let f satisfy (1), (2) and (3). Then, if f_n is uniformly bounded in H^k, for some $k > 0$, f_n converges to ω_K in any $H^{k'}$ with $k' < k$, and*

$$\|f_n - \omega_K\|_{H^{k'}} \leq C(N, s, k, k') \left(\frac{1}{n}\right)^{\frac{(k-k')(s-2)}{2k+N+2s}}.$$

Actually, we can improve the rate of convergence obtained in this simple and short way, by using another (quite classical) $n-$dependent splitting of I_n as shown in the following Section.

3 Proof of Theorem 1

Now, let us split I_n into integrals on high and low frequencies as follows

$$\begin{cases} I_n = H_n + L_n, \\ H_n = \displaystyle\int_{|\xi| \geq \eta\sqrt{n}} ... \, d\xi, \qquad L_n = \displaystyle\int_{|\xi| \leq \eta\sqrt{n}} ... \, d\xi \end{cases}$$

where $\eta > 0$ is a parameter to be determined. The regularity requirement (2) allows us to control the decay of high frequencies H_n while extra-moment condition (3) gives both estimates on the decay of low frequencies L_n, and determines the rate of convergence.

3.1 High frequencies estimates

First of all, let us deal with the high frequencies integral which decreases with arbitrarily fast rate as shown in the following claim.

Lemma 1 *Let f satisfy (1) and (2). Then, for any $\alpha > 0$, $n \geq M/2$, one has*

$$H_n \leq C/n^{\alpha}, \tag{12}$$

where C depends on $C_{M,k}$. If (2) is replaced by (4), then (12) holds in any H^k, $k \geq 0$.

Note that Lemma 1 only needs the regularity assumption (2) but does not require extra moment condition.

Proof. First, we simply evaluate

$$\begin{aligned} H_n &\le 2\int_{|\xi|\ge\eta\sqrt{n}} (1+|\xi|^2)^k |\widehat{f_n}(\xi)|^2\,d\xi + 2\int_{|\xi|\ge\eta\sqrt{n}} (1+|\xi|^2)^k |\widehat{\omega_K}(\xi)|^2\,d\xi \\ &\le 2\int_{|\xi|\ge\eta\sqrt{n}} (1+|\xi|^2)^k |\widehat{f}(\xi/\sqrt{n})|^{2n}\,d\xi + 2\int_{|\xi|\ge\eta\sqrt{n}} (1+|\xi|^2)^k |\widehat{\omega_K}(\xi/\sqrt{n})|^{2n}\,d\xi \end{aligned}$$

where we used (9) and the trivial property $|\widehat{\omega_K}(\xi)|^2 = |\widehat{\omega_K}(\xi/\sqrt{n})|^{2n}$. Hence, it suffices to deal with

$$\widetilde{H}_n = \int_{|\xi|\ge\eta\sqrt{n}} (1+|\xi|^2)^k |\widehat{\varphi}(\xi/\sqrt{n})|^{2n}\,d\xi = n^{N/2} \int_{|\zeta|\ge\eta} (1+n|\zeta|^2)^k |\widehat{\varphi}(\zeta)|^{2n}\,d\zeta,$$

φ verifying (1) and (2). We have

$$\begin{aligned} \widetilde{H}_n &= n^{N/2+k} \int_{|\zeta|\ge\eta} \left(\frac{(1+n|\zeta|^2)}{n(1+|\zeta|^2)}\right)^k (1+|\zeta|^2)^k |\widehat{\varphi}(\zeta)|^M \, |\widehat{\varphi}(\zeta)|^{2n-M}\,d\zeta, \\ &\le n^{N/2+k} \left(\|\widehat{\varphi}\|_{L^\infty(|\zeta|\ge\eta)}\right)^{2n-M} C_{M,k}. \end{aligned}$$

Let $\alpha > 0$. We claim that

$$\lim_{n\to\infty} n^{N/2+k+\alpha} \left(\|\widehat{\varphi}\|_{L^\infty(|\zeta|\ge\eta)}\right)^{2n-M} = 0, \tag{13}$$

which will prove (12). Indeed, (13) follows from the following elementary result.

Lemma 2 *i) Let $\psi \in L^1(\mathbb{R}^N, \mathbb{C})$ satisfy $\left|\int_{\mathbb{R}^N} \psi(x)\,dx\right| = \int_{\mathbb{R}^N} |\psi(x)|\,dx$. Then there exists $\theta \in (0, 2\pi)$ such that $\psi(x) = e^{-i\theta}|\psi(x)|$ a.e.*

ii) Let $\varphi \in L^1(\mathbb{R}^N)$ satisfy $\varphi \ge 0$ and $\int_{\mathbb{R}^N} \varphi(x)\,dx = 1$. Then, for all $\eta > 0$, one has $\sup_{|\xi|\ge\eta} |\widehat{\varphi}(\xi)| < 1$.

