Cross Ratios, Surface Groups, PSL($n, \mathbb{R}$) and Diffeomorphisms of the Circle

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Abstract

This article relates representations of surface groups to cross ratios. We first describe a n-connected component of the space of representations into PSL($n, \mathbb{R}$) – known as the *Hitchin component* – as the set of cross ratios on the boundary at infinity of the group which satisfy some functional relations. Similarly, we study some representations into $C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ associated cross ratios. and exhibit a "character variety" of these representations. We show that this character variety contains all n-Hitchin components as well as the set of negatively curved metrics on the surface.

1 Introduction

Let Σ be a closed surface with genus at least 2. We recall that the boundary at infinity $\partial_\infty \pi_1(\Sigma)$ of the fundamental group $\pi_1(\Sigma)$ is a one dimensional compact connected Hölder manifold – hence Hölder homeomorphic to the circle $\mathbb{T}$ – equipped with an action of $\pi_1(\Sigma)$ by Hölder homeomorphisms.

A *cross ratio* on $\partial_\infty \pi_1(\Sigma)$ is a Hölder function $\mathbb{B}$ defined on

$$\partial_\infty \pi_1(\Sigma)^{4*} = \{(x, y, z, t) \in \partial_\infty \pi_1(\Sigma)^4 \mid x \neq t \text{ and } y \neq z\},$$

invariant under the diagonal action of $\pi_1(\Sigma)$ and which satisfies some algebraic rules. Roughly speaking, these rules encode some symmetry and normalisation properties as well as multiplicative cocycle identities in some of the variables:

$$\begin{array}{rrcl}
\textsc{Symmetry:} & \mathbb{B}(x,y,z,t) & = & \mathbb{B}(z,t,x,y), \\
\textsc{Normalisation:} & \mathbb{B}(x,y,z,t) & = & 0 \quad \Leftrightarrow x = y \text{ or } z = t, \\
\textsc{Normalisation:} & \mathbb{B}(x,y,z,t) & = & 1 \quad \Leftrightarrow x = z \text{ or } y = t, \\
\textsc{Cocycle identity:} & \mathbb{B}(x,y,z,t) & = & \mathbb{B}(x,y,z,w)\mathbb{B}(x,w,z,t), \\
\textsc{Cocycle identity:} & \mathbb{B}(x,y,z,t) & = & \mathbb{B}(x,y,w,t)\mathbb{B}(w,y,z,t).
\end{array}$$

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The *period* of a non trivial element γ of $\pi_1(\Sigma)$ with respect to $\mathbb{B}$ is the following real number

$$\log|\mathbb{B}(\gamma^-, \gamma y, \gamma^+, y)| := \ell_{\mathbb{B}}(\gamma),$$

where γ^+ (respectively γ^-) is the attracting (respectively repelling) fixed point of γ on $\partial_\infty \pi_1(\Sigma)$ and y is any element of $\partial_\infty \pi_1(\Sigma)$; as the notation suggests, the period is independent of y – see Paragraph 3.3.

These definitions are closely related to those introduced by Otal in [27, 28] and discussed by Ledrappier in [25] from various view-points and by Bourdon in [4] in the context of CAT(-1)-spaces.

Whenever Σ is equipped with a hyperbolic metric, $\partial_\infty \pi_1(\Sigma)$ is identified with $\mathbb{RP}^1$ and inherits a cross ratio from this identification. Thus, every discrete faithful homomorphism from $\pi_1(\Sigma)$ to $\mathrm{PSL}(2, \mathbb{R})$ gives rises to a cross ratio – called *hyperbolic* – on $\partial_\infty \pi_1(\Sigma)$. The period of an element for a hyperbolic cross ratio is the length of the associated closed geodesic. These hyperbolic cross ratios satisfy the following further relation

$$1 - \mathbb{B}(x, y, z, t) = \mathbb{B}(t, y, z, x). \tag{1}$$

Conversely, every cross ratio satisfying Relation (1) is hyperbolic.

The purpose of this paper is to

- generalise the above construction to $\mathrm{PSL}(n, \mathbb{R})$
- give an n-asymptotic infinite dimensional version

A correspondence between cross ratios and Hitchin representations

Throughout this paper, a *representation* from a group to another group is a class of homomorphisms up to conjugation[1]. We denote by

$$\mathrm{Rep}(H, G) = \mathrm{Hom}(H, G)/G,$$

the *space of representations* from H to G, that is the space of homomorphisms from H to G identified up to conjugation by an element of G.

When G is a Lie group, $\mathrm{Rep}(\pi_1(\Sigma), G)$ is a remarkable space which has been studied from many view-points; classical references are [2, 14, 15, 20, 29].

In [23], we define an *n-Fuchsian* homomorphism to be a homomorphism ρ from $\pi_1(\Sigma)$ to $\mathrm{PSL}(n, \mathbb{R})$ which may be written as $\rho = \iota \circ \rho_0$, where ρ_0 is a discrete faithful homomorphism from $\pi_1(\Sigma)$ to $\mathrm{PSL}(2, \mathbb{R})$ and ι is the irreducible homomorphism from $\mathrm{PSL}(2, \mathbb{R})$ to $\mathrm{PSL}(n, \mathbb{R})$.

In [21], Hitchin proves the following remarkable result: *every connected component of the space of completely reducible representations from $\pi_1(\Sigma)$ to $\mathrm{PSL}(n, \mathbb{R})$ which contains an n-Fuchsian representation is diffeomorphic to a*

[1] We emphasise that this terminology is slightly non standard.

ball. Such a connected component is called a *Hitchin* component and denoted by

$$\mathrm{Rep}_H(\pi_1(\Sigma), \mathrm{PSL}(n, \mathbb{R})).$$

A representation which belongs to a Hitchin component is called an *n-Hitchin representation*. In other words, an n-Hitchin representation is a representation that may be deformed into an n-Fuchsian representation.

In [23], we give a geometric description of Hitchin representations, which is completed by Guichard [18] (*cf.* Section 2). In particular, we show in [23] that if ρ is a Hitchin representation and γ a nontrivial element of $\pi_1(\Sigma)$, then $\rho(\gamma)$ is real split (Theorem 1.5 of [23]). This allows us to define the *width* of a non trivial element γ of $\pi_1(\Sigma)$ with respect to a Hitchin representation ρ as

$$w_\rho(\gamma) = \log\left(\left|\frac{\lambda_{\max}(\rho(\gamma))}{\lambda_{\min}(\rho(\gamma))}\right|\right),$$

where $\lambda_{\max}(\rho(\gamma))$ and $\lambda_{\min}(\rho(\gamma))$ are respectively the eigenvalues of respectively maximum and minimum absolute values of the element $\rho(\gamma)$.

