A two-scale convergence result for a nonlinear scalar conservation law in one space variable

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Abstract

We prove a general result of two-scale convergence for BV functions. We then use this result to show the validity of nonlinear geometric optics for weak solutions to scalar conservation laws when the small perturbation of the initial datum is only BV and periodic with respect to the slow and the fast variable.

1 Introduction

At least in the one dimensional case, the justification of weakly nonlinear geometric optics (WNLGO) for systems [7],[12], [16], ... , naturally leads to considering the scalar case

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = 0 \qquad (1)$$

$$u^\varepsilon(0,x) = \underline{u} + \varepsilon u_0\left(x, \frac{x}{\varepsilon}\right) \qquad (2)$$

where $\underline{u}$ is a constant, u_0 is 1-periodic in $\theta := \dfrac{x}{\varepsilon}$ and $\varepsilon > 0$ is a given small parameter. Without loss of generality one may assume that $\underline{u} = 0$. Thanks to the ε factor, the high frequency oscillations propagate and the natural ansatz is ([12]) :

$$u^\varepsilon(t,x) = \varepsilon\sigma\left(t, x, \frac{x - f'(0)t}{\varepsilon}\right) + \cdots \qquad (3)$$

where $\sigma(t,x,\theta)$ is 1-periodic in θ.

The following cases have already been studied in the literature, see [5] for the case where $u_0(x,\theta) \equiv u_0(\theta)$ is a BV function, [3] for the case where $u_0(x,\theta)$ is smooth, and [15] when $u_0(x,\theta)$ has a compact support in x and θ and is Lipschitz in x with values in L^1_θ.

The goal of this paper is to rigorously justify (3) when u_0 is BV with respect to x and θ. This is a first step towards the study of the interaction between high frequency oscillations and strong shock waves for systems.

In this paper we treat the general case. We only assume f to be in $C^3(\mathbb{R},\mathbb{R})$. Then f is not necessarily convex. Note that our main result (Theorem 5) remains true when $f''(0) = 0$. Indeed, in this degenerate case, we obtain that the linearization of (1) gives a good approximation up to a time of order ε^{-1}.

The outline of the paper is as follows. After recalling a few basic facts in Section 2, we give a general result of two-scale convergence for BV functions in Section 3. Then in Section 4 we

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justify the asymptotic formula (3) with these very weak regularity assumptions, first for the case of Burgers equation and then for the general case. As in [12], one gets Burgers equation for σ after formal substitution of (3) into (1) and retaining only the terms of order O(1) and O(ε) for $\varepsilon > 0$.

2 Basic facts

2.1 BV-functions

We will use some properties of BV-functions. Let $\mathbb{T}$ be the unit circle, and H_1 (defined below) be the one-dimensional Hausdorff measure on $\mathbb{R}\times\mathbb{T}$. H_1 is the natural measure for one-dimensional subsets of $\mathbb{R}\times\mathbb{T}$ (see [13]).

For computations, we identify $\mathbb{T}$ with $[0,1[$.
Moreover, for convenience, we denote also any function $v(\theta)$ defined on $\mathbb{T}$ by $v(\widetilde{\theta})$ the same function defined on $\mathbb{R}$, and one-periodic with respect to $\widetilde{\theta}$, where $\widetilde{\theta}$ belongs to $\mathbb{R}$.

For any $X=(x,\theta)$, $Y=(y,\eta)\in\mathbb{R}\times\mathbb{T}$, we define $|X-Y| = \left(|x-y|^2+\min(|\theta-\eta|,1-|\theta-\eta|)^2\right)^{\frac{1}{2}}$.
For convenience we recall

Definition 1 (one-dimensional Hausdorff measure on $\mathbb{R}\times\mathbb{T}$.)
For any subset S of $\mathbb{R}\times\mathbb{T}$, define the diameter of S : $diam(S)=\sup\{|X-Y| \ : X,Y\in S\}$.
For any subset A of $\mathbb{R}\times\mathbb{T}$, we define the one-dimensional Hausdorff measure $H_1(A)$ by the following process. For δ small, cover A by countably many sets S_j with $diam(S_j)\le\delta$, add up all the $diam(S_j)$, and take the limit as $\delta\to 0$.

$$H_1(A) = \lim_{\delta\to 0} \inf_{\substack{A\subset\cup S_j \\ diam(S_j)\le\delta}} \sum_j diam(S_j)$$

For example, we have $H_1(\{0\}\times\mathbb{T})=1$ and $H_1(]0,1[\times\mathbb{T})=+\infty$. Then, if $H_1(A)=0$ the set is "very small": for any line D, $D\cap A$ is a set of measure zero with respect to the one-dimensional Lebesgue measure on the line D.

Here $C_c^\infty(\mathbb{R}\times\mathbb{T},\mathbb{R})$ denotes the space of C^∞ functions $\varphi(x,\theta)$ on $\mathbb{R}\times\mathbb{T}$, and the subscript $_c$ denotes, in all this paper, a space of functions with compact support.

Let u be a function in $L^1_{loc}(\mathbb{R}\times\mathbb{T})$. We say that u is in $BV(\mathbb{R}\times\mathbb{T})$ if there exists two constants C_x and C_θ such that for any $\varphi\in C_c^\infty(\mathbb{R}\times\mathbb{T},\mathbb{R})$:

$$\left|\int\int_{\mathbb{R}\times\mathbb{T}} u(x,\theta)\partial_x\varphi(x,\theta)dxd\theta\right| \le C_x\|\varphi\|_\infty, \tag{4}$$

$$\left|\int\int_{\mathbb{R}\times\mathbb{T}} u(x,\theta)\partial_\theta\varphi(x,\theta)dxd\theta\right| \le C_\theta\|\varphi\|_\infty. \tag{5}$$

We denote by $\left\|\frac{\partial u}{\partial x}\right\|_1$ (respectively $\left\|\frac{\partial u}{\partial \theta}\right\|_1$) the smallest constant C_x (respectively C_θ).
Moreover we have for any $u\in BV(\mathbb{R}\times\mathbb{T})(see[20])$:

$$\left\|\frac{\partial u}{\partial x}\right\|_1 := \int_{\mathbb{T}} TV_x u(.,\theta)d\theta = \|TV_x u\|_{L^1(\mathbb{T})} \tag{6}$$

$$\left\|\frac{\partial u}{\partial \theta}\right\|_1 := \int_{\mathbb{R}} TV_\theta u(x,.)dx = \|TV_\theta u\|_{L^1(\mathbb{R})} \tag{7}$$

where TV_x (respectively TV_θ) denotes the total variation on $\mathbb{R}$ (respectively one-period in θ). We will use the total variation of u, with $X = (x, \theta)$:

$$TVu := \left(\left\| \frac{\partial u}{\partial x} \right\|_1^2 + \left\| \frac{\partial u}{\partial \theta} \right\|_1^2 \right)^{\frac{1}{2}} = \sup_{H \in \mathbb{R} \times \mathbb{T} - \{(0,0)\}} \frac{1}{|H|} \int_{\mathbb{R} \times \mathbb{T}} |u(X+H) - u(X)| dX. \tag{8}$$

Here we only consider functions $u \in BV \cap L^1(\mathbb{R} \times \mathbb{T}, \mathbb{R})$.