Lemma 2 is very standard and we omit the proof. We only mention that *ii)* is a consequence of *i)*. Applying Lemma 2 *ii)* to $\varphi = f$ or ω_K gives $\|\widehat{\varphi}\|_{L^\infty(|\zeta|\ge\eta)} < 1$ for all $\eta > 0$, which in turn leads easily to (13). □

Remark 3 *Note that it is crucial to assume that the common density f lies in $L^1(\mathbb{R}^N)$; if f is a measure, Lemma 2 breaks down in general (recall for instance the case of the Dirac mass δ_0 which gives $\widehat{\delta_0} = 1$).*

3.2 Low frequencies estimates

In order to deal with the low frequencies integral, we are going to use the following claim where the distance introduced in [4] appears.

Lemma 3 *Let $s > 2$. If there exists $\eta > 0$ and $\sigma > 0$ such that, for $|\xi| \leq \eta$,*

$$|\widehat{f}(\xi)| \leq \exp\left\{-\frac{\sigma}{4}|\xi|^2\right\}, \tag{14}$$

then, for all $n \geq 2$, one has

$$\sup_{|\xi|\leq\eta\sqrt{n}} \left(\exp\left\{\frac{\sigma}{8}|\xi|^2\right\} \frac{\left|\widehat{f}_n(\xi) - \widehat{\omega_K}(\xi)\right|}{|\xi|^s} \right) \leq \left(\frac{1}{\sqrt{n}}\right)^{(s-2)} d_s(f, \omega_K).$$

Assume temporarily that the previous lemma holds true, and let us check that f satisfies the requirement (14). Indeed, the matrix K is clearly positive definite; we thus have $0 < \sigma I \leq K$, σ being the smallest eigenvalue of K. Obviously,

$$\widehat{\omega_K}(\xi) + \frac{\sigma}{8}|\xi|^2 \leq \exp\left\{-\frac{\sigma}{4}|\xi|^2\right\} \tag{15}$$

holds for $|\xi| \leq \epsilon$, $\epsilon > 0$ being sufficiently small. On the other hand, since f and ω_K have the same moments up to the order two, by expanding $\widehat{f}$, we get

$$\widehat{f}(\xi) = \widehat{\omega_K}(\xi) + o(|\xi|^2)$$

Combining this to (15), we deduce that there exists $0 < \eta < \epsilon$ such that

$$|\widehat{f}(\xi)| \leq \exp\left\{-\frac{\sigma}{4}|\xi|^2\right\}$$

holds for $|\xi| \leq \eta$.

Now, we can take advantage of the bound in Lemma 3 to estimate the decay of the low frequency integral and we obtain the announced rate of convergence. Indeed, by applying Lemma 3, one has

$$\begin{aligned}
L_n &= \int_{|\xi|\leq\eta\sqrt{n}} (1 + |\xi|^2)^k \left|\widehat{f}_n(\xi) - \widehat{\omega_K}(\xi)\right|^2 d\xi \\
&\leq \left(\frac{1}{\sqrt{n}}\right)^{2(s-2)} d_s^2(f, \omega_K) \int_{|\xi|\leq\eta\sqrt{n}} (1 + |\xi|^2)^k \exp\left\{-\frac{\sigma}{4}|\xi|^2\right\} |\xi|^{2s}\, d\xi \\
&\leq \frac{C}{n^{s-2}}\, d_s^2(f, \omega_K)
\end{aligned}$$

with $C := \int_{\mathbb{R}^N} |\xi|^{2s}(1+|\xi|^2)^k \exp\left\{-\frac{\sigma}{4}|\xi|^2\right\} d\xi$. Thus we proved

Lemma 4 *Let f satisfy (1) and (3). Then, one has*

$$L_n \leq C\left(\frac{1}{\sqrt{n}}\right)^{s-2},$$

where C depends on K, M and k.

To achieve the proof of Theorem 1, it remains to prove Lemma 3.