Generalising the situation for $n = 2$ which we briefly described in the previous paragraph, we identify n-Hitchin representations with certain types of cross ratios. More precisely, for every p let $\partial_\infty \pi_1(\Sigma)^p_*$ be the set of pairs

$$(e, u) = ((e_0, e_1, \ldots, e_p), (u_0, u_1, \ldots, u_p)),$$

of $p+1$-tuples of points in $\partial_\infty \pi_1(\Sigma)$ such that $e_j \neq e_i \neq u_0$ and $u_j \neq u_i \neq e_0$, whenever $j > i > 0$.

Let $\mathbb{B}$ be a cross ratio and let $\chi^p_{\mathbb{B}}$ be the map from $\partial_\infty \pi_1(\Sigma)^p_*$ to $\mathbb{R}$ defined by

$$\chi^p_{\mathbb{B}}(e, u) = \det_{i,j>0}((\mathbb{B}(e_i, u_j, e_0, u_0)).$$

A cross ratio $\mathbb{B}$ has *rank-n* if

- $\forall (e, u) \in \partial_\infty \pi_1(\Sigma)^n_*, \quad \chi^n_{\mathbb{B}}(e, u) \neq 0,$
- $\forall (e, u) \in \partial_\infty \pi_1(\Sigma)^{n+1}_*, \quad \chi^{n+1}_{\mathbb{B}}(e, u) = 0.$

Rank-2 cross ratios are precisely cross ratios satisfying Relation (1) (see Proposition 4.1). Our main result is the following.

Theorem 1.1 *There exists a bijection ϕ from the set of n-Hitchin representations to the set of rank n-cross ratios such that for any nontrivial element γ of $\pi_1(\Sigma)$*

$$\ell_{\mathbb{B}}(\gamma) = w_\rho(\gamma),$$

where $\ell_{\mathbb{B}}(\gamma)$ is the period of γ is given with respect to $\mathbb{B} = \phi(\rho)$, and $w_\rho(\gamma)$ is the width of γ with respect to ρ.

Cross ratios and representations are related through the *limit curve*, drawn in $\mathbb{P}(\mathbb{R}^n)$ and described in Paragraph 2.3. In [22], we use the relations between cross ratios and Hitchin representations to study the energy functional. In collaboration with McShane in [24], we use these relations to generalise McShane's identities [26]. In [22], cross ratios also appear in relation with maximal representations, as discussed in the work of Burger, Iozzi and Wienhard [8, 9] and Bradlow, García-Prada, Gothen, Mundet i Riera [6, 5, 7, 12, 17]. In [11], Fock and Goncharov describe combinatorially Hitchin representations.

A "Character variety" containing all Hitchin representations

Since Hitchin representations are irreducible – *cf* Lemma 10.1 in [23] – the natural embedding of $\mathrm{PSL}(n,\mathbb{R})$ in $\mathrm{PSL}(n+1,\mathbb{R})$ does not give rise to an embedding of the corresponding Hitchin component. Therefore, there is no natural algebraic way - by an injective limit procedure say - to build a limit when n goes to infinity of Hitchin components. However, it follows from the previous paragraph that all Hitchin components sit in the space of all cross ratios.

The second construction of this article refines this observation and tells us that *all Hitchin components lie in a "character variety" of $\pi_1(\Sigma)$ into an infinite dimensional group $H(\mathbb{T})$.*

The group $H(\mathbb{T})$ is defined as follows. Let $C^{1,h}(\mathbb{T})$ be the vector space of C^1-functions with Hölder derivatives on the circle $\mathbb{T}$, and let $\mathrm{Diff}^h(\mathbb{T})$ be the group of C^1-diffeomorphisms with Hölder derivatives of $\mathbb{T}$. We observe that $\mathrm{Diff}^h(\mathbb{T})$ acts naturally on $C^{1,h}(\mathbb{T})$ and we set

$$H(\mathbb{T}) = C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T}).$$

In Paragraph 6.1.2, we give an interpretation of $H(\mathbb{T})$ as a group of Hölder homeomorphisms of $J^1(\mathbb{T},\mathbb{R})$, the space of one-jets of real valued functions on the circle.

Our first result singles out a certain class of homomorphisms from $\pi_1(\Sigma)$ to $H(\mathbb{T})$ with interesting topological properties. Namely, we define in Paragraph 7.2 *∞-Hitchin homomorphisms* from $\pi_1(\Sigma)$ to $H(\mathbb{T})$. As part of the definition, the quotient $J^1(\mathbb{T})/\rho(\pi_1(\Sigma))$ is compact for an ∞-Hitchin homomorphism ρ. In Paragraph 9.2.2, we associate a real number $\ell_\rho(\gamma)$ – called the *ρ-length* of γ – to every nontrivial element γ of $\pi_1(\Sigma)$ and every ∞-Hitchin representation ρ. The ρ-length – considered as a map from $\pi_1(\Sigma)\setminus\{\mathrm{id}\}$ to $\mathbb{R}$ – is the *spectrum* of ρ. In Paragraph 9.2.1, we also associate to every ∞-Hitchin homomorphism a cross ratio and we relate the spectrum with the periods of this cross ratio.

We denote by Hom_H the set of all ∞-Hitchin homomorphisms. Let $Z(H(\mathbb{T}))$ be the one-dimensional centre of $H(\mathbb{T})$. Our first result describes the action on $H(\mathbb{T})$ on Hom_H.

Theorem 1.2 *The set* Hom_H *is open in* $\mathrm{Hom}(\pi_1(\Sigma), H(\mathbb{T}))$*. The group*

$$H(\mathbb{T})/Z(H(\mathbb{T})),$$

acts properly on Hom_H *and the quotient*

$$\mathrm{Rep}_H = \mathrm{Hom}(\pi_1(\Sigma), H(\mathbb{T}))/H(\mathbb{T}),$$

is Hausdorff. Moreover, two ∞-Hitchin homomorphisms with the same spectrum and the same cross ratio are conjugated.

Our next result exhibits an embedding of Hitchin components into this character variety Rep_H.