As in [19] (with a slight modification) we can define H_1-almost everywhere the *symmetric average of an arbitrary function* u by:

$$\overline{u}(x, \theta) := \lim_{r \to 0} \frac{1}{4r^2} \int_{-r}^{r} \int_{-r}^{r} u(x+y, \theta+\eta) dy d\eta, \quad H_1 \ a.e. \tag{9}$$

Finally, as in $BV \cap L^1(\mathbb{R}^2, \mathbb{R})$ ([6]), for any function u in $BV \cap L^1(\mathbb{R} \times \mathbb{T}, \mathbb{R})$, there exists a sequence of functions φ_n in $C_c^\infty(\mathbb{R} \times \mathbb{T}, \mathbb{R})$ such that

$$\lim_{n \to \infty} \|\varphi_n - u\|_{L^1} = 0, \tag{10}$$

$$\lim_{n \to \infty} \varphi_n(x, \theta) = \overline{u}(x, \theta), \quad H_1 - a.e \tag{11}$$

$$\lim_{n \to \infty} \left\| \frac{\partial \varphi_n}{\partial x} \right\|_1 = \left\| \frac{\partial u}{\partial x} \right\|_1, \tag{12}$$

$$\lim_{n \to \infty} TV(\varphi_n) = TV(u). \tag{13}$$

Moreover if u $\in BV \cap L^1 \cap L^\infty(\mathbb{R} \times \mathbb{T}, \mathbb{R})$ then

$$\forall n \in \mathbb{N}, \quad \|\varphi_n\|_\infty \leq \|u\|_\infty. \tag{14}$$

Of course, we recall that C_c^∞ is not dense in BV for the BV topology, since L^1 is a closed subspace of the space M_1 of bounded measures.

2.2 Kružkov estimates for scalar conservation law

For global existence and uniqueness of the solution to the initial value problem for a scalar conservation law we use the general result of S.N. Kružkov ([9]), which only requires that the flux is C^1 and the initial data is a bounded measurable function. We will often use the maximum principle, the stability with respect to the initial data in L^1 norm and the decay of the total variation if the initial data is a BV function. We recall this classical result, first for $BV \cap L^1 \cap L^\infty(\mathbb{R}^d, \mathbb{R})$ and then for $BV \cap L^1 \cap L^\infty(\mathbb{R} \times \mathbb{T}, \mathbb{R})$.

We denote by $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R})$ the space of distributions on $\mathbb{R}^+ \times \mathbb{R}^d$.

Let $\mathbf{f}$ be $\in C^1(\mathbb{R}, \mathbb{R}^d)$, $|.|$ is the Euclidean norm of $\mathbb{R}^d$ or $\mathbb{R}$, and $R > 0$. We write

$$Lip[\mathbf{f}, R] := \sup_{\substack{|u| \leq R \\ |v| \leq R \\ u \neq v}} \frac{|\mathbf{f}(u) - \mathbf{f}(v)|}{|u - v|}.$$

Theorem 1 (Entropy Solution ([9]))
Let $\mathbf{f} \in C^1(\mathbb{R}, \mathbb{R}^d)$, *and* $u_0, v_0 \in L^\infty(\mathbb{R}^d, \mathbb{R})$,

$$\begin{aligned} \frac{\partial u}{\partial t} + div\ \mathbf{f}(u) &= 0, && (15)\\ u(0,x) &= u_0(x), && (16)\\ v(0,x) &= v_0(x). && (17) \end{aligned}$$

If u *is a weak solution of (15), (16) and, for all* $k \in \mathbb{R}$,

$$\frac{\partial}{\partial t}|u-k| + div\ ((sgn(u-k)(\mathbf{f}(u)-\mathbf{f}(k))) \leq 0 \ in\ \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}), \tag{18}$$

then u *exists and is unique.* u *is called the entropy solution to (15), (16).*

Let v *be the entropy solution of (15), (17), and* $L = Lip[f, \max(\|u_0\|_\infty, \|v_0\|_\infty)]$. *Furthermore, we have*

$$u \in C^0(\mathbb{R}^+, L^1_{loc}(\mathbb{R}^d, \mathbb{R})), \tag{19}$$

$$if\ u_0 \in L^1(\mathbb{R}^d, \mathbb{R})\ then\ u \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^d, \mathbb{R})), \tag{20}$$

and, $\forall t \geq 0$:

$$\forall A > 0, \quad \int_{|x|<A+L\ t} |u(t,x) - v(t,x)| dx \leq \int_{|x|<A} |u_0(x) - v_0(x)| dx, \tag{21}$$

$$if\ u_0 \in L^1(\mathbb{R}^d, \mathbb{R})\ then \quad \int_{\mathbb{R}^d} |(u(t,x)| dx \leq \int_{\mathbb{R}^d} |(u_0(x)| dx, \tag{22}$$

$$if\ u_0 \in BV(\mathbb{R}^d, \mathbb{R})\ then \quad TV(u(t,.)) \leq TV(u_0(.)), \tag{23}$$

$$if\ u_0 \in L^\infty(\mathbb{R}^d, \mathbb{R})\ then \quad \|u(t,.)\|_\infty \leq \|u_0(.)\|_\infty. \tag{24}$$

Remark 1 *We can rewrite the Kružkov inequality in the two following ways:*

1. *For all* $k \in \mathbb{R}$, *for all* $\phi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^+)$, *we have*

$$\begin{aligned} 0 \ \geq\ & \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} \left(|u(t,x) - k| \frac{\partial \phi}{\partial t} + (sgn(u(t,x) - k)(\mathbf{f}(u(t,x)) - \mathbf{f}(k).\nabla_x \phi \right) dx dt \\ & + \int_{\mathbb{R}^d} |u_0(x) - k| \phi(0,x) dx. \end{aligned} \tag{25}$$

2. *For all* $k \in \mathbb{R}$, *for all* $\phi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^+)$, *we have for all* $t \geq 0$*:*

$$\begin{aligned} 0 \ \geq\ & \int_0^t \int_{\mathbb{R}^d} \left(|u(t,x) - k| \frac{\partial \phi}{\partial t} + (sgn(u(t,x) - k)(\mathbf{f}(u(t,x)) - \mathbf{f}(k).\nabla_x \phi \right) dx dt \\ & + \int_{\mathbb{R}^d} |u_0(x) - k| \phi(0,x) dx - \int_{\mathbb{R}^d} |u(t,x) - k| \phi(t,x) dx. \end{aligned} \tag{26}$$

We will use the two following lemmas to recall some classical results on stability with respect to the flux and the initial data.

Lemma 1 ([17]) *Let* u, v *be* $\in BV \cap L^1(\mathbb{R}^d, \mathbb{R})$, ϱ *a smooth nonnegative function with support included in* $\{x,\ |x| \leq \rho\}$, *and* $\int_{\mathbb{R}^d} \varrho(x)\ dx = 1$, *then*

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x) - v(y)| \varrho(y-x)\ dx dy - \int_{\mathbb{R}^d} |u(x) - v(x)| dx \right| \leq \rho TV(v),$$

where $TV(v) = \left(\sum_{i=1}^{d} \left\| \frac{\partial v}{\partial x_i} \right\|_1^2 \right)^{\frac{1}{2}}$.

