GEOMETRIC OPTICS WITH CRITICAL VANISHING VISCOSITY FOR ONE-DIMENSIONAL SEMILINEAR INITIAL VALUE PROBLEM

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Abstract. We study propagations of high frequency oscillations for one dimensional semi-linear hyperbolic system with small parabolic perturbation. We obtain a new degenerate parabolic system for profile, and valid an asymptotic development in a spirit of Joly, Métivier and Rauch.

key words: geometric optics, small viscosity, profile, phase, non stationary phase, maximum principle, energy estimates, interpolation, weakly coupled parabolic systems.

1. Introduction

Joly, Métivier and Rauch in [13] give a rigorous and deep description of solutions to one dimensional nonlinear hyperbolic equations with smooth and highly oscillatory initial data. On the other hand, numerous physical problems involve parabolic partial differential equations with small viscosity, for instance Navier-Stokes system with large Reynolds number. Then it is of interest to put the Joly, Métivier, Rauch framework with a small diffusion. The first step to study such multiphase expansions is the semilinear case. So we wish to investigate propagations of high frequency oscillations for following nonlinear parabolic system with highly oscillatory initial data u_0^ε and small positive viscosity ν:

$$-\nu\frac{\partial^2 u^\varepsilon}{\partial x^2}+\frac{\partial u^\varepsilon}{\partial t}+A(t,x)\frac{\partial u^\varepsilon}{\partial x} = F(t,x,u^\varepsilon) \tag{1.1}$$

$$u^\varepsilon(0,x) = u_0^\varepsilon(x). \tag{1.2}$$

The matrix $A(t,x)$ is a smooth $N\times N$ real matrix with N distinct real eigenvalues: $\lambda_1(t,x)<\lambda_2(t,x)<\cdots<\lambda_N(t,x)$, in such a way that $\partial_t+A(t,x)\partial_x$ is a *strictly hyperbolic operator*.

In this paper, we will focus our attention on the special case of a rapidly oscillating data with a given vectorial phase $\varphi^0(x)$, frequency $1/\varepsilon$ and a viscosity equals to the square of the wavelength:

$$u^\varepsilon(0,x)=u_0^\varepsilon(x) = U_0\left(x,\frac{\varphi^0(x)}{\varepsilon}\right), \tag{1.3}$$

$$\nu = \varepsilon^2. \tag{1.4}$$

So, with such oscillating data, the viscous coefficient has critical size (see next section), such that we expect to have $\varepsilon^2\partial_x^2 u^\varepsilon$ of order one. And we hope to see interactions of high oscillations with the viscous term. Indeed, it is the aim of this paper to justify a geometric optics expansion:

$$u^\varepsilon(t,x)=U\left(t,x,\frac{\varphi(t,x)}{\varepsilon}\right)+o(1),$$

where φ is a vectorial phase as in hyperbolic case (without viscosity) and the profile U satisfies a new degenerate parabolic modulation system. With such diagonal viscosity, we validate WKB expansion in L^2_x with initial data only bounded in $L^2_x \cap L^\infty_x$.

There is a rich literature devoted to nonlinear geometric optics expansions. The classical work here is [13]. For the quasilinear case restricted to constant background states and linear phases, see [4, 5, 11, 12, 18, 15, 16]. For the semilinear case, see [14, 13, 20, 21, 14]. For some results with oscillations and viscosity in multidimensional framework with one phase see [1, 7, 8], with a nonlinear instability result obtained trough WKB expansions in the second reference. To a reader interested by an overview on nonlinear geometric optics we refer to survey paper [6] and references given there.

The paper is organized as follows. In short section 2 we briefly explain the choose of viscosity (1.4) on a simple example. In section 3 we repeat relevant material from [13], give new parabolic profile systems and state our main results. Section 4 set up notation for the almost diagonal system, and reviews some of the standard estimates on parabolic equations. We conclude by a useful interpolation lemma. In section 5, we prove uniform existence in time for solutions of (1.1), (1.4) with initial data (1.2) uniformly bounded in $L^\infty \cap L^2$. Section 6 is devoted to the study of the profile system. Section 7 deals with the important linear scalar case. Then, in section 8, geometric optics expansion is proved. Finally , in section 9, we give few comments and comparisons with the inviscid case.

2. About viscosity size

Let us examine a simple example to get critical size of viscosity ν with ε-wavelength:

$$\partial_t u^\varepsilon + \lambda \partial_x u^\varepsilon = \nu \partial_x^2 u^\varepsilon, \qquad u^\varepsilon(0,x) = u_0(x/\varepsilon), \tag{2.1}$$

where $\nu > 0$, u_0 is a smooth one periodic function with zero mean. We have the classical decay rate in $L^\infty(\mathbb{R}_x)$ for one periodic solutions with zero mean:

$$\|u^\varepsilon(t,.)\|_\infty \leq \frac{\|u_0\|_\infty}{\sqrt{\pi t}} \frac{\varepsilon}{\sqrt{\nu}}. \tag{2.2}$$

So, if $\nu \gg \varepsilon^2$, diffusion is too strong and kills oscillations.
Now, let $v^\varepsilon(t,x) = u^\varepsilon(t,x) - u_0\left(\frac{x-\lambda t}{\varepsilon}\right)$. v^ε satisfies the following Cauchy problem:

$$\partial_t v^\varepsilon + \lambda \partial_x v^\varepsilon - \nu \partial_x^2 v^\varepsilon = \nu \varepsilon^{-2} u_0'', \qquad v^\varepsilon(0,x) = 0.$$

Using maximum principle for v^ε we conclude that:

$$\left\| u^\varepsilon(t,x) - u_0\left(\frac{x-\lambda t}{\varepsilon}\right) \right\|_\infty \leq t \|u_0''\|_\infty \frac{\nu}{\varepsilon^2}. \tag{2.3}$$

So, if $\nu \ll \varepsilon^2$,we have the same hyperbolic behavior when $\nu = 0$. Thus, the critical viscosity size is $\nu \sim \varepsilon^2$. Furthermore, for $\nu = \varepsilon^2$, inequalities (2.2), (2.3) give bounds of order one and we get an exact geometric optics expansion and a parabolic profile equation:

$$u^\varepsilon(t,x) = U\left(t, \frac{x-\lambda t}{\varepsilon}\right), \quad \text{where } -\partial_\theta^2 U + \partial_t U = 0, \quad U(0,\theta) = u_0(\theta).$$

3. Main Results

Viscosity ν is now fixed by (1.4). Our first goal is to prove that the exact solutions of (1.1), (1.2) exists on a domain independent of $\varepsilon \in (0, \varepsilon_0]$ for a positive ε_0. Let us begin with fixing an arbitrary $T_0 > 0$ for all the sequel. For such parabolic system, in order to avoid boundary value problems, we will work globally in the space variable and assume that the entries of matrix

$$A(t,x), \partial_x A(t,x), \partial_x^2 A(t,x) \text{ are bounded on } [0,T_0] \times \mathbb{R}_x. \tag{3.1}$$

In a same way, $F(t,x,u)$ is a smooth non linear function such that, for any positive T, and any compact $K \subset \mathbb{R}_u^N$,

$$F(t,x,u) \text{ and } \partial_u F(t,x,u) \text{ are bounded on } [0,T_0] \times \mathbb{R}_x \times K_u. \tag{3.2}$$

Furthermore, to have energy estimates, we also assume that

$$F(t,x,0) \in C^0([0,T_0]; L^2(\mathbb{R}_x, \mathbb{R}^N)). \tag{3.3}$$

The following preliminary result gives sufficient conditions on the Cauchy data u_ε^0 to get existence of a solution on a domain independent of ε.