Theorem 1.3 *There exists a continuous injective map ψ from every Hitchin component into Rep_H such that if ρ is an n-Hitchin representation, then for any γ in $\pi_1(\Sigma)$*

$$\ell_{\psi(\rho)}(\gamma) = w_\rho(\gamma), \tag{2}$$

where $\ell_{\psi(\rho)}$ is the $\psi(\rho)$-length and w_ρ is the width with respect to ρ. Moreover, the cross ratios associated to ρ and $\psi(\rho)$ coincide.

This suggests that the group $C^{1,h}(\mathbb{T})\rtimes\mathrm{Diff}^h(\mathbb{T})$ contains an n-asymptotic version of $\mathrm{PSL}(n,\mathbb{R})$.

This character variety contains yet another interesting space. Let $\mathcal{M}$ be the space of negatively curved metrics on the surface Σ, up to diffeomorphisms isotopic to the identity. For every negatively curved metric g and every non trivial element γ of $\pi_1(\Sigma)$, we denote by $\ell_g(\gamma)$ the length of the closed geodesic freely homotopic to γ. Then

Theorem 1.4 *There exists a continuous injective map ψ from $\mathcal{M}$ to Rep_H, such that for any γ in $\pi_1(\Sigma)$*

$$\ell_g(\gamma) = \ell_{\psi(g)}(\gamma),$$

where $\ell_{\psi(g)}$ is the $\psi(g)$-length of γ.

The injectivity in this Theorem uses a result by Otal [27]. Theorems 1.3 and 1.4 are both consequences of Theorem 11.3, a general conjugation result.

We finish this general introduction by stating a question about our construction: can one characterise in Rep_H the closure $\mathcal{H}_\infty$ of the union of images of Hitchin components ?

$$\mathcal{H}_\infty = \overline{\bigcup_n \mathrm{Rep}_H(\pi_1(\Sigma), \mathrm{PSL}(n,\mathbb{R}))}.$$

For instance does this closure $\mathcal{H}_\infty$ contains the space of all metrics $\mathcal{M}$?

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Contents

1 Introduction **1**

2 Curves and hyperconvex representations **7**

2.1 Hyperconvex and Hitchin representations 8

2.2 Hyperconvex representations and Frenet curves 8

2.2.1 A smooth map 10

3 Cross ratio, definitions and first properties **10**
3.1 Cross ratios and weak cross ratios 10

4 Examples of cross ratio **12**
4.1 Cross ratio on the projective line 12
4.2 Weak cross ratios and curves in projective spaces 14
4.3 Negatively curved metrics and comparisons 15
4.4 A symplectic construction 16
4.4.1 Polarised cross ratio 17
4.4.2 Curves 18
4.4.3 Projective spaces 18
4.4.4 Appendix: Periods, action difference and triple ratios . . 20

5 Hitchin representations and cross ratios **21**
5.1 Rank-n weak cross ratios 22
5.1.1 Nullity of χ. 23
5.2 Hyperconvex curves and rank-n cross ratios 23
5.3 Proof of Theorem 5.3 23
5.3.1 Cross ratio associated to curves 23
5.3.2 Strict cross ratio and Frenet curves 24
5.3.3 Curves associated to cross ratios 25
5.4 Hyperconvex representations and cross ratios 27
5.5 Proof of the main Theorem 1.1 28

6 The jet space $J^1(\mathbb{T},\mathbb{R})$ **28**
6.1 Description of the jet space 28
6.1.1 Projections 28
6.1.2 Action of $C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ 29
6.1.3 Foliation 29
6.1.4 Canonical flow 30
6.1.5 Contact form 30
6.2 A Geometric Characterisation of $C^\infty(\mathbb{T}) \rtimes \mathrm{Diff}^\infty(\mathbb{T})$ 30
6.2.1 A preliminary proposition 31
6.2.2 Proof of Proposition 6.1 31
6.3 $\mathrm{PSL}(2,\mathbb{R})$ and $C^\infty(\mathbb{T}) \rtimes \mathrm{Diff}^\infty(\mathbb{T})$ 32
6.3.1 Proof of Proposition 6.3 33
6.3.2 Alternate description of J 33

7 Anosov and ∞-Hitchin homomorphisms **34**
7.1 ∞-Fuchsian homomorphism and H-Fuchsian actions 34
7.2 Anosov homomorphisms 34

8 Stability of ∞-Hitchin homomorphisms **35**
8.1 Filtered Spaces and Holonomy. 36
8.1.1 Laminations and Filtered Space 36
8.1.2 Holonomy Theorem 38

8.1.3 Completeness of affine structure along leaves 40
8.2 A Stability Lemma for Anosov Homomorphisms 41
8.2.1 Minimal action on the circle 41
8.2.2 Proof of Lemma 8.10 42

9 Cross ratios and properness of the action 43
9.1 Corollaries of the Stability Lemma 8.10 43
9.2 Cross ratio, Spectrum and Ghys deformations 44
9.2.1 Cross ratio associated to ∞-Hitchin representations . . . 44
9.2.2 Spectrum 45
9.2.3 Coherent representations and Ghys deformations 46
9.3 Action of $H(\mathbb{T})$ on Hom_H 47
9.3.1 Characterisation of $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ 47
9.3.2 First step: Conjugation in $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ 48
9.3.3 Second step: from $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ to $H(\mathbb{T})$ 48
9.3.4 Proof of Theorem 9.10 49

10 A characterisation of $C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$. 50
10.1 C^∞-exact symplectomorphisms 51
10.2 π-Symplectic Homeomorphisms and π-curves 52
10.3 Width 54

11 A Conjugation Theorem 56
11.1 Contracting the leaves (bis) 57
11.2 A Conjugation Lemma 58
11.3 Proof of Theorem 11.3 60
11.3.1 A preliminary lemma 60
11.3.2 Proof of Theorem 11.3 61

12 Negatively curved metrics 63
12.1 The boundary at infinity and the geodesic flow 63
12.2 Proof of Theorem 12.1 64

13 Hitchin component 65
13.1 Hyperconvex curves 65
13.2 Proof of Theorem 13.1 66
13.2.1 Proof 66
13.2.2 Compact quotient 67
13.2.3 Contracting the leaves 69

2 Curves and hyperconvex representations

We recall results and definitions from [23].