Proof of Lemma 3. We write

$$\begin{aligned}
\frac{\left|\widehat{f_n}(\xi)-\widehat{\omega_K}(\xi)\right|}{|\xi|^s} &= \left(\frac{1}{\sqrt{n}}\right)^s \frac{\left|\widehat{f}(\xi/\sqrt{n})^n-\widehat{\omega_K}(\xi/\sqrt{n})^n\right|}{|\xi/\sqrt{n}|^s}\\
&= \left(\frac{1}{\sqrt{n}}\right)^s \frac{\left|\widehat{f}(\xi/\sqrt{n})-\widehat{\omega_K}(\xi/\sqrt{n})\right|}{|\xi/\sqrt{n}|^s}\left|\sum_{k=0}^{n-1}\widehat{f}(\xi/\sqrt{n})^k\widehat{\omega_K}(\xi/\sqrt{n})^{n-1-k}\right|.
\end{aligned}$$

By hypothesis, if $|\xi|\leq \eta\sqrt{n}$, we can estimate

$$\begin{aligned}
\left|\sum_{k=0}^{n-1}\widehat{f}(\xi/\sqrt{n})^k\widehat{\omega_K}(\xi/\sqrt{n})^{n-1-k}\right| &\leq \sum_{k=0}^{n-1}\left|\widehat{f}(\xi/\sqrt{n})^k\widehat{\omega_K}(\xi/\sqrt{n})^{n-1-k}\right|\\
&\leq \sum_{k=0}^{n-1}\left\{\exp\left(-\frac{\sigma}{4}|\xi|^2\frac{k}{n}\right)\exp\left(-\frac{\sigma}{2}|\xi|^2\frac{n-1-k}{n}\right)\right\}\\
&\leq n\exp\left(-\frac{\sigma}{4}|\xi|^2\frac{n-1}{n}\right).
\end{aligned}$$

Since $\frac{n-1}{n}\geq\frac{1}{2}$ if $n\geq 2$, we get for $|\xi|\leq\eta\sqrt{n}$,

$$\frac{\left|\widehat{f_n}(\xi)-\widehat{\omega_K}(\xi)\right|}{|\xi|^s}\leq\left(\frac{1}{\sqrt{n}}\right)^{s-2}\exp\left\{-\frac{\sigma}{8}|\xi|^2\right\}\frac{\left|\widehat{f}(\xi/\sqrt{n})-\widehat{\omega_K}(\xi/\sqrt{n})\right|}{|\xi/\sqrt{n}|^s}$$

which implies the result. □

Remark 4 *The evaluation of the optimal constant in the estimate of Theorem 1, remains open. On the other hand, let us give an example showing optimality of the obtained rate in the case $N = 1$ and $\delta = 1$. We set*

$$\begin{cases} f(x) = \overline{\widehat{\phi}}(x) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} e^{ix\xi}\phi(\xi)\,d\xi, \\ \phi(\xi) = e^{-|\xi|^2}(1 + i\alpha\xi^3 + \alpha\xi^4) \end{cases}$$

where $0 < \alpha << 1$. Obviously, moments of f up to order 2 are those of $e^{-x^2/2}$ and $f(x)$ takes value in $\mathbb{R}^+$ (since f behaves as $C\alpha e^{-x^2/4}x^4$ as $|x|$ is large, see [5] p. 382 and p. 1093, and, then, we choose α small enough). Clearly, discussion on optimality of the obtained decay relies on the study of the low frequencies integral L_n (since the other term decay faster). Therefore, optimality of the asserted rate of convergence follows from the following property

$$\lim_{n\to\infty}\Gamma_n = \lim_{n\to\infty}\left(n\int_0^{n^{1/6}}\left|\phi^n(\xi/\sqrt{n}) - e^{-|\xi|^2}\right|^2 d\xi\right) = C > 0. \tag{16}$$

Indeed, set $u = \alpha\dfrac{\xi^3}{n\sqrt{n}}\left(i + \dfrac{\xi}{\sqrt{n}}\right)$ and write

$$\phi^n(\xi/\sqrt{n}) - e^{-|\xi|^2} = e^{-|\xi|^2}((1+u)^n - 1) = e^{-|\xi|^2}\; nu\int_0^1 (1+su)^{n-1}\,ds.$$

Then, we get

$$\begin{aligned} \Gamma_n &= n\int_0^{n^{1/6}}\left|e^{-|\xi|^2} nu\int_0^1 (1+su)^{n-1}\,ds\right|^2 d\xi \\ &= \int_0^1 \exp(-2n^{1/3}\zeta^{2/3})\; n^3\;\alpha^2\;(\zeta/n)^2\;\left|i + (\zeta/n)^{1/3}\right|^2 \\ &\quad \left|\int_0^1 \left(1 + s\alpha(\zeta/n)\;\left[i + (\zeta/n)^{1/3}\right]\right)^{n-1} ds\right|^2 \frac{n^{1/6}}{3\zeta^{2/3}}\,d\zeta \\ &= \int_0^1 f_n(\zeta)g_n^2(\zeta)\,d\zeta \end{aligned}$$

by using the change of variable $\zeta = \xi^3/\sqrt{n}$ and the notation

$$\begin{cases} f_n(\zeta) = (\alpha^2/3)\; n^{7/6}\;\exp(-2n^{1/3}\zeta^{2/3})\;\zeta^{4/3}\left|i + (\zeta/n)^{1/3}\right|^2, \\ g_n(\zeta) = \displaystyle\int_0^1 \left(1 + s\alpha(\zeta/n)\;\left[i + (\zeta/n)^{1/3}\right]\right)^{n-1} ds. \end{cases}$$