Proposition 3.1. [Uniform existence]
Under assumptions (3.1), (3.2), (3.3), if the family $(u_0^\varepsilon)_{0<\varepsilon\le 1}$ is bounded in $L^\infty \cap L^2(\mathbb{R}, \mathbb{R}^N)$, then, there exist $T > 0$ and $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, the system (1.1), (1.2), (1.4) has a unique weak solution u^ε in $E := C([0,T]; L^2(\mathbb{R}, \mathbb{R}^N)) \cap L^\infty([0,T] \times \mathbb{R}, \mathbb{R}^N)$. Furthermore the family $(u^\varepsilon)_{0<\varepsilon\le 1}$ is uniformly bounded in E independently of ε.

The proof of this result is given in section 5 and uses energy estimates for the linearized problem together with the maximum principal for scalar parabolic equations and an interpolation estimate.

After this preliminary result, we are going to find phases and the profile system. Typically in (1.3), $\varphi^0 \in C^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{m}})$ and $U_0(x,\theta)$ is smooth function, supported in $\{|x| < 1\}$ and $\mathbb{Z}^{\mathrm{m}}$-periodic with respect to $\theta \in \Theta^0 = \mathbb{R}^{\mathrm{m}}$, but we will also consider more general almost-periodic profiles as in [13]. In order to state the main result, we have to introduce some notations.

Phases and characteristic vector fields:
We note $\varphi^0 = (\varphi_1^0, \ldots, \varphi_{\mathrm{m}}^0)$ where $\varphi_j^0 \in C^\infty(\mathbb{R}; \mathbb{R})$, and assume that the $\varphi_1^0, \ldots, \varphi_{\mathrm{m}}$ are linearly independent. As in a pure hyperbolic problem, we use characteristic vector fields of the hyperbolic operator $\partial_t + A(t,x)\partial_x$

$$X_j = \partial_t + \lambda_j(t,x)\partial_x, \ j = 1, \ldots, N \tag{3.4}$$

and define the (vector valued) phases $\varphi_j \in C^\infty([0,T_0], \times\mathbb{R}; \mathbb{R}^{\mathrm{m}})$, $j = 1, \ldots, N$ by

$$X_j \varphi_j = 0, \qquad \varphi_j(0,x) = \varphi^0(x). \tag{3.5}$$

Then we call $\varphi = (\varphi_1, \ldots, \varphi_N) \in C^\infty([0,T_0] \times \mathbb{R}; (\mathbb{R}^{\mathrm{m}})^N)$ and introduce the two following linear spaces:

$$\Theta^0 := span\{\varphi^0(\mathbb{R})\} = \mathbb{R}^{\mathrm{m}}, \quad \Theta := span\{\varphi([0,T_0] \times \mathbb{R})\} \subset \mathbb{R}^{\mathrm{m}N}.$$

First equality $\Theta^0 = \mathbb{R}^{\mathrm{m}}$ comes from linear independence of φ_j^0, $j = 1, \ldots, \mathrm{m}$.
Strict inequality $dim(\Theta) < N\mathrm{m}$ holds if and only if there exists *resonance relations* between the phases φ_j, i.e. $R := \{\alpha \in \mathbb{R}^N;\ \alpha.\varphi \equiv 0 \text{ on } [0,T_0] \times \mathbb{R}\} \neq \{0\}$. Notice that $R^\perp = \Theta$.

Let's call $\theta = (\theta_1, \ldots, \theta_N)$ the independent variables in the space $\mathbb{R}^{\mathrm{m}} \times \cdots \times \mathbb{R}^{\mathrm{m}}$, where $\theta_j = (\theta_{j,1}, \ldots, \theta_{j,\mathrm{m}}) \in \mathbb{R}^{\mathrm{m}}$. Let Π_j be the linear projection on j-component: $\Pi_j : \Theta \to \mathbb{R}^{\mathrm{m}}$, $\Pi_j(\theta) = \theta_j$. Let us define $\Psi_j = \ker \Pi_j \cap \Theta$ and $\Theta_j := \Psi_j^\perp \cap \Theta$.

Now, to valid a geometric optics expansion, we need some assumption on phases. We use the more general *transversality* conditions from [13] (named precisely weak transversality condition in [13]):

$$\forall \alpha \in \Theta^*, \forall j, \left\{ \begin{array}{l} \text{either } X_j(\alpha.\varphi) \equiv 0 \text{ on } [0,T_0] \times \mathbb{R}, \\ \text{or } X_j(\alpha.\varphi) \neq 0 \text{ almost everywhere on } [0,T_0] \times \mathbb{R}. \end{array} \right. \tag{3.6}$$

In second case of (3.6), we said that $\alpha.\varphi$ is *transverse* to X_j. We also said that if (3.6) is satisfied, phases are *transverse*.

To avoid constant phases, we assume *non stationary* condition on initial phases:

$$\forall \beta \in \mathbb{R}^{\mathrm{m}}, \partial_x(\beta.\varphi^0) \neq 0 \text{ almost everywhere on } \mathbb{R}. \tag{3.7}$$

Notice that (see [13]) such conditions (3.6) and (3.7) avoid constant phases.

We also assume following *closedness* property:

$$\forall \alpha \in \Theta^*, \forall j, X_j(\alpha.\varphi) \equiv 0 \text{ on } [0,T_0] \times \mathbb{R} \Rightarrow \exists \beta \in \mathbb{R}^{\mathrm{m}}, \text{ such that } \alpha.\varphi \equiv \beta.\varphi_j. \tag{3.8}$$

Indeed, it's not really a compulsive assumption, since, without loss of generality, we can add new phases coming from resonance as new initial phases in φ^0. Thus, we increase m, the number of initial phases.

Almost periodic profiles:
As in the classical geometrical optics ([13]), one is naturally lead to work within the class of almost periodic functions. If Z is some finite dimensional real linear space, we call $C^0_{ap}(Z,\mathbb{R})$ the topological closure in $L^\infty(Z,\mathbb{R})$ of the linear subspace $span\{\xi \mapsto \exp(\iota\alpha.\xi),\ \alpha \in Z^*\}$. We also denote $C^\infty_{ap}(Z,\mathbb{R}) = C^0_{ap}(Z,\mathbb{R}) \cap C^\infty(Z,\mathbb{R})$.

Averaging operators:
We use the averaging operators already introduced by [13] which are defined as follow. Let μ_j be the Lebesgue measure on Ψ_j, Q_j a cubic subset of Ψ_j such that $\mu_j(Q_j) = 1$. We define the averaging operator $\mathbb{E}_j$ acting on $C^0_{ap}(\Theta,\mathbb{R})$, by the formula

$$(\mathbb{E}_j v)(\theta_j) = \lim_{T \to +\infty} \frac{1}{T^{\dim \Psi_j}} \int_{T.Q_j} v(\theta + \xi) d\mu_j(\xi) \ .$$

In the previous equation the right hand side depends actually only on the variable θ_j and

$$\mathbb{E}_j \ : C^0_{ap}(\Theta,\mathbb{R}) \longrightarrow C^0_{ap}(\Theta_j,\mathbb{R})$$

is a linear and continuous projector. This map can be also defined on elementary exponential functions by the following rule

$$\mathbb{E}_j(\exp(i\alpha.\theta)) = \left\{ \begin{array}{cc} \exp(i\alpha.\theta) & if\ X_j(\alpha.\varphi) = 0 \\ 0 & else \end{array} \right.$$

and then extended by density and continuity (cf [13]).