2.1 Hyperconvex and Hitchin representations

Definition 2.1 [Fuchsian and Hitchin homomorphisms] *An* n-Fuchsian *homomorphism from* $\pi_1(\Sigma)$ *to* $\mathrm{PSL}(n,\mathbb{R})$ *is a homomorphism* ρ *which may be written as* $\rho = \iota \circ \rho_0$*, where* ρ_0 *is a discrete faithful homomorphism with values in* $\mathrm{PSL}(2,\mathbb{R})$ *and* ι *is the irreducible homomorphism from* $\mathrm{PSL}(2,\mathbb{R})$ *to* $\mathrm{PSL}(n,\mathbb{R})$*. A homomorphism is* Hitchin *if it may be deformed into an* n*-Fuchsian homomorphism.*

Definition 2.2 [Hyperconvex map] *A continuous map* ξ *from a set* S *to* $\mathbb{P}(\mathbb{R}^n)$ *is* hyperconvex *if, for any pairwise distinct points* $(x_1, \ldots, x_p)$ *with* $p \leq n$*, the following sum is direct*

$$\xi(x_1) + \ldots + \xi(x_p).$$

In the applications, S will be a subset of $\mathbb{T}$.

Definition 2.3 [Hyperconvex representation and limit curve] *A homomorphism* ρ *from* $\pi_1(\Sigma)$ *to* $\mathrm{PSL}(n,\mathbb{R})$ *is* n-hyperconvex*, if there exists a* ρ*-equivariant hyperconvex map from* $\partial_\infty \pi_1(\Sigma)$ *in* $\mathbb{P}(\mathbb{R}^n)$*. Such a map is actually unique and is called the* limit curve *of the homomorphism.*

A representation is *n-Fuchsian* (respectively *Hitchin*, *hyperconvex*) if, as a class, it contains an n-Fuchsian (respectively Hitchin, hyperconvex) homomorphism.

As an example, we observe that the Veronese embedding is a hyperconvex curve equivariant under all Fuchsian homomorphisms. Therefore, a Fuchsian representation is hyperconvex. More generally, we prove in [23]

Theorem. *Every Hitchin homomorphism is discrete, faithful and hyperconvex. If* ρ *is a Hitchin homomorphism and if* γ *in* $\pi_1(\Sigma)$ *different from the identity, then* $\rho(\gamma)$ *is real split with distinct eigenvalues.*

We explain later a refinement of this result (Theorem 2.6). Conversely, completing our work, O. Guichard [18] has shown the following result

Theorem 2.4 [Guichard] *Every hyperconvex representation is Hitchin.*

2.2 Hyperconvex representations and Frenet curves

Definition 2.5 [Frenet curve and osculating flag] *A hyperconvex curve* ξ *defined from* $\mathbb{T}$ *to* $\mathbb{RP}^{n-1}$ *is a* Frenet curve *if there exists a family of maps* $(\xi^1, \xi^2, \ldots, \xi^{n-1})$ *defined on* $\mathbb{T}$*, called the* osculating flag curve*, such that*

- *for each* p*, the curve* ξ^p *takes values in the Grassmannian of* p*-planes of* $\mathbb{R}^n$*,*
- *for every* x *in* $\mathbb{T}$*,* $\xi^p(x) \subset \xi^{p+1}(x)$*,*

- *for every x in $\mathbb{T}$, $\xi(x) = \xi^1(x)$,*
- *for every pairwise distinct points $(x_1, \ldots, x_l)$ in $\mathbb{T}$ and positive integers $(n_1, \ldots, n_l)$ such that $\sum_{i=1}^{i=l} n_i \leq n$, then the following sum is direct*
$$\xi^{n_i}(x_i) + \ldots + \xi^{n_l}(x_l), \tag{3}$$
- *finally, for every x in $\mathbb{T}$ and positive integers $(n_1, \ldots, n_l)$ such that $p = \sum_{i=1}^{i=l} n_i \leq n$, then*
$$\lim_{(y_1,\ldots,y_l)\to x, y_i \textit{ all distinct}} \left(\bigoplus_{i=1}^{i=l} \xi^{n_i}(y_i)\right) = \xi^p(x). \tag{4}$$

We call ξ^{n-1} the osculating hyperplane*.*

Remarks:

1. By Condition (4), the osculating flag of a Frenet hyperconvex curve is continuous and the curve ξ^1 completely determines ξ^p.
2. Furthermore, if ξ^1 is C^∞, then $\xi^p(x)$ is completely generated by the derivatives of ξ^1 at x up to order $p-1$.
3. In general a Frenet hyperconvex curve is not C^∞.
4. However by Condition (4), its image is a C^1-submanifold and the tangent line to $\xi^1(x)$ is $\xi^2(x)$.

We list several properties of hyperconvex representations proved in [23].

Theorem 2.6 *Let ρ be an hyperconvex homomorphism from $\pi_1(\Sigma)$ to $\mathrm{PSL}(E)$ with limit curve ξ. Then:*

1. *The limit curve ξ is a hyperconvex Frenet curve.*
2. *The osculating hyperplane curve ξ^* is hyperconvex.*
3. *The osculating flag curve of ξ is Hölder.*
4. *Finally, if γ^+ is the attracting fixed point of γ in $\partial_\infty\pi_1(\Sigma)$, then $\xi(\gamma^+)$, (respectively $\xi^*(\gamma^+)$) is the unique attracting fixed point of $\rho(\gamma)$ in $\mathbb{P}(E)$ (respectively $\mathbb{P}(E^*)$).*

2.2.1 A smooth map

As a consequence of Theorem 2.6, we obtain the following

Proposition 2.7 *Let ρ be a hyperconvex representation. Let*

$$\xi = (\xi^1, \ldots, \xi^{n-1}),$$

be the limit curve of ρ. Then, there exist

- *A C^1 embedding with Hölder derivatives $\eta = (\eta_1, \eta_2)$ from $\mathbb{T}^2$ to $\mathbb{P}(\mathbb{R}^n) \times \mathbb{P}(\mathbb{R}^{*n})$,*
- *two representations ρ_1 and ρ_2 from $\pi_1(\Sigma)$ in $\mathrm{Diff}^h(\mathbb{T})$, the group of C^1-diffeomorphisms of $\mathbb{T}$ with Hölder derivatives,*
- *a Hölder homeomorphism κ of $\mathbb{T}$,*

such that

1. *$\eta_1(\mathbb{T}) = \xi^1(\partial_\infty \pi_1(\Sigma))$ and $\eta_2(\mathbb{T}) = \xi^{n-1}(\partial_\infty \pi_1(\Sigma))$,*
2. *each map η_i is ρ_i equivariant,*
3. *the sum $\eta_1(s) + \eta_2(t)$ is direct when $\kappa(s) \neq t$,*
4. *κ intertwines ρ_1 and ρ_2.*

PROOF: By the Frenet property, $\xi^1(\partial_\infty \pi_1(\Sigma))$ and $\xi^{n-1}(\partial_\infty \pi_1(\Sigma))$ are a C^1 one-dimensional manifold. Let η_1 (respectively η_2) be the arc-length parametrisation of $\xi^1(\partial_\infty \pi_1(\Sigma))$ (respectively $\xi^{n-1}(\partial_\infty \pi_1(\Sigma))$). Let

$$\kappa = (\eta_2)^{-1} \circ \xi^1 \circ (\xi^{n-1})^{-1} \circ \eta_1.$$

The result follows. Q.E.D.