Proof: Let $I := \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x) - v(y)| \varrho(y - x) \; dxdy - \int_{\mathbb{R}^d} |u(x) - v(x)| dx \right|$, then

$$I \;=\; \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|u(x) - v(y)| - |u(x) - v(x)|) \, \varrho(y - x) \; dxdy. \right|$$

$$I \;\leq\; \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v(y) - v(x)| \varrho(y - x) \; dxdy$$

We wish to use the classical inequality: $\int_{\mathbb{R}^d} |v(x + h) - v(x)| dx \leq TV(v)|h|$. If $|h| \leq \rho$ then we have

$$\begin{aligned} I &\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |v(x + h) - v(x)| dx \right) \varrho(h) dh \\ &\leq \int_{\mathbb{R}^d} |h| TV(v) \varrho(h) dh \\ &\leq TV(v) \int_{\mathbb{R}^d} \rho \varrho(h) dh = \rho TV(v). \end{aligned}$$

□

Lemma 2 ([17])

Let u, v *be* $\in BV \cap L^1 \cap L^\infty(\mathbb{R}^d, \mathbb{R})$, $\mathbf{h}$ *be* $\in C^1(\mathbb{R}, \mathbb{R}^d)$, $L := Lip[\mathbf{h}, \max(\|u\|_\infty, \|v\|_\infty)]$, ϱ *a smooth nonnegative function with support contained in* $\{x, \; |x| \leq \rho\}$, *and* $\int_{\mathbb{R}^d} \varrho(x) \; dx = 1$,

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (sgn(u(x) - v(y))(\mathbf{h}(u(x)) - \mathbf{h}(v(y))).\nabla_y \rho(x - y) \; dxdy \right| \leq L \; TV(v). \tag{27}$$

Proof: First, if u and v belong to $C_c^\infty(\mathbb{R}^d, \mathbb{R})$, we set

$$\begin{aligned} I &:= \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (sgn(u(x) - v(y))(\mathbf{h}(u(x)) - \mathbf{h}(v(y)).\nabla_y \rho(x - y) \; dxdy \right| \\ I &= \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (sgn(u(x) - v(y)) \rho(x - y) \nabla_y \mathbf{h}(v(y) \; dxdy \right| \\ &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x - y) |\nabla_y \mathbf{h}(v(y)| \; dxdy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(h) |\nabla_y \mathbf{h}(v(y)| \; dhdy \\ &= \int_{\mathbb{R}^d} |\nabla_y \mathbf{h}(v(y)| \; dy = TV(\mathbf{h} \; o \; v) \leq L \; TV(v). \end{aligned}$$

Second, if u, v only belong to $BV \cap L^1 \cap L^\infty(\mathbb{R}^d, \mathbb{R})$, we have the properties similar to (10), (11),(13), (14) in $BV \cap L^1 \cap L^\infty(\mathbb{R}^d, \mathbb{R})$ (see [6]). Therefore, there exist $(u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}}$ such that

$$
\begin{aligned}
\forall n \in \mathbb{N}, &\quad u_n, v_n \in C_c^\infty(\mathbb{R}^d, \mathbb{R}) &(28)\\
\forall n \in \mathbb{N}, &\quad \max(\|u_n\|_\infty, \|v_n\|_\infty) \leq \max(\|u\|_\infty, \|v\|_\infty) &(29)\\
\text{for almost all } x \in \mathbb{R}^d, &\quad \lim_{n\to\infty} u_n(x) = u(x) &(30)\\
\text{for almost all } y \in \mathbb{R}^d, &\quad \lim_{n\to\infty} v_n(y) = v(y) &(31)\\
&\quad \lim_{n\to\infty} \int_{\mathbb{R}^d} |u_n(x) - u(x)| dx = 0 &(32)\\
&\quad \lim_{n\to\infty} \int_{\mathbb{R}^d} |v_n(y) - v(y)| dy = 0 &(33)\\
&\quad \lim_{n\to\infty} TV(v_n) = TV(v). &(34)
\end{aligned}
$$

With this sequence we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |\mathbf{h}(u_n(x)) - \mathbf{h}(u(x))| dx = 0, \tag{35}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |\mathbf{h}(v_n(y)) - \mathbf{h}(v(y))| dy = 0. \tag{36}$$

Combining these relations to inequality (27) for u_n and v_n, we now pass to the limit. We use the following notations: $C = \|\nabla_y \rho\|_\infty$, $S(x,y) = sgn(u(x) - v(y))$, $S_n(x,y) = sgn(u_n(x) - v_n(y))$, $H(x,y) = \mathbf{h}(u(x)) - \mathbf{h}(v(y))$, $H_n(x,y) = \mathbf{h}(u_n(x)) - \mathbf{h}(v_n(y))$. By (35), (36), we have

$$\lim_{n\to\infty} \int\int_{|x-y|\leq\rho} |H(x,y) - H_n(x,y)| dx dy = 0,$$

and by (30), (31),

$$\text{for almost all } (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \lim_{n\to\infty} S_n(x,y) = S(x,y).$$

For the left hand side we have

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} S(x,y)H(x,y).\nabla_y \rho(x-y)\ dxdy - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} S_n(x,y)H_n(x,y).\nabla_y \rho(x-y)\ dxdy \right| \\
\leq\ & C \left| \int\int_{|x-y|\leq\rho} (S(x,y)H(x,y) - S_n(x,y)H_n(x,y))\ dxdy \right| \\
=\ & C \left| \int\int_{|x-y|\leq\rho} ((S(x,y) - S_n(x,y))H(x,y) + S_n(x,y)(H(x,y) - H_n(x,y)))\ dxdy \right| \\
\leq\ & C \int\int_{|x-y|\leq\rho} |H(x,y) - H_n(x,y)|\ dxdy + C \left| \int\int_{|x-y|\leq\rho} (S(x,y) - S_n(x,y))H(x,y)\ dxdy \right|.
\end{aligned}
$$

The last two terms vanish when n goes to $+\infty$. Then, we can pass to the limit in inequality (27) to obtain Lemma 2 in the general case. □

In Section 4, we will have to approximate the flux $f(u)$ by its Taylor expansion, and we will need the following stability result

Theorem 2 (Stability result with respect to the flux ([17]))
Let $\mathbf{f},\mathbf{g}\in C^1(\mathbb{R},\mathbb{R}^d)$, *and* $u_0,v_0\in BV\cap L^1\cap L^\infty(\mathbb{R}^d,\mathbb{R})$, u,v *be the entropy solutions to*

$$\left\{\begin{array}{lcl}\dfrac{\partial u}{\partial t}+div\,\mathbf{f}(u) & = & 0\\ u(0,x) & = & u_0(x)\end{array}\right. \qquad \left\{\begin{array}{lcl}\dfrac{\partial v}{\partial t}+div\,\mathbf{g}(v) & = & 0\\ v(0,x) & = & v_0(x)\end{array}\right. .$$

Then, with $L:=Lip[\mathbf{f}-\mathbf{g},\max(\|u_0\|_\infty,\|v_0\|_\infty)]$ *for every* $t\geq 0$

$$\int_{\mathbb{R}^d}|u(t,x)-v(t,x)|\,dx\leq\int_{\mathbb{R}^d}|u_0(x)-v_0(x)|\,dx+L\;t\;TV(v_0)$$

This result is based on the weak formulation of the entropy inequalities ([10]). For the sake of convenience we recall the proof of Theorem 2 following [17].
Proof: Let $(\sigma,Y)\in\mathbb{R}^+\times\mathbb{R}^d$ fixed, $\psi\in C_c^\infty(\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^d,\mathbb{R}^+)$.

$$\psi(s,\sigma,X,Y)=\omega_\alpha(\sigma-s)\Omega_\beta(Y-X)$$

where ω_α (respectively Ω_β) is an even approximation of the Dirac measure when α (respectively β) approaches 0.
We write Kružkov inequality (26) with $k=v(\sigma,Y)$, and $\phi(s,X)=\psi(s,\sigma,X,Y)$. Then we have $\dfrac{\partial\psi}{\partial s}+\dfrac{\partial\psi}{\partial\sigma}\equiv 0$ and $\nabla_X\psi+\nabla_Y\psi\equiv 0$.
The Kružkov inequality (26) for u reads:

$$\begin{array}{rl} & \displaystyle\int_{\mathbb{R}^d}|u(t,x)-v(\sigma,y)|\psi(t,\sigma,x,y)\;dx\\ \leq & \displaystyle\int_0^t\int_{\mathbb{R}^d}|u(s,x)-v(\sigma,y)|\frac{\partial\psi}{\partial s}\;dxds\\ + & \displaystyle\int_0^t\int_{\mathbb{R}^d}(sgn(u(s,x)-v(\sigma,y))(\mathbf{f}(u(s,x))-\mathbf{f}(v(\sigma,y)).\nabla_x\psi\;dxds\\ + & \displaystyle\int_{\mathbb{R}^d}|u_0(x)-v(\sigma,y)|\psi(0,\sigma,x,y)\;dx.\end{array}$$