The profile system:
Let $\Lambda(t,x)$ the diagonal matrix with components $(\lambda_1(t,x),\dots,\lambda_N(t,x))$. There exists a smooth invertible matrix $P(t,x)$ such that: $P^{-1}AP = \Lambda$. To get profile system, it suffices to make a WKB expansion for $v^\varepsilon(t,x) = P^{-1}(t,x)u^\varepsilon(t,x)$ with anzatz:

$$v^\varepsilon(t,x) = V\left(t,x,\frac{\varphi(t,x)}{\varepsilon}\right) + o(1).$$

Following this way, we formally obtain differential operator $\mathcal{D}_j$

$$\mathcal{D}_j = \sum_{k=1}^{\mathrm{m}} \frac{\partial \varphi_{jk}}{\partial x} \frac{\partial}{\partial \theta_{jk}} = \partial_x \varphi_j . \nabla_{\theta_j} . \tag{3.9}$$

We also obtain initial profile V_0 and new right hand side of (1.1) written in the new base where A is a diagonal matrix. We give profile equations in the next proposition.

$$\begin{array}{lrcl} (3.10) & V_0(x,\theta) & = & P^{-1}(0,x)U_0(x,\theta), \\ (3.11) & G(t,x,v) & := & P^{-1}(t,x)\left(F(t,x,P(t,x)v) - (\partial_t P(t,x) + \Lambda(t,x)\partial_x P(t,x))\, v\right). \end{array}$$

Proposition 3.2. [Equations and existence of profile]. *Under assumptions (3.1), (3.2), (3.3), and if U_0 belongs in $L^2_x(\mathbb{R}; C^0_{ap}(\Theta^0, \mathbb{R}^N)) \cap L^\infty(\mathbb{R}_x \times \Theta^0, \mathbb{R}^N)$, there exists a positive T such that, following system, with $j = 1, \ldots, N$, admits a unique solution in $C^0([0,T]; L^2_x(\mathbb{R}; C^0_{ap}(\Theta, \mathbb{R}^N))) \cap L^\infty([0,T] \times \mathbb{R} \times \Theta, \mathbb{R}^N)$:*

$$\begin{array}{lrcl} (3.12) & -\mathcal{D}_j^2 V_j + X_j V_j & = & \mathbb{E}_j G_j(t,x,V), \\ (3.13) & V_j(0,x,\theta) & = & V_{0,j}(x,\theta_j). \end{array}$$

Indeed we have $V_j(t,x,\theta) = V_j(t,x,\theta_j)$. And, we denote $U(t,x,\theta) = P(t,x)V(t,x,\theta)$. We can now state the main result of the paper which describe the propagation on the oscillations for the parabolic system.

Theorem 3.1. [Validity of geometric optics expansion]. *We assume that U_0 belongs to $L^2(\mathbb{R}_x; C^0_{ap}(\mathbb{R}^{\mathrm{m}}, \mathbb{R}^N)) \cap L^\infty(\mathbb{R}_x \times \mathbb{R}^{\mathrm{m}}, \mathbb{R}^N)$ and assumptions (3.1), (3.2), (3.3) are satisfied. Then there exist $T > 0$ and $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, the system (1.1), (1.4) with initial data (1.3) admits an unique solution u^ε in $C([0,T]; L^2(\mathbb{R}, \mathbb{R}^N)) \cap L^\infty([0,T] \times \mathbb{R}, \mathbb{R}^N)$, and the system (3.12), (3.13) admits an unique solution U in $C^0([0,T]; L^2_x(\mathbb{R}; C^0_{ap}(\Theta, \mathbb{R}^N))) \cap L^\infty([0,T] \times \mathbb{R} \times \Theta, \mathbb{R}^N)$.*
Furthermore, if phases are **transverse**, *we have the following geometric optics expansion:*

$$\lim_{\varepsilon \to 0} \left[u^\varepsilon(t,x) - U\left(t, x, \frac{\varphi(t,x)}{\varepsilon}\right)\right] = 0 \quad in \quad L^\infty([0,T]; L^2(\mathbb{R}, \mathbb{R}^N)).$$

4. Almost diagonal system, L^∞ and L^2 estimates

We diagonalize A. Then we write system (1.1), (1.4), in this new base. After we recall and give energy and uniform estimates used below.

The almost diagonal system:
We diagonalize the hyperbolic operator $\partial_t + A(t,x)\partial_x$ in system (1.1). Unfortunately, this procedure slightly couples first order derivatives. But, we will control this coupling effect with Lemma 4.1 in section 5. More precisely, let us give following notations.

$$\begin{array}{lrcl} (4.1) & v^\varepsilon(t,x) & := & P^{-1}(t,x)u^\varepsilon(t,x), \quad v^\varepsilon_0(x) := P^{-1}(t,x)u^\varepsilon_0(t,x), \\ (4.2) & G(t,x,v) & := & P^{-1}(t,x)\left(F(t,x,P(t,x)v) - (\partial_t P(t,x) + \Lambda(t,x)\partial_x P(t,x))\, v\right), \\ (4.3) & Q_1(t,x) & := & 2P^{-1}(t,x)\partial_x P(t,x), \quad Q_2(t,x) := P^{-1}(t,x)\partial_x^2 P(t,x) \\ (4.4) & G^\varepsilon(t,x,v) & := & G(t,x,v) + \varepsilon^2 Q_2(t,x)v, \\ (4.5) & \mathbb{L}^\varepsilon & = & -\varepsilon^2 \dfrac{\partial^2}{\partial x^2} - \varepsilon^2 Q_1(t,x)\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial t} + \Lambda(t,x)\dfrac{\partial}{\partial x} \end{array}$$

v^ε is the solution of the almost diagonal system with initial data

$$\mathbb{L}^\varepsilon v^\varepsilon = G^\varepsilon(t,x,v^\varepsilon), \qquad v^\varepsilon(0,x) = v_0^\varepsilon(x). \tag{4.6}$$

We use below the diagonal part of the linear operator $\mathbb{L}^\varepsilon$:

$$\mathbb{D}^\varepsilon \;=\; \mathbb{L}^\varepsilon + \varepsilon^2 Q_1 \frac{\partial}{\partial x} = -\varepsilon^2 \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t} + \Lambda(t,x)\frac{\partial}{\partial x} \tag{4.7}$$

Energy and L^∞ estimates:
First notice that Λ, P and P^{-1} also satisfy assumption (3.1). So $\partial_x \Lambda$, $\partial_x Q_1$ are bounded on $[0,T_0]\times\mathbb{R}$ and we have:

Theorem 4.1. [Classical energy estimates]. *Let $1 \geq \nu > 0$, $T_0 \geq T > 0$, L and Q_1 bounded matrix on $[0,T_0]\times\mathbb{R}$, $w(t,x)$ the unique solution in $C^0([0,T_0],L^2(\mathbb{R}_x,\mathbb{R}^N))$ of:*

$$\begin{aligned} -\nu\frac{\partial^2 w}{\partial x^2} - \nu Q_1(t,x)\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} + \Lambda(t,x)\frac{\partial w}{\partial x} &= L(t,x)w + h(t,x) \\ w(0,x) &= w_0(x) \end{aligned}$$

then, there exist constants $C = C(T) > 1$ and D depending only on T, T_0, $\|L\|_{L^\infty([0,T_0]\times\mathbb{R},\mathbb{R}^N)}$, $\|\partial_x\Lambda\|_{L^\infty([0,T_0]\times\mathbb{R},\mathbb{R}^N)}$, $\|Q_1\|_{L^\infty([0,T_0]\times\mathbb{R},\mathbb{R}^N)}$, such that

$$\|w\|_{L^\infty([0,T],L^2(\mathbb{R},\mathbb{R}^N))} \leq C\left(\|w_0\|_{L^2(\mathbb{R},\mathbb{R}^N)} + T\,\|h\|_{L^\infty([0,T],L^2(\mathbb{R},\mathbb{R}^N))}\right), \qquad \lim_{T\to 0} C(T) = 1.$$

$$\sqrt{\nu}\|\partial_x w\|_{L^2([0,T]\times\mathbb{R},\mathbb{R}^N)} + \|w\|_{L^\infty([0,T],L^2(\mathbb{R},\mathbb{R}^N))} \leq D\left(\|w_0\|_{L^2(\mathbb{R},\mathbb{R}^N)} + \|h\|_{L^2([0,T]\times\mathbb{R},\mathbb{R}^N)}\right).$$

We recall this fundamental result.

Theorem 4.2. [Maximum principle].
Let $\nu > 0$, $T_0 \geq T > 0$, $\lambda(t,x) \in C^1([0,T_0]\times\mathbb{R},\mathbb{R})$, $g \in L^\infty(]0,T[\times\mathbb{R},\mathbb{R})$, $w_0 \in L^\infty(\mathbb{R},\mathbb{R})$, and $w(t,x)$ the unique solution in $L^\infty([0,T_0]\times\mathbb{R},\mathbb{R})$ of the following scalar equation with initial data w_0 :

$$-\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial t} + \lambda(t,x)\frac{\partial w}{\partial x} = g(t,x), \qquad w(0,x) = w_0(x),$$

then

$$\|w\|_{L^\infty([0,T]\times\mathbb{R},\mathbb{R})} \;\leq\; \|w_0\|_{L^\infty(\mathbb{R},\mathbb{R})} + T\|g\|_{L^\infty([0,T]\times\mathbb{R},\mathbb{R})}.$$

We are going in next section to use the following interpolation result.

Lemma 4.1. *Let $\nu > 0$, $T > 0$, $\lambda(t,x) \in C^1([0,T]\times\mathbb{R},\mathbb{R})$, $g \in L^2([0,T]\times\mathbb{R},\mathbb{R})$, and $w(t,x)$ the unique solution in $C^0([0,T],L^2(\mathbb{R}_x,\mathbb{R}))$ of the scalar equation, with null initial data :*

$$-\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial t} + \lambda(t,x)\frac{\partial w}{\partial x} = g(t,x), \qquad w(0,x) = 0.$$

If $\partial_x\lambda$ is bounded on $[0,T]\times\mathbb{R}$ then there exists $c_0 > 0$ such that:

$$\nu^{1/4}\|w\|_{L^\infty([0,T]\times\mathbb{R},\mathbb{R})} \leq c_0\|g\|_{L^2([0,T]\times\mathbb{R},\mathbb{R}))}$$

Proof : From standard energy estimates (Theorem 4.1), there exists $d_1 = d_1(T) > 0$ such that:

$$\sup_{0\leq t\leq T}\|w(t,.)\|_{L^2(\mathbb{R}_x)} \;\leq\; d_1\|g\|_{L^2([0,T]\times\mathbb{R})}.$$

We now turn to establish that there exists $d_2 = d_2(T) > 0$ such that for each $t \in [0,T]$:

$$\sqrt{\nu}\|\partial_x w(t,.)\|_{L^2(\mathbb{R}_x)} \leq d_2\|g\|_{L^2([0,T]\times\mathbb{R})}. \tag{4.8}$$

Let us beginning with smooth g with compact support and conclude by density. Write $z = \partial_x w$. So z satisfies:

$$-\nu\partial_x^2 z + \partial_t z + \lambda\partial_x z = -\lambda_x z + g_x, \qquad z(0,x) = 0.$$

Following classical computations to get an energy estimate we have:

$$\begin{aligned}
\int_{\mathbb{R}} \nu(\partial_x z)^2 dx + \frac{d}{dt}\int_{\mathbb{R}} \frac{z^2}{2} dx &= -\int_{\mathbb{R}} \lambda z\partial_x z dx - \int_{\mathbb{R}} \lambda_x z^2 dx + \int_{\mathbb{R}} g_x z dx \\
&= -\frac{1}{2}\int_{\mathbb{R}} \lambda_x z^2 dx - \int_{\mathbb{R}} g\partial_x z dx \\
&\leq \frac{\|\lambda_x\|_\infty}{2}\int_{\mathbb{R}} z^2 dx + \frac{1}{2\nu}\int_{\mathbb{R}} g^2 dx + \frac{\nu}{2}\int_{\mathbb{R}} (\partial_x z)^2 dx \\
\Rightarrow \frac{d}{dt}\int_{\mathbb{R}} z^2 dx &\leq \|\lambda_x\|_\infty \int_{\mathbb{R}} z^2 dx + \frac{1}{\nu}\int_{\mathbb{R}} g^2 dx, \\
\Rightarrow \int_{\mathbb{R}} z^2(t,x) dx &\leq \|\lambda_x\|_\infty \int_0^t\int_{\mathbb{R}} z^2(s,x) ds dx + \frac{1}{\nu}\int_0^T\int_{\mathbb{R}} g^2 dx,
\end{aligned}$$

which gives (4.8) by Gronwall Lemma.
Then, by classical interpolation inequality

$$\|w(t,.)\|^2_{L^\infty(\mathbb{R}_x,\mathbb{R})} \leq 2\|w(t,.)\|_{L^2(\mathbb{R}_x,\mathbb{R})}\|\partial_x w(t,.)\|_{L^2(\mathbb{R}_x,\mathbb{R})},$$

used for each t, we conclude the proof. $\square$

5. Proof of the uniform existence

In this section we prove Proposition 3.1 by a Picard iteration combining energy and L^∞ estimates. Fix T_0 a positive number. Let M be a constant chosen later (indeed M will be greater than the profile norm) such that:

$$M > M_0 := \sup_{0<\varepsilon\leq 1} \|v_0^\varepsilon\|_{L^2\cap L^\infty}, \text{ where } \|v_0^\varepsilon\|_{L^2\cap L^\infty} := \max\left(\|v_0^\varepsilon\|_{L^2(\mathbb{R}_x,\mathbb{R}^N)}, \|v_0^\varepsilon\|_{L^\infty(\mathbb{R}_x,\mathbb{R}^N)}\right).$$

Let $\mathcal{B}^\varepsilon(T)$ be the ball of $C^0([0,T];L^2(\mathbb{R},\mathbb{R}^N)) \cap L^\infty([0,T]\times\mathbb{R},\mathbb{R}^N)$ defined by:

$$\mathcal{B}^\varepsilon(T) = \left\{w^\varepsilon,\ w^\varepsilon(0,x) = v_0^\varepsilon(x) \text{ and } \|w^\varepsilon\|_{C^0L^2\cap L^\infty)} \leq M\right\}.$$

On $\mathcal{B}^\varepsilon(T)$, we define the nonlinear operator Π^ε for any $w^\varepsilon \in \mathcal{B}^\varepsilon(T)$ by $\Pi^\varepsilon(w^\varepsilon)$ is the unique solution on $[0,T]\times\mathbb{R}$ of following initial value problem

$$\mathbb{L}^\varepsilon\left[\Pi^\varepsilon(w^\varepsilon)\right] = G^\varepsilon(t,x,w^\varepsilon), \qquad \left[\Pi^\varepsilon(w^\varepsilon)\right](0,x) = v_0^\varepsilon(x). \tag{5.1}$$

Notice that v^ε is the solution of (4.6), if and only if $\Pi^\varepsilon(v^\varepsilon) = v^\varepsilon$. We are going to prove that Π^ε is a contraction on $\mathcal{B}^\varepsilon(T)$ for a positive T with a Lipschitz constant independent of ε.