3 Cross ratio, definitions and first properties

3.1 Cross ratios and weak cross ratios

Let S be a metric space equipped with an action of a group Γ by Hölder homeomorphisms and

$$S^{4*} = \{(x, y, z, t) \in S^4 \mid x \neq t \text{ and } y \neq z\}.$$

Definition 3.1 [CROSS RATIO] *A* cross ratio *on S is a Hölder $\mathbb{R}$-valued function $\mathbb{B}$ on S^{4*}, invariant under the diagonal action of Γ and which satisfies the following rules*

$$\begin{aligned}
\mathbb{B}(x,y,z,t) &= \mathbb{B}(z,t,x,y), & (5)\\
\mathbb{B}(x,y,z,t) &= 0 \ \Leftrightarrow x = y \text{ or } z = t, & (6)\\
\mathbb{B}(x,y,z,t) &= 1 \ \Leftrightarrow x = z \text{ or } y = t, & (7)\\
\mathbb{B}(x,y,z,t) &= \mathbb{B}(x,y,z,w)\mathbb{B}(x,w,z,t), & (8)\\
\mathbb{B}(x,y,z,t) &= \mathbb{B}(x,y,w,t)\mathbb{B}(w,y,z,t). & (9)
\end{aligned}$$

Definition 3.2 [Weak cross ratio] *If $\mathbb{B}$ – non necessarily Hölder – satisfies all relations except (7), we say that $\mathbb{B}$ is a* weak cross ratio. *If a weak cross ratio satisfies Relation (7), we say that the weak cross ratio is* strict. *A weak cross ratio that is strict and Hölder is a genuine cross ratio.*

In this article, we shall only consider the case where $S = \mathbb{T}$, or $S = \partial_\infty\pi_1(\Sigma)$ equipped with the action of $\pi_1(\Sigma)$. In [24], we extend and specialise this definition to subsets of $\mathbb{T}$.

We give examples and constructions in Section 4.

Definition 3.3 [Period] *Let $\mathbb{B}$ be a weak cross ratio on $\partial_\infty\pi_1(\Sigma)$ and γ be a nontrivial element in $\pi_1(\Sigma)$. The* period $\ell_{\mathbb{B}}(\gamma)$ *is defined as follows. Let γ^+ (respectively γ^-) be the attracting (respectively repelling) fixed point of γ on $\partial_\infty\pi_1(\Sigma)$. Let y be an element of $\partial_\infty\pi_1(\Sigma)$. Let*

$$\ell_{\mathbb{B}}(\gamma, y) = \log|\mathbb{B}(\gamma^-, \gamma y, \gamma^+, y)|. \tag{10}$$

Relation (8) and the invariance under the action of γ imply that $\ell_{\mathbb{B}}(\gamma) = \ell_{\mathbb{B}}(\gamma, y)$ does not depend on y. Moreover, by Equation (5), $\ell_{\mathbb{B}}(\gamma) = \ell_{\mathbb{B}}(\gamma^{-1})$.

Remarks:

1. The definition given above **does not coincide** with the definition given by Otal in [28] and studied by Ledrappier, Bourdon and Hamenstädt in [25, 4, 19]. We in particular warn the reader that our cross ratios are not determined by their periods contrarily to Otal's [28]. Apart from the choice of multiplicative cocycle identities rather than additive and the requirement of the symmetry $\mathbb{B}(z, t, x, y) = \mathbb{B}(x, y, z, t)$ which both have mild consequences, we more crucially do not require that $\mathbb{B}(x, y, z, t) = \mathbb{B}(y, x, t, z)$. This allows more flexibility as we shall see in Definition 9.9 and breaks the period rigidity.

2. However, if $\mathbb{B}(x, y, z, t)$ is a cross ratio according to our definition, so is $\mathbb{B}^*(x, y, z, t) = \mathbb{B}(y, x, t, z)$, and finally $\log|\mathbb{B}\mathbb{B}^*|$ is a cross ratio according to Otal's definition. We explain the relation with negatively curved metrics discovered by Otal in Section 4.3.

3. Ledrappier's article [25] contains many view-points on the subject, in particular the link with Bonahon's geodesic currents [3] as well as an accurate bibliography.

4. *Triple ratios.* The remark made in this paragraph is not used in the article Let $\mathbb{B}$ be a cross ratio. For every quadruple of pairwise distinct points (x, y, z, t), the expression

$$\mathbb{B}(x, y, z, t)\mathbb{B}(z, x, y, t)\mathbb{B}(y, z, x, t),$$

is independent of the choice of t. We call such a function of (x, y, z) a *triple ratio*. Indeed, in some cases, it is related to the triple ratios introduced by A. Goncharov in [16]. A triple ratio satisfies the (multiplicative) cocycle identity and hence defines a bounded cohomology class in $H^2_b(\pi_1(\Sigma))$.

4 Examples of cross ratio

In this section, we recall the construction of the classical cross ratio on the projective line and explain that various structures such as curves in projective spaces and negatively curved metrics on surfaces give rise to cross ratios. In some cases, the periods are computed explicitly. We also explain a general symplectic construction.