Then, we can check easily that

$$\begin{cases} f_n(\zeta) \longrightarrow C\delta_{\zeta=0}, \qquad C = \alpha^2 \int_0^\infty e^{-2|\zeta|^2} |\zeta|^6 d\zeta, \\ g_n(\zeta) \longrightarrow \int_0^1 e^{i\alpha s\zeta} ds = g(\zeta) \qquad uniformly \end{cases}$$

as n goes to infinity. It follows that

$$\Gamma_n \longrightarrow Cg(0) = C > 0$$

which ends the proof of (16).

4 Stable laws

It is possible to extend our formalism and results to the case of stable laws. Let us consider the stricly stable random variable θ, associated to the characteristic function

$$\Theta(\xi) = e^{-\sigma|\xi|^p} = \widehat{f_\infty}(\xi), \qquad 0 < p \leq 2, \qquad 0 < \sigma.$$

For $p = 2$, one recovers the Gaussian law, for $p = 1$, the Cauchy law... The normal domain of attraction (NDA) is the set of densities f such that, when dealing with the iidrv. X_i distributed folowing f, the normalized sum $\frac{X_1+...+X_n}{n^{1/p}}$ converges to θ. The NDA is characterized by requiring that (see [3]), for all $\xi \in \mathbb{R}^N$,

$$\lim_{n\to\infty} n\big(\widehat{f}(\xi/n^{1/p}) - 1\big) = -\sigma|\xi|^p.$$

Then, we search for information on the rate of convergence for some metric, possibly, of course, at the cost of restricting to a part of the NDA, in the spirit of [9]. According to the previous section, one still consider the Fourier-based metric

$$d_p(f,g) = \sup_{\xi\neq 0} \frac{|\widehat{f}(\xi) - \widehat{g}(\xi)|}{|\xi|^p}.$$

This quantity is finite when f and g belong to the set

$$D_{\sigma,p} = \{f \in \mathcal{P},\ \widehat{f}(\xi) = 1 - \sigma|\xi|^p + \psi_f(\xi), \text{ with } \frac{\psi_f(\xi)}{|\xi|^p} \in L^\infty(\mathbb{R}^N)\}.$$

Introduction of such a set is best understood when one realizes that for the normal case ($p = 2$) the keypoint of the previous developments was an expansion $\widehat{f}(\xi) = 1 - \sigma|\xi|^2 +$

$\psi_f(\xi)$ with a favorable control on the remainder ψ_f: it has to decay faster than $|\xi|^2$. Notice also that

- f_∞ belongs to $D_{\sigma,p}$ (and actually the remainder goes to 0 faster than $|\xi|^p$),
- if $g \in D_{\sigma,p}$ and $d_p(f,g) < \infty$ then $f \in D_{\sigma,p}$.

Indeed, the second claim follws from the following observation

$$\begin{aligned}\left|\frac{\widehat{f}(\xi) - 1 + \sigma|\xi|^p}{|\xi|^p}\right| &\leq \left|\frac{\widehat{f}(\xi) - \widehat{g}(\xi)}{|\xi|^p}\right| + \left|\frac{\widehat{g}(\xi) - 1 + \sigma|\xi|^p}{|\xi|^p}\right| \\ &\leq d_p(f,g) + \psi_g(\xi),\end{aligned}$$

where $\psi_g(\xi) \in L^\infty(\mathbb{R}^N)$.

We naturally introduce the following restriction of $D_{\sigma,p}$, that we call the Fourier domain of attraction (FDA)

$$D^0_{\sigma,p} = \left\{ f \in D_{\sigma,p}, \quad \frac{\psi(\xi)}{|\xi|^p} \xrightarrow[\xi\to 0]{} 0 \right\}.$$

We first compare convergence in $D^0_{\sigma,p}$ for the d_p metric to the usual convergence and FDA to NDA.

Lemma 5 *If f_n is a sequence in $D^0_{\sigma,p}$ such that*

$$\begin{cases} \sup\limits_n \dfrac{|\psi_n(\xi)|}{|\xi|^p} \xrightarrow[\xi\to 0]{} 0, \\ d_p(f_n, f) \xrightarrow[n\to\infty]{} 0 \quad \text{for some } f \in \mathcal{P}. \end{cases}$$

Then, $f_n \rightharpoonup f$ and $f \in D^0_{\sigma,p}$.