Lemma 5.1. *There exist $T_1 \in (0,T_0]$ and $\varepsilon_0 \in (0,1]$ such that for all $\varepsilon \in (0,\varepsilon_0]$, $\Pi^\varepsilon(\mathcal{B}^\varepsilon(T_1)) \subset \mathcal{B}^\varepsilon(T_1)$.*

Proof : We first study $L^\infty L^2$ stability of Π^ε.
Since $G^\varepsilon(t,x,w) = G(t,x,w) + \varepsilon^2 Q_2(t,x)w$, there exists c_1 such that for all $w \in \mathcal{B}^\varepsilon(T_0)$:

$$\|G^\varepsilon(t,x,w(t,x))\|_{L^\infty L^2} \leq c_1(1 + \|w\|_{L^\infty L^2}).$$

By Theorem 4.1 we get:

$$\begin{aligned}\|\Pi^\varepsilon(w^\varepsilon)\|_{L^\infty([0,T];L^2)} &\leq C(T)\left(\|w_0^\varepsilon\|_{L^2} + T\|G^\varepsilon(t,x,w^\varepsilon(t,x))\|_{L^\infty L^2}\right)\\ &\leq C(T)(M_0 + Tc_1(1+M)).\end{aligned}$$

Since $\lim_{T\to 0} C(T) = 1$, there exists T', $0 < T' \leq T_0$ such that $\|\Pi^\varepsilon(w^\varepsilon)\|_{L^\infty([0,T'];L^2)} \leq M$.

We now study L^∞-stability of Π^ε.
Let $z^\varepsilon = \Pi^\varepsilon(w^\varepsilon)$ and $E_2 := \sup_{0<\varepsilon\leq 1} \sup_{w^\varepsilon \in \mathcal{B}^\varepsilon(T_0)} \|G^\varepsilon(t,x,w^\varepsilon(t,x))\|_{L^2([0,T_0]\times\mathbb{R})} < \infty$.
According to Theorem 4.1 we have:

$$\|\varepsilon\partial_x z^\varepsilon\|_{L^2([0,T_0]\times\mathbb{R})} \leq D(M_0 + E_2).$$

By linearity, we decompose $z^\varepsilon = z_1^\varepsilon + z_2^\varepsilon$ in such way that:

$$\begin{aligned}\mathbb{D}^\varepsilon z_1^\varepsilon &= G^\varepsilon(t,x,w^\varepsilon), && z_1^\varepsilon(0,x) = v_0^\varepsilon(x),\\ \mathbb{D}^\varepsilon z_2^\varepsilon &= \varepsilon\left(Q_1(t,x)\varepsilon\frac{\partial z^\varepsilon}{\partial x}\right), && z_2^\varepsilon(0,x) = 0.\end{aligned}$$

Since z_1^ε system is decoupled, we apply Theorem 4.2 for each component of z_1^ε and we have:

$$\|z_1^\varepsilon\|_{L^\infty([0,T_0]\times\mathbb{R})} \leq \|v_0^\varepsilon\|_{L^\infty(\mathbb{R})} + T \sup_{0\leq t\leq T_0,\ x\in\mathbb{R},\ |w|\leq M} |G^\varepsilon(t,x,w)| \leq M_0 + TE_\infty.$$

According to Lemma 4.1 component by component we have:

$$\|z_2^\varepsilon\|_{L^\infty([0,T]\times\mathbb{R})} \leq c_0\sqrt{\varepsilon}\left(\|Q_1\|_{L^\infty([0,T_0]\times\mathbb{R})} D(M_0 + E_2)\right).$$

Then, we can choose $0 < T_1 \leq T'$ and ε_0 such that for all $\varepsilon \in (0, \varepsilon_0]$:

$$\|z^\varepsilon\|_{L^\infty([0,T_1]\times\mathbb{R})} \leq \|z_1^\varepsilon\|_{L^\infty([0,T_1]\times\mathbb{R})} + \|z_2^\varepsilon\|_{L^\infty([0,T_1]\times\mathbb{R})} \leq M.$$

□

Lemma 5.2. *There exists T_2 such that $0 < T_2 \leq T_1$ and Π^ε is a contraction on $\mathcal{B}^\varepsilon(T_2)$) for the $L^\infty([0,T_2];L^2(\mathbb{R},\mathbb{R}^N))$ norm with a same Lipschitz constant, for all $\varepsilon \in (0,\varepsilon_0]$.*

Proof : Let us define $z_1^\varepsilon = \Pi^\varepsilon(w_1^\varepsilon), z_2^\varepsilon = \Pi^\varepsilon(w_2^\varepsilon), z^\varepsilon = z_1^\varepsilon - z_2^\varepsilon, w^\varepsilon = w_1^\varepsilon - w_2^\varepsilon$. z^ε is the solution of the following linear system : $\mathbb{L}^\varepsilon z^\varepsilon = G^\varepsilon(t,x,w_1^\varepsilon) - G^\varepsilon(t,x,w_2^\varepsilon)$, $\quad z^\varepsilon(0,x) = 0$. Obviously, there exists a constant g such that

$$\|G^\varepsilon(t,x,w_1^\varepsilon) - G^\varepsilon(t,x,w_2^\varepsilon)\|_{L^\infty([0,T_0];L^2)} \leq g\||w^\varepsilon|\|_{L^\infty([0,T_0];L^2)},$$

then $\|z^\varepsilon\|_{L^\infty([0,T_0];L^2)} \leq CTg\|w^\varepsilon\|_{L^\infty([0,T_0];L^2)}$ so we can choose $0 < T_2 \leq T_1$ such that Π^ε is a contraction for the $L^\infty L^2$ norm on $\mathcal{B}^\varepsilon(T_2)$ with a Lipschitz constant independent of ε. □

Proof of Proposition 3.1 : Let $v_{n+1}^\varepsilon = \Pi^\varepsilon(v_n^\varepsilon)$, $n \in \mathbb{N}$ and $v_{n=0}^\varepsilon(t,x) = v_0^\varepsilon(x)$. Lemmas 5.1 and 5.2 imply the uniform existence of $(v^\varepsilon)_{0<\varepsilon\leq\varepsilon_0}$, on $[0,T_2]\times\mathbb{R}$ and $\lim_{n\to+\infty} v_n^\varepsilon = v^\varepsilon$ in C^0L^2. Then v^ε also belongs in $\mathcal{B}^\varepsilon(T_2)$. □