We obtain a similar inequality for v where (s,X) is fixed. Integrating the first inequality with respect to (σ,Y), the second with respect to (s,X) and summing up we obtain

$$\begin{array}{rl} & \displaystyle\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|u(t,x)-v(\sigma,y)|\psi(t,\sigma,x,y)\;dxdyd\sigma\\ + & \displaystyle\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|u(s,x)-v(t,y)|\psi(t,\sigma,x,y)\;dxdyds\\ \leq\;- & \displaystyle\int_0^t\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}\omega_\alpha J(s,\sigma,x,y)).\nabla_y\Omega_\beta\;dxdsdyd\sigma\\ + & \displaystyle\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|u_0(x)-v(\sigma,y)|\psi(0,\sigma,x,y)\;dxdyd\sigma\\ + & \displaystyle\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|u(s,x)-v_0(y)|\psi(s,0,x,y)\;dydxds.\end{array}$$

where $J(s,\sigma,X,Y):=(sgn(u(s,X)-v(\sigma,Y))(h(u(s,X))-h(v(\sigma,Y)).$

Since u, v are in $C^0(\mathbb{R}^+, L^1(\mathbb{R}\times\mathbb{T},\mathbb{R}))$, when α approaches 0, we get

$$
\begin{aligned}
& 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |u(t,x)-v(t,y)|\psi(t,t,x,y)\ dxdy && (37)\\
\leq\ -\ & \int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(s,s,x,y)).\nabla_y\Omega_\beta\ dxdyds && (38)\\
+\ \ & 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |u_0(x)-v_0(y)|\psi(0,0,x,y)\ dxdyd\sigma && (39)
\end{aligned}
$$

We use Lemma 2 to control the term on the second line:

$$\left|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(s,s,x,y)).\nabla_y\Omega_\beta\ dxdyds\right| \leq \int_0^t |L\ TV(v)| \leq L\ t\ Tv(v)$$

Then, we use Lemma 1 when β approaches 0 to obtain the result of Theorem 3:

$$2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |u(t,x)-v(t,y)|\psi(t,t,x,y)\ dxdy = 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |u(t,x)-v(t,y)|\Omega_\beta(Y-X)\ dxdy.$$

Moreover, by Lemma 1 when β approaches 0 we have:

$$\lim_{\beta\to 0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |u(t,x)-v(t,y)|\Omega_\beta(Y-X)\ dxdy = \int_{\mathbb{R}^d} |u(t,x)-v(t,x)|dx$$

We conclude in the same way for the last term $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |u_0(x)-v_0(y)|\psi(0,0,x,y)\ dxdyd\sigma$.

□

For another recent stability result with respect to the flux, see [2] with a slightly weaker assumption on the flux.

In $\mathbb{R}\times\mathbb{T}$ we have the similar results. Before, we need the following Lemma.

Lemma 3
Let u be in $L^\infty(\mathbb{R}\times\mathbb{T},\mathbb{R})\cap L^1(\mathbb{R}\times\mathbb{T},\mathbb{R})$, then for all $C\geq 0$

$$\lim_{n\to+\infty}\frac{1}{2n}\int_{\sqrt{x^2+\widetilde{\theta}^2}\leq n+C} u(x,\widetilde{\theta})dxd\widetilde{\theta} = \int_{\mathbb{R}\times\mathbb{T}} u(x,\theta)dxd\theta.$$

Proof: For almost all x, define $w_n(x) = \dfrac{1}{2n}\displaystyle\int_{\widetilde{\theta}^2\leq(n+C)^2-x^2} u(x,\widetilde{\theta})d\widetilde{\theta}$. For fixed x, taking n sufficient large so that $(n+C)^2-x^2>n$, we have

$$
\begin{aligned}
\left|w_n(x)-\int_{\mathbb{T}} u(x,\theta)d\theta\right| &= \left|w_n(x)-\frac{1}{2n}\int_{-n}^{n} u(x,\widetilde{\theta})d\widetilde{\theta}\right|\\
&= \frac{1}{2n}\left|\int_n^{\sqrt{(n+C)^2-x^2}} u(x,\widetilde{\theta})d\widetilde{\theta} + \int_{\sqrt{(n+C)^2-x^2}}^{-n} u(x,\widetilde{\theta})d\widetilde{\theta}\right|\\
&\leq \frac{1}{2n}\|u\|_\infty 2(\sqrt{(n+C)^2-x^2}-n) = O\left(\frac{1}{n}\right),
\end{aligned}
$$

since $\sqrt{(n+C)^2-x^2}-n = O(1)$. Therefore, for almost all x

$$\lim_{n\to+\infty} w_n(x) = \int_{\mathbb{T}} u(x,\theta)d\theta.$$

Let $v(x) = \int_{\mathbb{T}} |u(x,\theta)|d\theta$. Then v belongs to $L^1(\mathbb{R},\mathbb{R})$ since u belongs to $L^1(\mathbb{R}\times\mathbb{T},\mathbb{R})$. Taking n sufficient large so that $n > C$, we have for almost all x

$$\begin{aligned} |w_n(x)| &\leq \frac{1}{2n}\int_{\widetilde{\theta}^2 \leq (n+C)^2 - x^2} |u(x,\widetilde{\theta})|d\widetilde{\theta} \leq \frac{1}{2n}\int_{|\widetilde{\theta}|\leq n+C} |u(x,\widetilde{\theta})|d\widetilde{\theta} \\ &\leq \frac{1}{2n}\int_{|\widetilde{\theta}|\leq 2n} |u(x,\widetilde{\theta})|d\widetilde{\theta} = 2v(x). \end{aligned}$$

Therefore $\lim_{n\to+\infty}\int_{\mathbb{R}} w_n(x)dx = \int_{\mathbb{R}\times\mathbb{T}} u(x,\theta)dxd\theta$, and this equality: $\frac{1}{2n}\int_{\sqrt{x^2+\widetilde{\theta}^2}\leq n+C} u(x,\widetilde{\theta})dxd\widetilde{\theta} = \int_{\mathbb{R}} w_n(x)dx$ concludes the proof of the lemma. □

Corollary 1 (entropy solution)
Let $\mathbf{f} \in C^1(\mathbb{R},\mathbb{R}^d)$, *and* $u_0, v_0 \in BV \cap L^1 \cap L^\infty(\mathbb{R}\times\mathbb{T},\mathbb{R})$, u *be the entropy solution to (40), (41), and* v *be the entropy solution to (40), (42):*

$$\frac{\partial u}{\partial t} + div\,\mathbf{f}(u) = 0, \tag{40}$$

$$u(0,x,\widetilde{\theta}) = u_0(x,\widetilde{\theta}), \tag{41}$$

$$v(0,x,\widetilde{\theta}) = v_0(x,\widetilde{\theta}). \tag{42}$$

Then u, v *are one periodic with respect to* $\widetilde{\theta}$, *and* $u, v \in L^\infty(\mathbb{R}^+\times\mathbb{R}\times\mathbb{T},\mathbb{R})$.
Furthermore, we have for all $k \in \mathbb{R}$,

$$\frac{\partial}{\partial t}|u-k| + div\ \left((sgn(u-k)(\mathbf{f}(u)-\mathbf{f}(k))\right) \leq 0 \ in\ \mathcal{D}'(\mathbb{R}^+\times\mathbb{R}\times\mathbb{T},\mathbb{R}), \tag{43}$$

and,

$$u \in C^0(\mathbb{R}^+, L^1(\mathbb{R}\times\mathbb{T},\mathbb{R})), \tag{44}$$

$$\int_{\mathbb{R}\times\mathbb{T}} |u(t,x)|dx \leq \int_{\mathbb{R}\times\mathbb{T}} |u_0(x)|dx \forall t \geq 0, \tag{45}$$

and, for all $t \geq 0$:

$$\forall A > 0, \int_{|x|<A+L\ t}\int_{\mathbb{T}} |u(t,x,\theta)-v(t,x,\theta)|dxd\theta \leq \int_{|x|<A}\int_{\mathbb{T}} |u_0(x,\theta)-v_0(x,\theta)|dxd\theta, \tag{46}$$

$$TV(u(t,.)) \leq TV(u_0(.)), \tag{47}$$

$$\|u(t,.)\|_\infty \leq \|u_0(.)\|_\infty. \tag{48}$$

Proof: This Corollary is essentially a consequence of the fundamental Kružkov Theorem, namely Theorem 1.