4.1 Cross ratio on the projective line

Let E be a vector space with $\dim(E) = 2$. The *classical* cross ratio on $\mathbb{P}(E)$, identified with $\mathbb{R} \cup \{\infty\}$ using projective coordinates, is defined by

$$\mathbf{b}(x, y, z, t) = \frac{(x-y)(z-t)}{(x-t)(z-y)}.$$

The classical cross ratio is a cross ratio according to Definition (3.1). It also satisfies the following rule

$$1 - \mathbf{b}(x, y, z, t) = \mathbf{b}(t, y, z, x). \tag{11}$$

This extra relation completely characterises the classical cross ratio. Moreover, this (simple) relation is equivalent to a more sophisticated one which we may generalise to higher dimensions

Proposition 4.1 *For a cross ratio $\mathbb{B}$, Relation (11) is equivalent to*

$$(\mathbb{B}(f, v, e, u) - 1)(\mathbb{B}(g, w, e, u) - 1) = (\mathbb{B}(f, w, e, u) - 1)(\mathbb{B}(g, v, e, u) - 1). \tag{12}$$

Furthermore, if a cross ratio $\mathbb{B}$ on S satisfies Relation (12), then there exists an injective Hölder map φ of S in $\mathbb{RP}^1$, such that

$$\mathbb{B}(x, y, z, t) = \mathbf{b}(\varphi(x), \varphi(y), \varphi(z), \varphi(t)).$$

Proof: Suppose first that $\mathbb{B}$ satisfies Relation (11). By Relation (8), it also satisfies (12):

$$\begin{aligned}
&(\mathbb{B}(f, v, e, u) - 1)(\mathbb{B}(g, w, e, u) - 1) \\
&= \mathbb{B}(u, v, e, f)\mathbb{B}(u, w, e, g) \quad \text{by (11)} \\
&= \big(\mathbb{B}(u, v, e, w)\mathbb{B}(u, w, e, f)\big)\big(\mathbb{B}(u, v, e, g)\mathbb{B}(u, w, e, v)\big) \quad \text{by (8)} \\
&= \big(\mathbb{B}(u, v, e, w)\mathbb{B}(u, w, e, v)\big)\big(\mathbb{B}(u, v, e, g)\mathbb{B}(u, w, e, f)\big) \\
&= \mathbb{B}(u, w, e, f)\mathbb{B}(u, v, e, g) \quad \text{by (8)}
\end{aligned}$$

$$= (\mathbb{B}(f,w,e,u)-1)(\mathbb{B}(g,v,e,u)-1) \text{ by (11).}$$

Conversely, suppose that a cross ratio $\mathbb{B}$ satisfies Relation (12). We first prove that

$$\mathbb{B}(f,v,k,z) = \frac{(\mathbb{B}(v,w,e,u)-\mathbb{B}(f,w,e,u))(\mathbb{B}(z,w,e,u)-\mathbb{B}(k,w,e,u))}{(\mathbb{B}(v,w,e,u)-\mathbb{B}(k,w,e,u))(\mathbb{B}(z,w,e,u)-\mathbb{B}(f,w,e,u))}. \tag{13}$$

Indeed, we first have

$$\mathbb{B}(f,v,e,u) = \frac{(\mathbb{B}(f,w,e,u)-1)(\mathbb{B}(g,v,e,u)-1)}{\mathbb{B}(g,w,e,u)-1}+1.$$

Setting $g=v$, we obtain

$$\begin{aligned}\mathbb{B}(f,v,e,u) &= \frac{1-\mathbb{B}(f,w,e,u)}{\mathbb{B}(v,w,e,u)-1}+1\\ &= \frac{\mathbb{B}(v,w,e,u)-\mathbb{B}(f,w,e,u)}{\mathbb{B}(v,w,e,u)-1}.\end{aligned} \tag{14}$$

We have

$$\begin{aligned}\mathbb{B}(f,v,k,z) &= \frac{\mathbb{B}(f,v,e,z)}{\mathbb{B}(k,v,e,z)} \text{ by (9)}\\ &= \left(\frac{\mathbb{B}(f,v,e,u)}{\mathbb{B}(f,z,e,u)}\right)\left(\frac{\mathbb{B}(k,z,e,u)}{\mathbb{B}(k,v,e,u)}\right) \text{ by (8)}\\ &= \frac{\mathbb{B}(f,v,e,u)\mathbb{B}(k,z,e,u)}{\mathbb{B}(k,v,e,u)\mathbb{B}(f,z,e,u)}.\end{aligned} \tag{15}$$

Applying Relation (14) to the four right terms of Equation (15), we obtain Equation (13).

Let (w,e,u) be a triple of pairwise distinct points in S, and let φ be the map from S to $\mathbb{RP}^1$ – identified with $\mathbb{R}\cup\{\infty\}$ – defined by

$$\varphi = \varphi_{(w,e,u)}\left\{\begin{array}{ccc} S & \to & \mathbb{R}\cup\{\infty\},\\ x & \mapsto & \mathbb{B}(x,w,e,u),\end{array}\right.$$

where by convention $\mathbb{B}(w,w,e,u)=\infty$.

We now observe that φ has the same regularity as $\mathbb{B}$. This is obvious in $S\setminus\{w\}$, and for x in the neighbourhood of w we use that

$$\mathbb{B}(x,u,e,w) = \frac{1}{\mathbb{B}(x,w,e,u)}.$$

The normalisation Relation (7) implies that φ is injective. Indeed

$$\mathbb{B}(x,w,e,u)=\mathbb{B}(y,w,e,u) \implies \mathbb{B}(x,w,y,u)=1 \implies x=y.$$

By construction and Equation (13), we have

$$\mathbb{B}(x,y,z,t) = \mathbf{b}(\varphi(x),\varphi(y),\varphi(z),\varphi(t)).$$

This in turn implies that $\mathbb{B}$ satisfies Relation (11) and completes the proof of the proposition Q.E.D.

4.2 Weak cross ratios and curves in projective spaces

We now extend the previous discussion to higher dimension. Let E be an n-dimensional vector space. Let ξ and ξ^* be two curves from a set S to $\mathbb{P}(E)$ and $\mathbb{P}(E^*)$ respectively, such that

$$\xi(y) \in \ker \xi^*(z) \Leftrightarrow z = y. \tag{16}$$

We observe that a hyperconvex Frenet curve ξ and its osculating hyperplane ξ^* satisfy Condition 16. In the applications, S will be a subset of $\mathbb{T}$.