Proof. By definition, one has $|\widehat{f_n}(\xi) - \widehat{f}(\xi)| \leq d_p(f_n, f)\ |\xi|^p$. This implies that $\widehat{f_n}(\xi)$ converges to $\widehat{f}(\xi)$ for $\xi \neq 0$ when $n \to \infty$. On the other hand, since f_n and f belong to $\mathcal{P}$, one has $\widehat{f_n}(0) = 1 = \widehat{f}(0)$. One deduces that $f_n \rightharpoonup f$.

Next, let $\varepsilon > 0$, $|\xi| = 1$ and $t > 0$. Then, the quantity

$$\begin{aligned}\frac{|\widehat{f_n}(t\xi) - \widehat{f}(t\xi)|}{t^p\ |\xi|^p} &= \frac{|1 - \sigma t^p|\xi|^p + \psi_n(t\xi) - \widehat{f}(t\xi)|}{t^p\ |\xi|^p} \\ &= \left|\frac{1 - \sigma t^p|\xi|^p - \widehat{f}(t\xi)}{t^p\ |\xi|^p} + \frac{\psi_n(t\xi)}{t^p\ |\xi|^p}\right| \\ &\leq d_p(f_n, f)\end{aligned}$$

is $\leq \varepsilon$ provided $n \geq N(\varepsilon)$ is large enough. Therefore, it follows that

$$\begin{aligned}\frac{|\widehat{f}(t\xi) - (1-\sigma t^p|\xi|^p)|}{t^p\ |\xi|^p} &= \frac{|\psi_f(t\xi)|}{t^p\ |\xi|^p} = \frac{|\widehat{f}(t\xi) - \widehat{f}_n(t\xi) + \psi_n(t\xi)|}{t^p\ |\xi|^p} \\ &\leq d_p(f_n, f) + \sup_n \frac{|\psi_n(t\xi)|}{t^p\ |\xi|^p} \leq 2\varepsilon\end{aligned}$$

holds by choosing $n \geq N(\varepsilon)$ and then $|t| \leq \eta(\varepsilon)$ to control the second term. This proves that $f \in D^0_{\sigma,p}$.

It is worth remarking that we cannot remove the uniform control on the remainder, otherwise, the limit f does not belong to $D^0_{\sigma,p}$. Indeed, assume that there exist $\varepsilon > 0$, a sequence ξ_n that tends to 0, and a subsequence, still labelled by n, such that $|\psi_n(\xi_n)|/|\xi_n|^p \geq 2\varepsilon$. Then, we are led to

$$\begin{aligned}\frac{|\widehat{f}(\xi_n) - (1-\sigma|\xi_n|^p)|}{|\xi_n|^p} &= \frac{|\widehat{f}(\xi_n) - \widehat{f}_n(\xi_n) + \psi_n(\xi_n)|}{|\xi_n|^p} \\ &\geq \left| \frac{|\psi_n(\xi_n)|}{|\xi_n|^p} - \frac{|\widehat{f}(\xi_n) - \widehat{f}_n(\xi_n)|}{|\xi_n|^p} \right| \\ &\geq 2\varepsilon - d_p(f_n, f)\end{aligned}$$

that remains $> \varepsilon$ for large enough n's. □

Lemma 6 *If $f \in D^0_{\sigma,p}$ then, $f \in NDA$. If $N = 1$, one actually has $D^0_{\sigma,p} = NDA$.*

Proof. Let $f \in D^0_{\sigma,p}$. Then, one has

$$\begin{aligned}\sigma|\xi|^p + n\big(\widehat{f}(\xi/a_n) - 1\big) &= \sigma|\xi|^p - n\sigma\big(|\xi|/a_n\big)^p + n\psi(\xi/a_n) \\ &= \sigma|\xi|^p\Big(1 - \frac{n}{a_n^p}\Big) + \frac{\psi(\xi/a_n)}{\big(|\xi|/a_n\big)^p}\ \frac{n}{a_n^p}\ |\xi|^p \xrightarrow[n\to\infty]{} 0\end{aligned}$$

for any $\xi \in \mathbb{R}^N\backslash\{0\}$ and any sequence $a_n \sim n^{1/p}$. Therefore $f \in NDA$.