- Proof of the periodicity: Theorem 1 implies that there exists a unique solution of (40), (41) satisfying (18). But v defined by $v(t,x,\widetilde{\theta}) = u(t,x,\widetilde{\theta}+1)$also satisfies (40), (41), (18). Then, by the uniqueness of the entropy solution, $u(t,x,\widetilde{\theta}) = u(t,x,\widetilde{\theta}+1)$ and we can consider that u belongs to $L^\infty(\mathbb{R}^+\times\mathbb{R}\times\mathbb{T},\mathbb{R})$.

- Proof of (43): We proove that the entropy solution of (40), (41) satisfies (43).
Let $n \in \mathbb{N}$, $\phi(t,x,\theta) \in C^\infty(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{T}, \mathbb{R})$. We choose $\phi_n \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R})$ such that :
$$\phi_n(t,x,\widetilde{\theta}) = \begin{cases} \phi(t,x,\widetilde{\theta}) & \text{if } |\widetilde{\theta}| \le n, \\ 0 & \text{if} |\widetilde{\theta}| > n+1. \end{cases}$$
There exists $C \le 0$ such that $\|\nabla\phi_n\|_\infty \le C\|\phi\|_\infty + \|\nabla\phi\|_\infty$.

If $v(t,x,\widetilde{\theta})$ is bounded and one-periodic with respect to $\widetilde{\theta}$, we have:
$$\frac{1}{2n}\int_{\mathbb{R}^+\times\mathbb{R}^2} v\frac{\partial\phi_n}{\partial x} dtdxd\widetilde{\theta} = \int_{\mathbb{R}^+\times\mathbb{R}\times\mathbb{T}} v\frac{\partial\phi}{\partial x} dtdxd\theta + \mathrm{O}\left(\frac{1}{n}\right),$$
and similar estimates for other partial derivatives of ϕ_n.
For ϕ_n we write Kružkov inequalities (25)
$$\begin{aligned} 0 \le \quad & \int_{\mathbb{R}^+\times\mathbb{R}^2} |u-k|\frac{\partial\phi_n}{\partial t}\ dxd\widetilde{\theta}dt \\ + \quad & \int_{\mathbb{R}^+\times\mathbb{R}^2} (sgn(u-k)(f_1(u)-f_1(k))\frac{\partial\phi_n}{\partial x}\ dxd\widetilde{\theta}dt \\ + \quad & \int_{\mathbb{R}^+\times\mathbb{R}^2} (sgn(u-k)(f_2(u)-f_2(k))\frac{\partial\phi_n}{\partial\widetilde{\theta}}\ dxd\widetilde{\theta}dt \\ + \quad & \int_{\mathbb{R}^2} |u_0(x,\widetilde{\theta})-k|\phi(0,x,\widetilde{\theta})\ dxd\widetilde{\theta}. \end{aligned}$$
Dividing by $2n$ we obtain:
$$\begin{aligned} 0 \le \quad & \int_{\mathbb{R}^+\times\mathbb{R}\times\mathbb{T}} |u-k|\frac{\partial\phi}{\partial t}\ d\theta dxdt \\ + \quad & \int_{\mathbb{R}^+\times\mathbb{R}\times\mathbb{T}} (sgn(u-k)(f_1(u)-f_1(k))\frac{\partial\phi}{\partial x}\ d\theta dxdt \\ + \quad & \int_{\mathbb{R}^+\times\mathbb{R}\times\mathbb{T}} (sgn(u-k)(f_2(u)-f_2(k))\frac{\partial\phi}{\partial\theta}\ d\theta dxdt \\ + \quad & \int_{\mathbb{R}\times\mathbb{T}} |u_0(x,\theta)-k|\phi(0,x,\theta)\ d\theta dx \\ + \quad & \mathrm{O}\left(\frac{1}{n}\right). \end{aligned}$$
Passing to the limit when $n \to +\infty$ we obtain (43).

- Proof of (44): By (19) we have immediately (44).

- Proof of (46): To have (46) we use (21):
$$\int_{|X|<A+L\ t}\int_{\mathbb{T}} |u(t,x,\widetilde{\theta})-v(t,x,\widetilde{\theta})|dxd\widetilde{\theta} \le \int_{|X|<A}\int_{\mathbb{T}} |u_0(x,\widetilde{\theta})-v_0(x,\widetilde{\theta})|dxd\widetilde{\theta},$$
with $X = (x,\widetilde{\theta})$, $|X| = \sqrt{x^2+\widetilde{\theta}^2}$, $L = Lip[f, \max(\|u_0\|_\infty, \|v_0\|_\infty)]$. With $A = n$ and dividing by $2n$ we have:
$$\frac{1}{2n}\int_{|X|<n+L\ t}\int_{\mathbb{T}} |u(t,x,\widetilde{\theta})-v(t,x,\widetilde{\theta})|dxd\widetilde{\theta} \le \frac{1}{2n}\int_{|X|<n}\int_{\mathbb{T}} |u_0(x,\widetilde{\theta})-v_0(x,\widetilde{\theta})|dxd\widetilde{\theta}.$$
Using Lemma 3, with $C = L\ t$ for the left hand side, and with $C = 0$ for the right hand side, we have (46).

- Proof of (45): We use (46) with $v_0 \equiv 0$.
- Proof of (47): We use (46) with $v_0(X) = u_0(X + H)$, then, dividing by $|H|$:

$$\frac{1}{|H|}\int_{\mathbb{R}\times\mathbb{T}} |u(t,x,\theta) - v(t,x,\theta)|dxd\theta \leq \int_{\mathbb{R}\times\mathbb{T}} |u_0(x,\theta) - v_0(x,\theta)|dx,\theta$$

We conclude by (8).

- Proof of (48): This is a consequence of (24).

□

Theorem 3
Let $\mathbf{f}, \mathbf{g}$ *be* $\in C^1(\mathbb{R}, \mathbb{R}^2)$, $u_0, v_0 \in BV \cap L^1 \cap L^\infty(\mathbb{R}\times\mathbb{T}, \mathbb{R})$, u, v *be the entropy solutions to*

$$\left\{\begin{array}{rcl} \dfrac{\partial u}{\partial t} + div\, \mathbf{f}(u) & = & 0 \\ u(0,x,\theta) & = & u_0(x,\theta) \end{array}\right. \qquad \left\{\begin{array}{rcl} \dfrac{\partial v}{\partial t} + div\, \mathbf{g}(v) & = & 0 \\ v(0,x,\theta) & = & v_0(x,\theta) \end{array}\right.$$

then, for every $t \geq 0$, *with* $L := Lip[\mathbf{f} - \mathbf{g}, \|u_0\|_\infty]$,

$$\int_{\mathbb{R}\times\mathbb{T}} |u(t,x,\theta) - v(t,x,\theta)|\, dxd\theta \leq \int_{\mathbb{R}\times\mathbb{T}} |u_0(x,\theta) - v_0(x,\theta)|\, dxd\theta + L\ t\ TV(u_0).$$

The proof of Theorem 3 is essentially the same proof as that of Theorem 2, upon replacing $\mathbb{R}^d$ with $\mathbb{R}\times\mathbb{T}$ wich is possible by Corollary 1.