Definition 4.2 [Weak cross ratio associated to curves] *For every x in S, we choose a nonzero vector $\hat{\xi}(x)$ (respectively $\hat{\xi}^*(x)$) in the line $\xi(x)$ (respectively $\xi^*(x)$). The* weak cross ratio associated to (ξ, ξ^*) *is the function $\mathbb{B}_{\xi,\xi^*}$ on S^{4*} defined by*

$$\mathbb{B}_{\xi,\xi^*}(x,y,z,t) = \frac{\langle \hat{\xi}(x), \hat{\xi}^*(y)\rangle \langle\langle \hat{\xi}(z), \hat{\xi}^*(t)\rangle}{\langle \hat{\xi}(z), \hat{\xi}^*(y)\rangle \langle \hat{\xi}(x), \hat{\xi}^*(t)\rangle}.$$

Remarks:

- The definition of $\mathbb{B}_{\xi,\xi^*}$ does not depend on the choice of $\hat{\xi}$ and $\hat{\xi}^*$.
- $\mathbb{B}_{\xi,\xi^*}$ is a weak cross ratio.
- Let x, z be distinct points S and $V = \xi(x) \oplus \xi(z)$. For any m in S, let $\zeta(m) = \ker \xi^*(m) \cap V$. Let $\mathbf{b}_V$ be the classical cross ratio on $\mathbb{P}(V)$, then

$$\mathbb{B}_{\xi,\xi^*}(x,y,z,t) = \mathbf{b}_V(\xi(x), \zeta(y), \xi(z), \zeta(t)).$$

- As a consequence, $\mathbb{B}_{\xi,\xi^*}$ is strict if for all quadruple of pairwise distinct points (x, y, z, t),

$$\ker(\xi^*(z)) \cap \big(\xi(x) \oplus \xi(y)\big) \neq \ker(\xi^*(t)) \cap \big(\xi(x) \oplus \xi(y)\big).$$

Finally, we have

Lemma 4.3 *Let (ξ, ξ^*) and (η, η^*) be two pairs of maps from S to $\mathbb{P}(\mathbb{R}^n) \times \mathbb{P}(\mathbb{R}^{*n})$, satisfying Condition (16), such that ξ^* and η^* are hyperconvex. Assume that*

$$\mathbb{B}_{\eta,\eta^*} = \mathbb{B}_{\xi,\xi^*}.$$

Then there exists a linear map A such that $\xi = A \circ \eta$.

Proof: We assume the hypotheses of the lemma. Let $(x_0, x_1, \ldots, x_n)$ be a tuple of $n+1$ pairwise distinct points of S. Let u_0 be a nonzero vector in $\xi^*(x_0)$ and let $\mathcal{U} = (\hat{u}_1, \ldots, \hat{u}_n)$ be the basis of E^* such that $\hat{u}_i \in \xi^*(x_i)$ and $\langle u_0, \hat{u}_i\rangle = 1$. The hyperconvexity of ξ^* guaranty the existence of $\mathcal{U}$.

The projective coordinates of $\xi(y)$ in the dual basis of $\mathcal{U}$ are

$$\begin{aligned}
[\ldots : \langle \xi(y), \hat{u}_i \rangle : \ldots] &= [\ldots : \frac{\langle \xi(y), \hat{u}_i \rangle}{\langle \xi(y), \hat{u}_1 \rangle} : \ldots] \\
&= [\ldots : \frac{\langle \xi(y), \hat{u}_i \rangle}{\langle \xi(y), \hat{u}_1 \rangle} \frac{\langle u_0, \hat{u}_1 \rangle}{\langle u_0, \hat{u}_i \rangle} : \ldots] \\
&= [\ldots : \mathbb{B}_{\xi,\xi^*}(y, x_i, x_0, x_1) : \ldots]. \qquad (17)
\end{aligned}$$

Symmetrically, let v_0 be a non zero vector in $\eta^*(x_0)$ and let $\mathcal{V} = (\hat{v}_1, \ldots, \hat{v}_n)$ be the basis of E^* such that $\hat{v}_i \in \eta^*(x_i)$ and $\langle v_0, \hat{v}_i \rangle = 1$.

Let A be the linear map that sends the dual basis of $\mathcal{V}$ to the dual basis of $\mathcal{U}$. By Equation 17, for all $y \in S$, we have that $\xi(y) = A(\eta(y))$. The result now follows Q.E.D.

4.3 Negatively curved metrics and comparisons

We now explain Otal's construction of cross ratios associated to negatively curved metrics on surfaces [27, 28]. Let g be a negatively curved metric on Σ which we lift to the universal cover $\tilde{\Sigma}$ of Σ. Let (a_1, a_2, a_3, a_4) be a quadruple of pairwise distinct points of $\partial_\infty \pi_1(\Sigma) = \partial_\infty \tilde{\Sigma}$. Let c_{ij} be the unique geodesic from a_i to a_j. We choose nonintersecting horoballs H_i centred at each point a_i and set

$$O_{ij} = \tilde{\Sigma} \setminus (H_i \cup H_j).$$

Let ℓ_{ij} be the length of the geodesic arc $c_{ij} \cap O_{ij}$. *Otal's cross ratio* of the quadruple (a_1, a_2, a_3, a_4) is

$$O_g(a_1, a_2, a_3, a_4) = \ell_{12} - \ell_{23} + \ell_{34} - \ell_{41}.$$

This number is independent on the choice of the horoballs H_i and is a cross ratio according to Otal's definition.

Definition 4.4 [Cross ratio associated to negatively curved metric] *The* cross ratio associated to g *according to definition is defined as follows. For a cyclically ordered 4-tuple of distinct, we take the exponential of Otal's cross ratio*

$$\mathbb{B}_g(a_1, a_2, a_3, a_4) = e^{O_g(a_1, a_2, a_3, a_4)}.$$

For noncyclically ordered 4-tuple, we introduce a sign compatible with the sign of the classical cross ratio.

Remarks:

1. The function $\mathbb{B}_g$ satisfies the rules 3.1 and the period of an element γ is the length of the closed geodesic of Σ in the free homotopy class of γ.

2. The cross ratio of a negatively curved metric satisfies the extra symmetry

$$\mathbb{B}(x,y,z,t) = \mathbb{B}(y,x,t,z).$$

3. If the metric g is hyperbolic, $\partial_\infty \pi_1(\Sigma)$ is identified with $\mathbb{RP}^1$ and the cross ratio $\mathbb{B}_g$ coincides with the classical cross ratio.

4. This construction is yet another instance of the "symplectic construction" explained in Section 4.4. Indeed, the metric identifies the tangent space of the universal cover of a negatively curved manifold with the cotangent space. Therefore, this tangent space inherits a symplectic structure and its symplectic reduction with respect to the multiplicative $\mathbb{R}$-action is the space of geodesics which thus admits a symplectic structure. Identified with the space of pairs of distinct points of the boundary at infinity, the space of geodesics also admits a product structure. The cross ratio defined according to Section 4.4 coincides with the cross ratio described above.

5. More generally, Otal's construction can be extended to Anosov flows on unit tangent bundle of surfaces of whom geodesic flows of a negatively curved metrics are special cases [1]. For a cross associated to an Anosov flow, the period of an element γ is the length of the closed orbit in the free homotopy class of γ.