6. On profile equations

Lemma 6.1 (L^∞ and L^2 estimates for linear scalar profile equation)**.**
Let j be fixed, $V_{0j} \in L^2(\mathbb{R}; C^0_{ap}(\Theta^0, \mathbb{R})) \cap L^\infty(\mathbb{R}\times\Theta^0, \mathbb{R})$, $f_j \in C^0([0,T]; L^2(\mathbb{R}; C^0_{ap}(\Theta^j, \mathbb{R}))) \cap L^\infty([0,T]\times\mathbb{R}\times\Theta_j, \mathbb{R})$, and $V_j(t,x,\theta_j)$ the unique solution of the scalar equation, with initial data :

$$-\mathcal{D}_j^2 V_j + X_j V_j = f_j, \quad V_j(0,x,\theta_j) = V_{0j}(0,x,\theta_j).$$

Then, there exists C such that:

$$\begin{aligned}
\|V_j\|_{L^\infty([0,T]\times\mathbb{R}\times\Theta_j,\mathbb{R})} &\leq \|V_{0j}\|_{L^\infty(\mathbb{R}\times\Theta^0,\mathbb{R})} + T\|f_j\|_{L^\infty([0,T]\times\mathbb{R}\times\Theta_j,\mathbb{R})}, \\
\|V_j\|_{C^0([0,T],L^2(\mathbb{R};C^0_{ap}(\Theta_j)))} &\leq C\left(\|V_{0j}\|_{L^2(\mathbb{R};C^0_{ap}(\Theta^0))} + T\,\|f_j\|_{C^0([0,T],L^2(\mathbb{R};C^0_{ap}(\Theta)))}\right).
\end{aligned}$$

Proof : The first inequality is the classical maximum principle. So, we have only to prove the second one. Let w_j be the solution of :

$$-\mathcal{D}_j^2 w_j + X_j w_j = \overline{f}_j = \|f_j(t,x,.)\|_{L^\infty(\Theta)}, \quad w_j(0,x,\theta_j) = \overline{V}_{0j} = \|V_{0j}(0,x,.)\|_{L^\infty(\Theta^j)}.$$

By maximum principle $|V_j(t,x,\theta_j)| \leq |w_j(t,x,\theta_j)|$ and, by uniqueness, w_j does not depend of θ. In fact, w_j is solution of : $X_j w_j = \overline{f}_j, \quad w_j(0,x,\theta_j) = \overline{V}_{0j}$. It is a linear hyperbolic equation, so the classical energy inequality for w_j give us the result. □

Now we are able to prove **Proposition 3.2:** By a standard Picard iteration, $V^0(t,x,\theta) = V_0(x,\theta)$, and, for $n \in \mathbb{N}$, $j = 1,\ldots,N$, V_j^{n+1} is given by:

$$-\mathcal{D}_j^2 V_j^{n+1} + X_j V_j^{n+1} = \mathbb{E}_j G_j(t,x,V^n), \quad V_j^{n+1}(0,x,\theta_j) = V_{0,j}(x,\theta_j).$$

Thanks to Lemma 6.1, and since $|||\mathbb{E}_j||| = 1$ we get classically a contraction for small positive T. □

A straightforward computations gives us the following useful Lemma:

Lemma 6.2 (periodic exponential solution)**.** *Let j be fixed, T positive, $\alpha \in \Theta^*$, $b_\alpha(t,x) \in C^0([0,T]; L^2(\mathbb{R},\mathbb{R})))$, $a^0_\alpha(t,x) \in L^2(\mathbb{R},\mathbb{R})$. Let V be the unique solution of:*

$$-\mathcal{D}_j^2 V + X_j V = b_\alpha(t,x)\exp(i\alpha.\theta), \quad V(0,x,\theta) = a^0_\alpha(x)\exp(i\alpha.\theta).$$

Then $V(t,x,\theta) = a_\alpha(t,x)\exp(i\alpha.\theta)$, where a_α is the solution of:

$$X_j a_\alpha + |\alpha.\partial_x\phi_j|^2 a_\alpha = b_\alpha(t,x), \quad a_\alpha(0,x) = a^0_\alpha(x).$$

Then we can compute Fourier expansion of V. Indeed a_α has the same smoothness as a^0_α and b_α. Furthermore, if data are compactly supported w.r.t. x, a_α and V too.

7. The linear scalar case

In this section, we validate the geometric optics expansion for the linear scalar case. The last result comes in useful to prove the main result of this paper.

Lemma 7.1. *Let $0 < \varepsilon \leq 1$, $h \in L^2(\mathbb{R}; C^0_{ap}(\Theta^0, \mathbb{R}))$, and $j \in \{1,\cdots,N\}$.*
Let u^ε and U be the unique solutions of equations (7.1) ,(7.2) on $[0,T]\times\mathbb{R}$,

$$-\varepsilon^2\partial_x^2 u^\varepsilon + X_j u^\varepsilon = 0, \qquad u^\varepsilon(0,x) = h\left(x, \frac{\varphi^0}{\varepsilon}\right), \tag{7.1}$$

$$-\mathcal{D}_j^2 U + X_j U = 0, \qquad U(0,x,\theta_j) = h(x,\theta_j), \tag{7.2}$$

then $u^\varepsilon(t,x) = U\left(t,x,\frac{\varphi_j}{\varepsilon}\right) + O(\varepsilon)$ *in* $L^\infty([0,T];L^2(\mathbb{R},\mathbb{R}))$

Proof : First, we assume that $h \in C_c^\infty(\mathbb{R}_x; C_{ap}^\infty(\Theta^0,\mathbb{R}))$, where the subscript c means that h has compact support with respect to x. Let $v^\varepsilon(t,x) = U\left(t,x,\frac{\varphi_j(t,x)}{\varepsilon}\right)$. Then, a direct computation give us with $S^\varepsilon(t,x) := -\varepsilon^2\partial_x^2 U - \varepsilon(\partial_x \mathcal{D}_j)U = \mathrm{O}(\varepsilon) \in L^\infty([0,T];L^2(\mathbb{R},\mathbb{R}))$:

$$-\varepsilon^2\partial_x^2 v^\varepsilon + X_j v^\varepsilon = S^\varepsilon - \mathcal{D}_j^2 U + X_j U = S^\varepsilon, \qquad v^\varepsilon(0,x) \;=\; u^\varepsilon(0,x).$$

Since $-\varepsilon^2\partial_x^2 u^\varepsilon + X_j u^\varepsilon = 0$, we can apply energy estimate to $u^\varepsilon - v^\varepsilon$ to conclude this smooth case. We finish the proof using density of $C_c^\infty(\mathbb{R}_x; C_{ap}^\infty(\Theta^0,\mathbb{R}))$ in $L^2(\mathbb{R}; C_{ap}^0(\Theta^0,\mathbb{R}))$. $\square$

We recall and slightly extend on unbounded domain and with a linear term a non stationary phases Lemma from [13]:

Lemma 7.2. *Let* $0 < \varepsilon \le 1$, $a \in C^0([0,T];L^2(\mathbb{R},\mathbb{R}))$, $b \in L^\infty([0,T]\times\mathbb{R},\mathbb{R})$, $\psi \in C^\infty([0,T]\times\mathbb{R},\mathbb{R})$ *and* $j \in \{1,\cdots,N\}$. *Assume that* ψ *is transverse to* X_j *i.e.* $X_j\psi \neq 0$ *a.e., and* u^ε *is the unique solution of (7.3) on* $[0,T]\times\mathbb{R}$,

$$X_j u^\varepsilon + b(t,x)u^\varepsilon = a(t,x)\exp\left(i\frac{\psi(t,x)}{\varepsilon}\right), \qquad u^\varepsilon(0,x) = 0, \tag{7.3}$$

then $\lim_{\varepsilon\to 0} u^\varepsilon(t,x) = 0$ *in* $L^\infty([0,T];L^2(\mathbb{R},\mathbb{R}^N))$.