Conversely, for $f \in NDA$, we set $\psi(\xi) = \widehat{f}(\xi) - 1 + \sigma|\xi|^p$, so that $n\psi(\xi/n^{1/p}) = |\xi|^p \frac{\psi(\xi/n^{1/p})}{|\xi|^p/n}$ tends to 0, for any $\xi \neq 0$ when $n \to \infty$. Therefore, the problem reduces to deduce from $\lim_{n\to\infty} g(s/n) = 0$ for any $s > 0$ that $\lim_{s\to 0} g(s) = 0$. Of course this is false in general, but here one knows that g is continuous on $\mathbb{R}^{*+}$ since it is obtained from the Fourier transform of a probability measure. Suppose that $g(s)$ does not tend to 0 when s vanishes: there exist $\varepsilon > 0$ and a sequence s_n such that $\lim_{n\to 0} s_n = 0$ while $|g(s_n)| > \varepsilon$. Since g is continuous in $s_1 > 0$, there exists $\gamma_1 > 0$ such that $|g(x)| > \varepsilon/2$

for any $x \in [s_1; s_1+\gamma_1] = F_1$. Set $n_1 = 1 = k_1$, so that for all $x \in F_1$ we have $|g(x/n_1)| \geq \varepsilon/2$. Next, by using $\lim_{n\to\infty} s_n = 0$, we can find some $s_{k_2} \in]0, \min(s_{k_1}, \gamma_1)[$. One has $|g(s_{k_2})| > \varepsilon$ and, by continuity, there exists some $\gamma_2 > 0$ such that $|g(x)| > \varepsilon/2$ for any $x \in [s_{k_2}; s_{k_2}+\gamma_2]$. Furthermore, $\mathbb{R}$ being archimedian, there exists an integer $n_2 = \left[s_{k_1}/s_{k_2}\right]$ verifying $s_{k_1} \leq n_2 s_{k_2} \leq s_{k_1} + s_{k_2} \leq s_{k_1} + \gamma_1$. Hence, we can choose $\gamma_2 > 0$ small enough to garantee that $s_{k_1} \leq n_2 s_{k_2} \leq n_2(s_{k_2}+\gamma_2) < s_{k_1}+\gamma_1$. Then, we set $F_2 = F_1 \cap [n_2 s_{k_2}; n_2(s_{k_2}+\gamma_2)]$. Notice that the interior of F_2 is non empty and for any $x \in F_2$, one has $|g(x/n_2)| \geq \varepsilon/2$. Repeating this reasoning, one constructs

- a decreasing subsequence $\left(s_{k_\ell}\right)_{\ell\in\mathbb{N}^*}$ which tends to 0 as $\ell \to \infty$,
- a sequence of positive reals γ_ℓ,
- a sequence of closed intervals F_ℓ having non empty interiors and verifying $F_{\ell+1} \subset F_\ell$,
- and a sequence of integers n_ℓ (given by $\left[n_{\ell-1}s_{k_{\ell-1}}/s_{k_\ell}\right]$) that increases to $+\infty$ as $\ell \to \infty$,

such that for all $\ell \in \mathbb{N}^*$, all $x \in F_\ell$, $|g(x/n_\ell)| \geq \varepsilon/2$. However $F = \bigcap_\ell F_\ell$ is a non empty interval. Thus, for $x \in F$, one has $|g(x/n_\ell)| \geq \varepsilon/2$ that contradicts the assumption $\lim_{n\to 0} g(x/n) = 0$. □

Rate of convergence will be discussed by introducing the set, for $\delta > 0$,

$$D^\delta_{\sigma,p} = \{f \in D^0_{\sigma,p},\ |\psi(\xi)|/|\xi|^p \leq |\xi|^\delta\}.$$

Remark 5 $\widehat{f_\infty}(\xi) = e^{-\sigma|\xi|^p} \in D^\delta_{\sigma,p}$ *if and only if* $0 \leq \delta \leq p$

Remark 6 *If* $\delta > 0$ *and* f_n *is a sequence bounded in* $D^\delta_{\sigma,p}$*, such that* $d_p(f_n, f) \xrightarrow[n\to\infty]{} 0$ *for some* $f \in \mathcal{P}$*. Then,* $f_n \rightharpoonup f$ *and* $f \in D^\delta_{\sigma,p}$.

One naturally asks for a easy-to-check criterion ensuring that a given density f lies in $D^\delta_{\sigma,p}$. A simple criterion relies on finiteness of some pseudo-moment. For the normal case $p=2$, f_∞ has all moments finite, in turn integrability of $x^\alpha(f-f_\infty)$ is equivalent to integrability for $x^\alpha\, f$. This does not apply when $p<2$. Note also that, for $p=2$, the set considered in the previous section satisfies

$$\{f \in \mathcal{P}, \int (1, x, x\otimes x)\, f(x)\, dx = \int (1, x, x\otimes x)\, \omega_K(x)\, dx, \int |x|^{2+\delta}\, f(x)\, dx < \infty\} \subset D^\delta_{\sigma,2}$$

with a strict inclusion. Let us discuss in the following lemmata some connections between moments and distance $d_{p+\delta}$, in the general case.