3 Two-scale convergence for BV functions

We are going to prove the following general result

Theorem 4
Let $u \in BV_c \cap L^\infty(\mathbb{R}\times\mathbb{T}, \mathbb{R})$, *then for every* $\varepsilon > 0$

$$\left|\int_{\mathbb{R}} \overline{u}\left(x, \frac{x}{\varepsilon}\right) dx - \int\int_{\mathbb{R}\times\mathbb{T}} u(x,\theta) dxd\theta\right| \leq C\varepsilon \left\|\frac{\partial u}{\partial x}\right\|_1 \tag{49}$$

where C *is a constant independent of* u *and* ε.

This result has been obtained independently by two slightly different methods in [8] and in [18]. Here we give a third proof motivated by a remark of J. Rauch.
The main point here is the estimation of the convergence as in the C^1 case.
Proof: Let $v(x,\theta) := u(x,\theta) - \int_{\mathbb{T}} u(x,\theta)d\theta$. The above function is defined almost everywhere. It is easy to show that *for every* x the symmetric average $\overline{v}$ satisfies $\int_{\mathbb{T}} \overline{v}(x,\theta)d\theta = 0$ for every x, and

$$\left\|\frac{\partial v}{\partial x}\right\|_1 \leq 2\left\|\frac{\partial u}{\partial x}\right\|_1. \tag{50}$$

We first prove Theorem 4 for $v \in C_c^\infty(\mathbb{R}\times\mathbb{T},\mathbb{R})$. For every $j \in \mathbb{Z}$, we have:

$$\begin{aligned}\left|\int_{j\varepsilon}^{(j+1)\varepsilon} v\left(x,\frac{x}{\varepsilon}\right)dx\right| &= \left|\int_0^{\varepsilon} v\left(j\varepsilon+y,\frac{y}{\varepsilon}\right)dy\right| = \left|\varepsilon\int_0^1 v(j\varepsilon+\varepsilon z,z)dz\right| \\ &= \left|\varepsilon\int_0^1 \left[v(j\varepsilon+\varepsilon z,z)-v(j\varepsilon,z)\right]dz\right| \\ &= \left|\varepsilon\int_0^1\int_{j\varepsilon}^{j\varepsilon+\varepsilon z}\frac{\partial v}{\partial x}(x,z)dxdz\right| \le \varepsilon\int_0^1\int_{j\varepsilon}^{(j+1)\varepsilon}\left|\frac{\partial v}{\partial x}(x,z)\right|dxdz.\end{aligned}$$

Summing over j, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}} v\left(x,\frac{x}{\varepsilon}\right)dx\right| &= \left|\sum_{j\in\mathbb{Z}}\int_{j\varepsilon}^{(j+1)\varepsilon} v\left(x,\frac{x}{\varepsilon}\right)dx\right| \le \left|\sum_{j\in\mathbb{Z}}\varepsilon\int_0^1\int_{j\varepsilon}^{(j+1)\varepsilon}\left|\frac{\partial v}{\partial x}(x,z)\right|dxdz\right| \\ &\le \varepsilon\int_0^1\int_{\mathbb{R}}\left|\frac{\partial v}{\partial x}(x,z)\right|dxdz = \varepsilon\left\|\frac{\partial v}{\partial x}\right\|_1.\end{aligned}$$

Then we have in the smooth case

$$\left|\int_{\mathbb{R}}\overline{v}\left(x,\frac{x}{\varepsilon}\right)dx\right| \le \varepsilon\left\|\frac{\partial v}{\partial x}\right\|_1, \tag{51}$$

since $v \equiv \overline{v}$.

Now if v is arbitrary in $BV\cap L^1$, there exists a sequence of functions (φ_n) in $C_c^\infty(\mathbb{R}\times\mathbb{T},\mathbb{R})$ which satisfies (10), (11), (12), as well as (51). Passing to the limit, when $n\to+\infty$, we show that v satisfies (51). By the definition of v we also have

$$\int_{\mathbb{R}}\overline{v}\left(x,\frac{x}{\varepsilon}\right)dx = \int_{\mathbb{R}}\left[\overline{u}\left(x,\frac{x}{\varepsilon}\right)-\int_{\mathbb{T}}\overline{u}(x,\theta)d\theta\right]dx = \int_{\mathbb{R}}\overline{u}\left(x,\frac{x}{\varepsilon}\right)dx - \int\int_{\mathbb{R}\times\mathbb{T}}u(x,\theta)dxd\theta.$$

Then, using inequality (50), we obtain inequality (49) with $C=2$, and therefore the proof of Theorem 4 is complete. □

4 Aymptotics

We are going to examine the propagation of high frequency oscillations using Ansatz (3) (see also [7], [5], [3])

Let us state the main result of this paper

Theorem 5
Let $f\in C^3(\mathbb{R},\mathbb{R})$, u_0 in $BV_c\cap L^\infty(\mathbb{R}\times\mathbb{T},\mathbb{R})$. We assume that u_0 satisfies

$$\sup_{0<\varepsilon\le 1} TV_x\left(x\mapsto\varepsilon\overline{u}_0\left(x,\frac{x}{\varepsilon}\right)\right)<\infty. \tag{52}$$

Let u^ε be the entropy solution to

$$\left\{\begin{array}{rcl}\partial_t u^\varepsilon+\partial_x f(u^\varepsilon) &=& 0,\\ u^\varepsilon(0,x) &=& \varepsilon\overline{u}_0\left(x,\dfrac{x}{\varepsilon}\right).\end{array}\right. \tag{53}$$

Let $a = f'(0)$, $b = f''(0)$ and σ be the entropy solution to

$$\left\{ \begin{array}{rcl} \partial_t \sigma + a\partial_x \sigma + b\partial_\theta \left(\dfrac{\sigma^2}{2}\right) & = & 0, \\ \sigma(0,x,\theta) & = & \overline{u}_0\,(x,\theta)\,. \end{array} \right. \tag{54}$$

Then there exists a positive constant $C > 0$ such that

$$\int_{\mathbb{R}} \left| u^\varepsilon(t,x) - \varepsilon\overline{\sigma}\left(t,x,\frac{x-at}{\varepsilon}\right)\right| dx \le C\varepsilon^2(1+t), \quad for\ a.e\ \varepsilon \in]0,1], \quad \forall t > 0. \tag{55}$$

Remark 2 *If $f(u)$ is a quadratic function and u_0 has compact support, we do not need assumption (52).*

Remark 3 *The asymptotic expansion is valid for all ε if u_0 is sufficiently smooth.*
Let $D_\varepsilon := \left\{ \left(x, \dfrac{x}{\varepsilon}\right),\ x \in \mathbb{R} \right\}$. If u_0 has a trace on every D_ε, $\varepsilon \in]0,\varepsilon_0]$ then equation (55) is valid for all $\varepsilon \in]0,\varepsilon_0]$.

Remark 4 *If $f''(0) = 0$, Theorem 5 asserts that the linearization of equation (1) gives an approximation of order $O(\varepsilon^2 t)$. In the generic case: $f''(0) \neq 0$, the linearization only gives an approximation of order $O(\varepsilon t)$.*

Remark 5 *One gets (54) using Anzats (3) and only retaining (after the substitution of (3) into (1), (2)) the $O(1)$ and $O(\varepsilon)$ terms as $\varepsilon \to 0$.*

Proof: Using Theorem 2 and assumption (52), we are first going to show that u^ε is well approximated by v^ε, the solution to a suitable Burgers equation. Next, using Theorem 4 and Theorem 2, we are going to justify (3) for this function v^ε.