We now apply the previous Lemma to a weakly parabolic equation.

Lemma 7.3. *Let* $0 < \varepsilon \le 1$, $a \in C^0([0,T];L^2(\mathbb{R},\mathbb{R}))$, $j \in \{1,\cdots,N\}$.
Let u^ε, U *the unique solution of the following equations on* $[0,T]\times\mathbb{R}$

$$\begin{array}{rclrcl} -\varepsilon^2\partial_x^2 u^\varepsilon + X_j u^\varepsilon &=& a(t,x)\exp\left(i\alpha.\frac{\varphi(t,x)}{\varepsilon}\right), & u^\varepsilon(0,x) &=& 0,\\ -\mathcal{D}_j^2 U + X_j U &=& a(t,x)\mathbb{E}_j\left(\exp(i\alpha.\theta)\right), & U(0,x,\theta_j) &=& 0, \end{array}$$

then $\lim_{\varepsilon\to 0}\left[u^\varepsilon(t,x) - U\left(t,x,\frac{\varphi_j(t,x)}{\varepsilon}\right)\right] = 0$ *in* $L^\infty([0,T];L^2(\mathbb{R},\mathbb{R}^N))$.

Proof : First, we assume that $a \in C_c^2([0,T]\times\mathbb{R},\mathbb{R}))$. After, we extend the result to more general data by density.
Let $\psi :\equiv \alpha.\varphi$. By transversality assumptions on phase φ, we have only two cases:

1. $X_j\psi \equiv 0$: then we have $\mathbb{E}_j\left(\exp(i\alpha.\theta)\right) = \exp(i\alpha.\theta)$. As in Lemma 7.1 proof, let us consider $v^\varepsilon(t,x) := U\left(t,x,\frac{\varphi_j(t,x)}{\varepsilon}\right)$, then we have:

   $$-\varepsilon^2\partial_x^2 v^\varepsilon + X_j v^\varepsilon = a(t,x)\exp\left(i\alpha.\frac{\varphi(t,x)}{\varepsilon}\right) + O(\varepsilon), \qquad v^\varepsilon(0,x) = 0,$$

   and we conclude this case by applying an energy estimate to $u^\varepsilon - v^\varepsilon$.
2. ψ is transverse to X_j : then we have $\mathbb{E}_j\exp(i\alpha.\theta) \equiv 0$ and $U \equiv 0$. Let w^ε be the solution of

$$X_j w^\varepsilon = a(t,x)\exp\left(i\alpha.\frac{\varphi(t,x)}{\varepsilon}\right), \qquad w^\varepsilon(0,x) = 0. \tag{7.4}$$

From Lemma 7.2 we get $w^\varepsilon \to 0$ in $L_t^\infty L_x^2$. In the same way, we are going to see that $\varepsilon\partial_x w^\varepsilon$ and $\varepsilon^2\partial_x^2 w^\varepsilon$ also converge towards 0 in L^2. So, at first, we derive (7.4) with respect to $\varepsilon\partial_x$ and then with respect to $\varepsilon^2\partial_x^2$ to get :

$$X_j z^\varepsilon + b z^\varepsilon = \beta \exp\left(i\frac{\psi}{\varepsilon}\right) + \mathrm{O}(\varepsilon), \quad z^\varepsilon(0,x) = 0,$$

first with $z^\varepsilon := \varepsilon\partial_x w^\varepsilon$, and after with $z^\varepsilon := \varepsilon^2\partial_x^2 w^\varepsilon$. Then, we conclude with Lemma 7.2. More precisely, we obtain following equations:

$$\begin{aligned} X_j\left(\varepsilon\partial_x w^\varepsilon\right) + \left(\partial_x\lambda_j\right)\left(\varepsilon\partial_x w^\varepsilon\right) &= ia\partial_x\psi \exp\left(i\frac{\psi}{\varepsilon}\right) + \varepsilon \times \left(\partial_x a \exp\left(i\frac{\psi}{\varepsilon}\right)\right) \\ X_j\left(\varepsilon^2\partial_x^2 v^\varepsilon\right) + \left(\partial_x\lambda_j\right)\left(\varepsilon^2\partial_x^2 w^\varepsilon\right) &= ia(\partial_x\psi)^2 \exp\left(i\frac{\psi}{\varepsilon}\right) + \mathrm{O}(\varepsilon), \end{aligned}$$

where $\mathrm{O}(\varepsilon) = \varepsilon\left[\left(ia\partial_x^2\psi) + 2(\partial_x a)(\partial_x\psi) + \varepsilon\partial_x^2 a\right)\exp(i\psi/\varepsilon) - (\partial_x^2\lambda_j)(\varepsilon\partial_x w^\varepsilon)\right].$

Now, let $R^\varepsilon := u^\varepsilon - w^\varepsilon$. We can conclude by using energy estimate since R^ε is the solution of: $-\varepsilon^2\partial_x^2 R^\varepsilon + X_j R^\varepsilon = \varepsilon^2\partial_x^2 w^\varepsilon, \qquad w^\varepsilon(0,x) = 0.$ □

Proposition 7.1. *Let* $0 < \varepsilon \leq 1$, $h \in L^2(\mathbb{R}; C^0_{ap}(\Theta^0, \mathbb{R}))$, $f \in C^0([O,T]; L^2(\mathbb{R}; C^0_{ap}(\Theta, \mathbb{R})))$, *and let* u^ε, U *be the unique solutions of the following equations on* $[0,T]$

$$\begin{array}{rclrcl} -\varepsilon^2\partial_x^2 u^\varepsilon + X_j u^\varepsilon &=& f\left(t,x,\dfrac{\varphi}{\varepsilon}\right), & u^\varepsilon(0,x) &=& h\left(x,\dfrac{\varphi^0}{\varepsilon}\right) \\ -\mathcal{D}_j^2 U + X_j U &=& \mathbb{E}_j f(t,x,\theta), & U(0,x,\theta_j) &=& h(x,\theta_j) \end{array}$$

then $\displaystyle\lim_{\varepsilon\to 0}\left[u^\varepsilon(t,x) - U\left(t,x,\frac{\varphi_j(t,x)}{\varepsilon}\right)\right] = 0$ *in* $L^\infty([0,T]; L^2(\mathbb{R},\mathbb{R}^N))$.