Lemma 7 *Let $0 \le p < 1$ and $0 < \delta \le \min(p, 1-p)$. Then, the following embedding holds*

$$\{f \in \mathcal{P}, \int |x|^{p+\delta} |f(x) - f_\infty(x)| \, dx < \infty\} \subset D^\delta_{\sigma,p}.$$

Let $1 \le p < 2$ and $0 \le \delta \le 2 - p$. Then

$$\{f \in \mathcal{P}, \int |x|^{p+\delta} |f(x) - f_\infty(x)| \, dx < \infty, \int x \, (f(x) - f_\infty(x)) \, dx = 0\} \subset D^\delta_{\sigma,p}.$$

Proof. The lemma follows from the following claim

$$\left\{ \begin{array}{l} \text{Let } g \in L^1(\mathbb{R}^N) \text{ and } 0 < \gamma < 1 \text{ such that } |x|^\gamma g \in L^1(\mathbb{R}^N), \text{ then} \\ \widehat{g}(\xi) = \widehat{g}(0) + \psi_g(\xi), \qquad \psi_g(\xi)/|\xi|^\gamma \underset{\xi \to 0}{\longrightarrow} 0. \end{array} \right. \tag{17}$$

If $\gamma = 1$, then $\widehat{g}$ is C^1 with bounded gradient, and we can write a similar expansion, but the remainder only satisfies $\psi_g(\xi)/|\xi|$ bounded. Indeed, applying the property (17) with $\gamma = p + \delta$, $g = f - f_\infty$ gives the first embedding. The second one follows from the same reasoning, with an expansion at an higher order since now we deal with C^1 functions.

Proof of (17) relies on the following simple remark

$$\begin{aligned} \left| \frac{\widehat{g}(\xi) - \widehat{g}(0)}{|\xi|^\gamma} \right| &= \left| \int_{\mathbb{R}} g(x) |x|^\gamma \, \frac{e^{-ix\cdot\xi} - 1}{(|x| \; |\xi|)^\gamma} \, dx \right| \\ &\le \left| \int_{|x||\xi| \le R} |g(x)| \; |x|^\gamma \, \frac{e^{-ix\cdot\xi} - 1}{(|x| \; |\xi|)^\gamma} \, dx \right| + \left| \int_{|x||\xi| \ge R} |g(x)| \; |x|^\gamma \, \frac{e^{-ix\cdot\xi} - 1}{(|x| \; |\xi|)^\gamma} \, dx \right| \\ &\le \int_{|x| \le R/|\xi|} |g(x)| \; |x|^\gamma \, \frac{|e^{-ix\cdot\xi} - 1|}{(|x| \; |\xi|)^\gamma} \, dx + \frac{2}{R^\gamma} \int_{\mathbb{R}^N} |g(x)| \; |x|^\gamma \, dx \end{aligned}$$

Let $\varepsilon > 0$. Choose $R = R(\varepsilon)$ so that the second term becomes $\le \varepsilon$. Then, by using the Lebesgue theorem, one sees that there exists $h(\varepsilon) > 0$ so that for $|\xi| \le \eta(\varepsilon)$, the first term is also $\le \varepsilon$. In turn, one gets

$$\left| \frac{\widehat{g}(\xi) - \widehat{g}(0)}{|\xi|^\gamma} \right| \le 2\varepsilon$$

provided $|\xi| \le \eta(\varepsilon)$. □

Lemma 8 *If $d_{p+\delta}(f, f_\infty) < \infty$ with $p + \delta > 1$, then f has a vanishing first moment in the weak sense that*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} x f(x) \; e^{-\varepsilon x^2/2} \, dx = 0.$$

Proof. By a parity argument, the integral of $xf_\infty(x)\ e^{-\varepsilon x^2/2}$ vanishes. Therefore, we will show the result for $g = f - f_\infty$ that fulfills $|\widehat{g}(\xi)| \le C|\xi|^s$ for $s = p + \delta > 1$. Of course, this gives $\widehat{g}(0) = 0 = \int g(x)\,dx$ but we also have

$$\frac{|\widehat{g}(\xi) - \widehat{g}(0)|}{|\xi|} \le C|\xi|^{s-1} \xrightarrow[|\xi|\to 0]{} 0,$$

which says that $\widehat{g}$ is derivable in 0 with $\nabla\widehat{g}(0) = 0$. Then, we use the relation $\widehat{ixf} = \nabla(\widehat{f})$ which holds in $\mathcal{S}'$ with the mollifiers $\varphi_\varepsilon(\xi) = (2\pi\varepsilon)^{-N/2}\ e^{-\xi^2/(2\varepsilon)}$ as test functions. One gets