More precisely, let u^ε be the entropy solution to

$$\left\{ \begin{array}{rcl} \partial_t u^\varepsilon + \partial_x[Q(u^\varepsilon) + R(u^\varepsilon)] & = & 0, \\ u^\varepsilon(0,x) & = & \varepsilon\overline{u}_0\left(x, \dfrac{x}{\varepsilon}\right), \end{array} \right. \tag{56}$$

where $Q(v) := av + \frac{1}{2}bv^2$, and $R(v) := f(v) - Q(v) = \mathrm{O}(v^3)$ for $v \to 0$.
Neglecting the remainder, we define v^ε as the entropy solution to

$$\left\{ \begin{array}{rcl} \partial_t v^\varepsilon + \partial_x Q(v^\varepsilon) & = & 0 \\ v^\varepsilon(0,x) & = & \varepsilon\overline{u}_0\left(x, \dfrac{x}{\varepsilon}\right) \end{array} \right. \tag{57}$$

We can rewrite $R(v) = \displaystyle\int_0^v \frac{(v-s)^2}{2} f'''(s)ds$ then $R'(v) = \displaystyle\int_0^v (v-s)f'''(s)ds$. We have

$$L^\varepsilon = Lip[R, \|\varepsilon u_0\|_\infty] \le \sup_{|s|\le\|\varepsilon u_0\|_\infty} |f'''(s)| \int_0^v (v-s)ds \le \sup_{|s|\le\|u_0\|_\infty} |f'''(s)| \frac{\|\varepsilon u_0\|_\infty^2}{2} = O(\varepsilon^2).$$

By Theorem 2 we have

$$\int_{\mathbb{R}} |u^\varepsilon(t,x) - v^\varepsilon(t,x)|\, dx \le L^\varepsilon\ t\ TV(v^\varepsilon(0,.)) \le D\varepsilon^2 t$$

where $D := \sup_{|s|\leq\|u_0\|_\infty} |f'''(s)| \sup_{0<\varepsilon\leq 1} TV_x v^\varepsilon(0,.)$ is finite, in view of (52). Therefore

$$\|u^\varepsilon(t,.) - v^\varepsilon(t,.)\|_1 \leq D\varepsilon^2 t \tag{58}$$

Now we are going to show that v^ε satisfies (55). First, we introduce, as in [15], the *intermediate profile* σ^ε, namely the entropy solution to

$$\left\{\begin{array}{rcl} \partial_t\sigma^\varepsilon + a\partial_x\sigma^\varepsilon + b\partial_\theta\left(\dfrac{(\sigma^\varepsilon)^2}{2}\right) & = & -\varepsilon b\partial_x\left(\dfrac{(\sigma^\varepsilon)^2}{2}\right) \\ \sigma^\varepsilon(0,x,\theta) & = & \overline{u}_0(x,\theta). \end{array}\right. \tag{59}$$

Observe that $\sigma = \sigma^\varepsilon|_{\varepsilon=0}$. It is more convenient to introduce a new variable: $z := x - at - \varepsilon\theta$, and to define

$$w^\varepsilon(t,x,z) := \varepsilon\sigma^\varepsilon\left(t, x, \frac{x-at-z}{\varepsilon}\right) = \varepsilon\sigma^\varepsilon(t,x,\theta).$$

w^ε is ε periodic with respect to z. Define $\varepsilon\mathbb{T}$ the circle of length ε identify to $[0,\varepsilon[$. We compute

$$\partial_t w^\varepsilon + \partial_x Q(w^\varepsilon) \;=\; \varepsilon\partial_t\sigma^\varepsilon - a\partial_\theta\sigma^\varepsilon + \varepsilon a\partial_x\sigma^\varepsilon + a\partial_\theta\sigma^\varepsilon + b\varepsilon^2\partial_x\left(\frac{(\sigma^\varepsilon)^2}{2}\right) + b\varepsilon\partial_\theta\left(\frac{(\sigma^\varepsilon)^2}{2}\right) = 0.$$

Then w^ε is the solution to

$$\left\{\begin{array}{rcl} \partial_t w^\varepsilon + \partial_x Q(w^\varepsilon) & = & 0 \\ w^\varepsilon(0,x,z) & = & \varepsilon\overline{u}_0\left(x, \dfrac{x-z}{\varepsilon}\right). \end{array}\right. \tag{60}$$

The change of variables has been performed in order to obtain only one space derivative.

At this level, we can view w^ε either as the entropy solution to the two-dimensional Burgers equation (60), or as the entropy solution $\widetilde{w}^\varepsilon$ to a family of one-dimensional Burgers equations (60) parameterized by z. We have the following result:

Lemma 4
For every $\varepsilon > 0$, for every t,for almost every x, z, $w^\varepsilon(t,x,z) = \widetilde{w}^\varepsilon(t,x,z)$.

Proof: Let ε be fixed. First, we prove that $\widetilde{w}^\varepsilon$ is a entropy solution of (57). Let $k \in \mathbb{R}$, $\psi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R} \times \varepsilon\mathbb{T}, \mathbb{R}^+)$. For all z, $\psi(.,.,z) \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$, and $\widetilde{w}^\varepsilon(.,.,z)$ is the entropy solution of the one-dimensional Burgers equations (60). Then, by the Kružkov inequality we have

$$\begin{array}{rl} 0 \leq & \displaystyle\int_{\mathbb{R}^+}\int_{\mathbb{R}} |\widetilde{w}^\varepsilon(t,x,z) - k| \frac{\partial}{\partial t}\psi(t,x,z)\; dxdt \\ + & \displaystyle\int_{\mathbb{R}^+}\int_{\mathbb{R}} (sgn(\widetilde{w}^\varepsilon(t,x,z) - k)(Q(\widetilde{w}^\varepsilon(t,x,z)) - Q(k))\frac{\partial}{\partial x}\psi(t,x,z)\; dxdt \\ + & \displaystyle\int_{\mathbb{R}} |u_0(x,z) - k|\psi(0,x,z)\; dx. \end{array}$$

Integrating with respect to z, we deduce that $\widetilde{w}^\varepsilon$ is entropy solution of the two-dimensional Burgers equation (60). Then $\widetilde{w}^\varepsilon$ is equal to w^ε almost everywhere on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{T}$.

In fact $w^\varepsilon \in C^0(\mathbb{R}^+, L^1(\mathbb{R} \times \varepsilon\mathbb{T}, \mathbb{R})$ (see (44)),we are able to prove the same regularity for $\widetilde{w}^\varepsilon$ and conclude this lemma. Let $I(s,t,z) = \int_{\mathbb{R}} |\widetilde{w}^\varepsilon(s,x,z) - \widetilde{w}^\varepsilon(t,x,z)| dx$.

For all z, $\widetilde{w}^\varepsilon(.,.,z) \in C^0(\mathbb{R}^+, L^1(\mathbb{R},\mathbb{R})$ (see (20)), then $\lim_{s\to t} I(s,t,z) = 0$.
Furthermore $0 \le I(s,t,z) \le \int_{\mathbb{R}} |\widetilde{w}^\varepsilon(s,x,z)|dx + \int_{\mathbb{R}} |\widetilde{w}^\varepsilon(t,x,z)|dx \le 2\int_{\mathbb{R}} \left|\varepsilon\overline{u}_0\left(x, \frac{x-z}{\varepsilon}\right)\right| dx = g(z)$ by (22). Since $g \in L^1(\mathbb{R},\mathbb{R})$ we are able to pass to the limit:
$\lim_{s\to t} \int_{\mathbb{R}\times\varepsilon\mathbb{T}} |\widetilde{w}^\varepsilon(s,x,z) - \widetilde{w}^\varepsilon(t,x,z)|dxdz = 0$ Then $\widetilde{w}^\varepsilon \in C^0(\mathbb{R}^+, L^1(\mathbb{R}\times\varepsilon\mathbb{T},\mathbb{R}))$ like w^ε, and therefore, $\widetilde{w}^\varepsilon(t,.,.)$ is equal to $w^\varepsilon(t,.,.)$ for all t in $L^1(\mathbb{R}\times\varepsilon\mathbb{T},\mathbb{R})$.