Proof : By linearity of previous P.D.E. we can decompose $u^\varepsilon = u_1^\varepsilon + u_2^\varepsilon$. We make an analogous decomposition for U. And u_1^ε, u_2^ε, U_1, U_2 are solutions of following equations

$$\begin{array}{rclrcl} -\varepsilon^2\partial_x^2 u_1^\varepsilon + X_j u_1^\varepsilon &=& 0, & u_1^\varepsilon(0,x) &=& h\left(x,\frac{\varphi^0}{\varepsilon}\right), \\ -\varepsilon^2\partial_x^2 u_2^\varepsilon + X_j u_2^\varepsilon &=& f\left(t,x,\frac{\varphi}{\varepsilon}\right), & u_2^\varepsilon(0,x) &=& 0, \\ -\mathcal{D}_j^2 U_1 + X_j U &=& 0, & U_1(0,x,\theta) &=& h(x,\theta), \\ -\mathcal{D}_j^2 U_2 + X_j U &=& \mathbb{E}_j f(t,x,\theta), & U_2(0,x,\theta) &=& 0. \end{array}$$

By Lemma 7.1, we obtain $u_1^\varepsilon = U_1(t,x,\varphi/\varepsilon) + \mathrm{O}(\varepsilon)$. For u_2^ε we use the density of trigonometric polynomial, as in [13]. Then, we replace f by a trigonometric polynomial, and, by linearity, we can use Lemma 7.3. Then, the proof is complete. □

8. Proof of geometric optics expansion

The proof is based on the following observation as in [14]. For a contraction mapping Π with a Lipschitz constant ρ and the fixed point u, we recall a classical posteriori estimate:

$$\|u - v\| \leq \frac{1}{1-\rho}\|\Pi(v) - v\|.$$

So, let $w^\varepsilon(t,x) = V\left(t, x, \dfrac{\varphi(t,x)}{\varepsilon}\right)$. With stationary and non stationary phases Lemmas we are going to prove that $\Pi^\varepsilon(w^\varepsilon) = w^\varepsilon + o(1)$. Since $(\Pi^\varepsilon)_{0<\varepsilon\leq\varepsilon_0}$ is a family of contraction with same Lipschitz constant we have: $v^\varepsilon = w^\varepsilon + o(1)$.

Precisely, we use notations from section 5. Let T_3 satisfying Proposition 3.2: a time existence of the profile. Fix $M > \|V\|_{L^\infty([0,T_3];L^2(\mathbb{R}:C^0_{ap}(\Theta,\mathbb{R}^N)))} + \|V\|_{L^\infty([0,T_3]\times\mathbb{R}\times\Theta,\mathbb{R}^N)}$. We now choose T_4 such that v^ε, V live on $[0,T_4]$, $\Pi^\varepsilon(\mathcal{B}^\varepsilon(T_4)) \subset \mathcal{B}^\varepsilon(T_4)$, and Π^ε are a uniform $L^\infty L^2$ contraction on $\mathcal{B}^\varepsilon(T_4)$.
First w^ε belongs to $\mathcal{B}^\varepsilon(T_4)$. Let $z^\varepsilon = \Pi^\varepsilon(w^\varepsilon)$ and j fixed. Notice that z^ε belongs to $\mathcal{B}^\varepsilon(T_4)$. z^ε_j satisfies following equation:

$$-\varepsilon^2\partial_x^2 z_j^\varepsilon + X_j z_j^\varepsilon \quad = \quad G_j\left(t, x, V\left(t, x, \frac{\varphi(t,x)}{\varepsilon}\right)\right) + r_j^\varepsilon$$

where $r^\varepsilon = \varepsilon^2 Q_2 w^\varepsilon + \varepsilon^2 Q_1 \partial_x z^\varepsilon = O(\varepsilon)$ in $L^\infty L^2$. Then, neglecting r^ε, and according to Proposition 7.1, we have $z_j^\varepsilon(t,x) = W_j\left(t, x, \dfrac{\varphi_j}{\varepsilon}\right) + o(1)$, where W_j is the solution of

$$(-\mathcal{D}_j^2 + X_j)U = \mathbb{E}_j G_j\left(t, x, V\left(t, x, \theta\right)\right) \text{ and } W_j(0,.,.) = V_j(0,.,.).$$

So W_j is exactly V_j, $z_j^\varepsilon = w_j^\varepsilon + o(1)$ and :

$$\Pi^\varepsilon(w^\varepsilon) = w^\varepsilon + o(1) \quad in \quad L^\infty([0,T_4]; L^2(\mathbb{R},\mathbb{R}^N))$$

Since Π^ε is a $L^\infty([0,T_4];L^2(\mathbb{R},\mathbb{R}^N))$ contraction on $\mathcal{B}^\varepsilon(T_4)$ with a Lipschitz constant independent of $\varepsilon \in (0,\varepsilon_0]$, we have

$$v^\varepsilon = w^\varepsilon + o(1) \quad in \quad L^\infty([0,T_4]; L^2(\mathbb{R},\mathbb{R}^N)),$$

which completes the proof of Theorem 3.1.

9. Comparisons with the inviscid case

Let us first investigate the standard Lemma of non stationary phase. Could this Lemma be improved by the presence of small viscosity? Unfortunately the answer is no as you can see on the following simple example.

$$-\varepsilon^2\partial_x^2 u^\varepsilon + \partial_t u^\varepsilon = \exp\left(i\phi(t,x)/\varepsilon\right), \qquad u^\varepsilon(0,x) = 0, \tag{9.1}$$

$$\partial_t v^\varepsilon = \exp\left(i\phi(t,x)/\varepsilon\right), \qquad v^\varepsilon(0,x) = 0, \tag{9.2}$$

with $X = \partial_t + 0 \times \partial_x = \partial_t$.
Assume $X\phi \neq 0$ a.e., for instance with $\phi(t,x) = t$, viscosity plays no role since $u^\varepsilon(t,x) = v^\varepsilon(t,x) = i\varepsilon(1 - \exp(it/\varepsilon)) = \mathrm{O}(\varepsilon)$. Furthermore, if $\phi(t,x) = \psi(t)$, we always have $u^\varepsilon(t,x) = v^\varepsilon(t,x)$. So, if ψ' vanishes on a discrete set we have no rate of convergence. For instance if $\psi(t) \sim t^\alpha$ when t goes to 0 and $\alpha > 1$, we have $u^\varepsilon(t,x) \sim C\varepsilon^{1/\alpha}$ with $C \neq 0$. So we cannot expect any correctors for multiphase expansions with such such general transversality conditions on phases. This construction follows [13], Remark 1.2, p. 114-115.

The new feature in our paper is to get and valid parabolic equations for profiles. These equations are only related to phases transported by the hyperbolic operator $\partial_t + A(t,x)\partial_x$. For instance, solutions of (9.1),(9.2) with $\phi(t,x) = x$, $(X\phi \equiv 0)$, are $u^\varepsilon(t,x) = (1 - e^{-t})\exp(ix/\varepsilon)$, $v^\varepsilon(t,x) = t\exp(ix/\varepsilon)$. Profiles satisfy the following equations:

$$-\partial_\theta^2 U + \partial_t U = \exp\left(i\theta\right), \qquad U(0,x,\theta) = 0,$$
$$\partial_t V = \exp\left(i\theta\right), \qquad V(0,x,\theta) = 0.$$

Then viscous profile is $U(t,x,\theta) = (1-e^{-t})e^{i\theta}$ and hyperbolic profile is $V(t,x,\theta) = te^{i\theta}$. So, except for small time, u^ε and v^ε have different behaviors.

We conclude this section by few remarks about large time. If $(u^\varepsilon)_{0<\varepsilon\leq 1}$ and U are bounded in $L^\infty([0,T],L^2(\mathbb{R}_x)) \cap L^\infty([0,T]\times\mathbb{R}_x)$ then geometric expansions stated in theorem 3.1 is still valid on $[0,T]$. But, in general, the maximal existence time of the family of exact solutions is not bounded from below by the existence time of profile. Indeed, to have such result we have to valid WKB expansion in $L^\infty([0,T]\times\mathbb{R}_x)$. For this purpose we have to strengthen transversality assumptions on phases and smoothness of data and solutions. For a fuller treatment of existence times in the inviscid case we refer the reader to [13].

Acknowledgments: The author wishes to express his thanks to **Olivier GUES** for suggesting the problem and for many stimulating conversations.

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