$$\begin{aligned}
\langle \widehat{ixf}, \varphi_\varepsilon\rangle &= \int_{\mathbb{R}^N} ixf(x)\ \widehat{\varphi_\varepsilon}(x)\,dx = \int_{\mathbb{R}^N} ixf(x)\ (2\pi)^{-N/2}\ e^{-\varepsilon x^2/2}\,dx \\
&= \langle \widehat{f}, \nabla\varphi_\varepsilon\rangle = \int_{\mathbb{R}^N} \widehat{f}(\xi)\ (2\pi\varepsilon)^{-N/2}\ e^{-\xi^2/(2\varepsilon)}\ (-\xi/\varepsilon)\,d\xi \\
&= \int_{\mathbb{R}^N} \widehat{f}(\sqrt{\varepsilon}\zeta)\ (2\pi)^{-N/2}\ e^{-\zeta^2/2}\ (-\zeta/\sqrt{\varepsilon})\,d\zeta
\end{aligned}$$

that can be estimated by

$$C\ \sqrt{\varepsilon}^{p+\delta-1} \quad \int_{\mathbb{R}^N} |\zeta|^{p+\delta+1}\ (2\pi)^{-N/2}\ e^{-\zeta^2/2}\,d\zeta = C^{st}\ \varepsilon^{(s-1)/2},$$

with $s > 1$. □

Let us now prove the corresponding convergence result. The stable law satisfies

$$\widehat{f_\infty}(\xi) = e^{-\sigma|\xi|^p} = 1 - \sigma|\xi|^p + \psi_\infty(\xi), \qquad \psi_\infty(\xi)/|\xi|^p \xrightarrow[\xi\to 0]{} 0.$$

Then, for $|\xi| \le \varepsilon$ small enough, one has $\widehat{f_\infty}(\xi) + \sigma|\xi|^p/8 = 1 - 7\sigma|\xi|^p/8 + \psi_\infty(\xi) \le 1 - \sigma|\xi|^p/4 \le e^{-\sigma|\xi|^p/4}$. Let $f \in D^0_{\sigma,p}$. One gets $\widehat{f}(\xi) = 1 - \sigma|\xi|^p + \psi_f(\xi) = \widehat{f_\infty}(\xi) + r_f(\xi)$ where $r_f(\xi)/|\xi|^p \xrightarrow[\xi\to 0]{} 0$. Then, we can choose $0 < \eta < \varepsilon$ sufficiently small to ensure that

$$|\widehat{f}(\xi)| \le \exp\left\{-\frac{\sigma}{4}|\xi|^p\right\}$$

holds for any $|\xi| \le \eta$. Lemma 3 generalizes readily to

$$\frac{|\widehat{f_n}(\xi) - \widehat{f_\infty}(\xi)|}{|\xi|^{p+\delta}} \le \frac{1}{n^{\delta/p}}\ d_{p+\delta}(f, f_\infty)\ \exp\left\{-\frac{\sigma}{8}|\xi|^p\right\}.$$

for $n \geq 2$, $|\xi| \leq \eta n^{1/p}$. One deduces the estimate on the low frequencies integral

$$\begin{aligned}
L_n &= \int_{|\xi|\leq \eta n^{1/p}} (1+|\xi|^2)^k \left|\widehat{f_n}(\xi) - \widehat{f_\infty}(\xi)\right|^2 d\xi \\
&\leq \frac{1}{n^{2\delta/p}} d_{p+\delta}^2(f, f_\infty) \int_{|\xi|\leq \eta n^{1/p}} (1+|\xi|^2)^k \exp\left\{-\frac{\sigma}{4}|\xi|^p\right\} |\xi|^{2(p+\delta)} d\xi \\
&\leq \frac{C}{n^{2\delta/p}} d_{p+\delta}^2(f, f_\infty)
\end{aligned}$$

with $C := \int_{\mathbb{R}^N} |\xi|^{2(p+\delta)}(1+|\xi|^2)^k \exp\left\{-\frac{\sigma}{4}|\xi|^p\right\} d\xi$. Since the estimates on the high frequencies apply *mutadis mutandis*, we are led to the following statement.

Theorem 4 *Let $f \in D^\delta_{\sigma,p}$ satisfy (2), for some $\delta \in]0,p]$. Then $\|f_n - f_\infty\|_{H^k} \leq C\ n^{-\delta/p}$.*

Remark 7 *With above assumptions we get easily the same rate of convergence for the fourier based disance: $d_{p+\delta}(f_n, f_\infty) \leq D\ n^{-\delta/p}$.*

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