□

In fact, in order to use Theorem 4, we should have

$$\overline{w}^\varepsilon(t,x,0) = v^\varepsilon(t,x) \text{ in } C^0((0,+\infty), L^1(\mathbb{R})). \tag{61}$$

By construction, we have $\widetilde{w}^\varepsilon(t,x,0) = v^\varepsilon(t,x)$ in $C^0((0,+\infty), L^1(\mathbb{R}))$. Unfortunately, (61) is false in general, since Burgers equation does not propagate symmetric averages, i.e. $\overline{w}^\varepsilon(t,.,0) \ne \widetilde{w}^\varepsilon(t,.,0)$. The following example illustrates this phenomenon: let h be the entropy solution of

$$\begin{cases} \partial_t h + \partial_x(h^2/2) & = & 0 \\ h(0,x,z) & = & \begin{cases} \pm 1 & if & \pm xz > 0, \\ 0 & if & z = 0. \end{cases} \end{cases} \tag{62}$$

and let $\widetilde{h}$ be the entropy solution to a family of one-dimensional Burgers equations parametrized by z. Solving (62) we have:

$$\begin{array}{lll} if\ z = 0\ then & \widetilde{h}(t,x,0) = \ 0 & \\ if\ z > 0\ then & \widetilde{h}(t,x,z) = \begin{cases} \dfrac{x}{t} & for\ |x| \le t \\ \pm 1 & for\ \pm x \ge t \end{cases} & \text{is a rarefaction wave at } z \text{ fixed,} \\ if\ z < 0\ then & \widetilde{h}(t,x,z) = \ \mp 1\ if\ \ \pm x > 0 & \text{is a shock wave at } z \text{ fixed.} \end{array}$$

Then, an easy computation yields :

$$\overline{h}(t,x,0) = \begin{cases} \dfrac{1}{2}\left(\dfrac{x}{t} - 1\right) & for\ 0 < x < t, \\ \dfrac{1}{2}\left(\dfrac{x}{t} + 1\right) & for\ -t < x < 0, \\ 0 & for\ |x| \ge t. \end{cases}$$

Therefore, $\widetilde{h}(t,x,0) \ne \overline{h}(t,x,0)$ for $0 < |x| < t$.

However the traces are propagated: if $\widetilde{w}^\varepsilon(0,.,.)$ has a trace at $\{z=0\}$ then $\overline{w}^\varepsilon(t,.,0) = \widetilde{w}^\varepsilon(t,.,0)$, $\forall t \ge 0$.
Indeed, we have the following result

Lemma 5
For almost every $\varepsilon \in]0,1]$, *for every* t, *we have* $\widetilde{w}^\varepsilon(t,x,0) = \overline{w}^\varepsilon(t,x,0)$ *for almost every* x.

Proof: Let $D_\varepsilon := \left\{\left(x, \frac{x}{\varepsilon}\right),\ x \in \mathbb{R}\right\}$, and for any BV-function $g(x,\theta)$ let $E(g)$ be the subset of ε in $]0,1]$ such that g has a trace on D_ε. The measure of $(]0,1] - E(u_0))$ is zero, since u_0 is a BV functions, (this is essentially a consequence of Fubini's Theorem and of the one-dimensional measure of the singularities of u_0 (see [20])).

Thanks to the L^1 stability of solutions for one-dimensional scalar conservation laws with respect to the initial data (see Theorem 3), we have:

$$\int_{\mathbb{R}} |\widetilde{w}^\varepsilon(t,x,z) - \widetilde{w}^\varepsilon(t,x,0)|dx \leq \varepsilon \int_{\mathbb{R}} \left| \overline{u}_0\left(x, \frac{x-z}{\varepsilon}\right) - \overline{u}_0\left(x, \frac{x}{\varepsilon}\right)\right| dx.$$

Let r be any positive number. We integrate with respect to z on $[-r,r]$, multiply by $\frac{1}{2r}$, and take the limit when r goes to 0. We see that if $\varepsilon \in E(u_0)$, the right hand side tends to 0. Therefore $E(u_0)$ is a subset of $E(\widetilde{w}^\varepsilon(t,.,0))$ for every t, which completes the proof of the lemma. □

Before finishing the proof of Theorem 5, we now prove the following estimate

$$\int\int_{\mathbb{R}\times\mathbb{T}} |\sigma^\varepsilon(t,x,\theta) - \sigma(t,x,\theta)|dxd\theta = 0(\varepsilon t). \tag{63}$$

Indeed by (59), (48) and (47), we have for every $\varepsilon > 0$, $\|\sigma^\varepsilon\|_{L^\infty} \leq \|u_0\|_{L^\infty}$ and $TV(\sigma^\varepsilon) \leq TV(u_0)$. Using Theorem 2 with $d = 2$ and the uniform estimate of σ^ε in $L^\infty \cap BV$, we obtain (63). We also have

$$TV|\sigma^\varepsilon(t,.,.) - \sigma(t,.,.)| \leq TV(\sigma^\varepsilon(t,.,.) - \sigma(t,.,.)) \leq TV(\sigma^\varepsilon(t,.,.)) + TV(\sigma(t,.,.)) \leq 2TV u_0.$$

We can now complete the proof of Theorem 5. Using Theorem 3, we have for any ε in $E(u_0)$

$$\begin{aligned}
\int_{\mathbb{R}} \left| v^\varepsilon(t,x) - \varepsilon\overline{\sigma}\left(t,x,\frac{x-at}{\varepsilon}\right)\right| dx &= \int_{\mathbb{R}} \left| \widetilde{w}^\varepsilon(t,x,0) - \varepsilon\overline{\sigma}\left(t,x,\frac{x-at}{\varepsilon}\right)\right| dx \\
&= \int_{\mathbb{R}} \left| \overline{w}^\varepsilon(t,x,0) - \varepsilon\overline{\sigma}\left(t,x,\frac{x-at}{\varepsilon}\right)\right| dx \\
&= \int_{\mathbb{R}} \left| \varepsilon\overline{\sigma}^\varepsilon\left(t,x,\frac{x-at}{\varepsilon}\right) - \varepsilon\overline{\sigma}\left(t,x,\frac{x-at}{\varepsilon}\right)\right| dx \\
&= \varepsilon \int_{\mathbb{R}} \overline{|\sigma^\varepsilon - \sigma|}(\left(t,x,\frac{x-at}{\varepsilon}\right) dx \\
&\leq \varepsilon \int\int_{\mathbb{R}\times\mathbb{T}} |\sigma^\varepsilon(t,x,\theta) - \sigma(t,x,\theta)|dxd\theta + 4\varepsilon^2 TV(u_0) \\
&= \mathrm{O}(\varepsilon^2 t) + 4\varepsilon^2 TV(u_0) = \mathrm{O}(\varepsilon^2(t+1)).
\end{aligned}$$

□

Theorem 5 justifies the Anzatz (3) for time of order ε^{-1}. This is sufficient, since it is strictly better than the linearization: therefore adding the third nonlinear term in (54) has improved the accuracy of the approximation. In [5], with one scale ($\partial_x u_0 \equiv 0$), the asymptotic is uniform in $t > 0$. To obtain a uniform estimate we need to have a sufficient decay of the total variation of u^ε and σ^ε. In fact, for example, it suffices to have estimates similar to the following:

$$TV_x(v^\varepsilon(t,.)) = O\left(\frac{1}{1+t}\right), \quad TV(\sigma(t,.,.)) = O\left(\tfrac{1}{1+t}\right), \quad TV(\sigma^\varepsilon(t,.,.)) = O\left(\frac{1}{1+t}\right).$$

To our knowledge, such a decay result has not yet been proved under the assumptions considered here.

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