6. Conversely a cross ratio gives rise to a flow as it is explained in Section 3.4.1 of [22]. The flow identity is equivalent to the Cocycle Identity (8) in the definition of cross ratios. More precisely, given a cross ratio $\mathbb{B}$ on $\partial_\infty \pi_1(\Sigma)^{4*}$ and any real number t, we define in [22] a map φ_t from $\partial_\infty \pi_1(\Sigma)^{3+}$ – see Definition (12.3) – to itself by

$$\varphi_t(x_-, x_0, x_+) = (x_-, x_t, x_+),$$

where

$$\mathbb{B}(x_+, x_0, x_-, x_t) = e^t.$$

Equation (8) implies that $t \mapsto \varphi_t$ is a one parameter group, that is

$$\varphi_{t+s} = \varphi_t \circ \varphi_s.$$

In [25], François Ledrappier gives an excellent overview of the various aspects of Otal's cross ratios and in particular their enlightening interpretation as Bonahon's geodesic currents [3].

4.4 A symplectic construction

All the examples of cross ratio that we have defined may be interpreted from the following "symplectic" construction. Let V and W be two manifolds of the same dimension and O be an open set of $V \times W$ equipped with an *exact* symplectic structure, or more generally an exact two-form ω. We assume furthermore that the two foliations coming from the product structure

$$\begin{aligned}\mathcal{F}^+_w &= O \cap (V \times \{w\}),\\ \mathcal{F}^-_v &= O \cap (\{v\} \times W),\end{aligned}$$

satisfy the following properties:

1. Leaves are connected.
2. The first cohomology groups of the leaves are reduced to zero.
3. ω restricted to the leaves is zero.
4. Squares are homotopic to zero, where *squares* are closed curves $c = c_1^+ \cup c_1^- \cup c_2^+ \cup c_2^-$ with $c_i^\pm$ along $\mathcal{F}^\pm$.

Remark: When ω is symplectic, every leaf $\mathcal{F}^\pm$ is Lagrangian by definition, and a standard observation shows that they carries a flat affine structure. If every leaf is simply connected and complete from the affine view-point, condition (4) above is satisfied: one may "straighten" the edges of the square, that is deforming them into geodesics of the affine structure, then use these straightening to define a homotopy.

4.4.1 Polarised cross ratio

Let

$$U = \{(e, u, f, v) \in V \times W \times V \times W | (e, u), (f, u), (e, v), (f, v) \in O\},$$

Definition 4.5 [Polarised cross ratio] *The* polarised cross ratio *is the function defined* $\mathbb{B}$ *defined on* U *by*

$$B(e, u, f, v) = e^{\frac{1}{2}\int G^*\omega}.$$

where G *is a map from the square* $[0, 1]^2$ *to* O *such that*

- *the image of the vertexes* $(0, 0)$, $(0, 1)$, $(1, 1)$, $(1, 0)$ *are respectively* (e, u), (f, u), (e, v), (f, v),
- *the image of every edge on the boundary of the square lies in a leaf of* $\mathcal{F}^+$ *or* $\mathcal{F}^-$.

The definition of B does not depend on the choice of the specific map G. Let ψ be an diffeomorphism of O preserving ω and isotopic to the identity. Let α and ζ be two fixed points of ψ and c a curve joining α and ζ. Since ψ is isotopic to the identity, it follows that $c \cup \psi(c)$ bounds a disc D.

4.4.2 Curves

Let $\phi = (\rho, \bar{\rho})$ be a representation from $\pi_1(\Sigma)$ in the group of diffeomorphisms of O which preserve ω and are restrictions of elements of $\mathrm{Diff}(V) \times \mathrm{Diff}(W)$. Let (ξ, ξ^*) be a ϕ-equivariant map of $\partial_\infty \pi_1(\Sigma)$ to O.

Then, we have the following immediate

Proposition 4.6 *The function $\hat{\mathbb{B}}_{\xi,\xi^*}$ defined by*

$$\mathbb{B}_{\xi,\xi^*}(x, y, z, t) = B(\xi(x)), \xi^*(y), \xi(z), \xi^*(t)),$$

satisfies

$$\begin{aligned} \mathbb{B}_{\xi,\xi^*}(x, y, z, t) &= \mathbb{B}(z, t, x, y), \\ \mathbb{B}_{\xi,\xi^*}(x, y, z, t) &= \mathbb{B}(x, y, z, w)\mathbb{B}(x, w, z, t), \\ \mathbb{B}_{\xi,\xi^*}(x, y, z, t) &= \mathbb{B}(x, y, w, t)\mathbb{B}(w, y, z, t). \end{aligned}$$

This function $\mathbb{B}_{\xi,\xi^*}$ may be undefined for $x = y$ and $z = t$. By the above proposition, it extends to a weak cross ratio provided that

$$\lim_{y \to x} \mathbb{B}_{\xi,\xi^*}(x, y, z, t) = 0.$$

4.4.3 Projective spaces

We concentrate on the case of projective spaces, although the construction can be extended to flag manifolds to produce a whole family of cross ratios associated to a hyperconvex curve. However in this case, we do not know how to characterise these cross ratios using functional relations, as we did in the case of curves in the projective space.

As a specific example of the previous situation, we relate the polarised cross ratio to the weak cross ratio associated to curves. Let E be a vector space. Let

$$\mathbb{P}^{2*} = \mathbb{P}(E) \times \mathbb{P}(E^*) \setminus \{(D, P) \mid D \subset P^\perp\}.$$

Using the identification of $T_{(D,P)}\mathbb{P}^{2*}$ with $\mathrm{Hom}(D, P^\perp) \oplus \mathrm{Hom}(P^\perp, D)$, let

$$\Omega((f, g), (h, j)) = \mathrm{tr}(f \circ j) - \mathrm{tr}(h \circ g).$$

Let L be bundle over $\mathbb{P}(n)^{2*}$, whose fibre at (D, P) is

$$L_{(D,P)} = \{u \in D, f \in P \mid \langle f, u\rangle = 1\}/\{+1, -1\}.$$

Note that L is a principal $\mathbb{R}$-bundle whose $\mathbb{R}$-action is given by $\lambda(u, f) = (e^{-\lambda}u, e^{\lambda}f)$, equipped with an action of $\mathrm{PSL}(n, \mathbb{R})$.

The following proposition summarise properties of this construction

Proposition 4.7 *The form Ω is symplectic and the polarised cross ratio associated to 2Ω is*

$$\mathbb{B}(u, f, v, g) = \frac{\langle \hat{f}, \hat{v}\rangle\langle \hat{g}, \hat{u}\rangle}{\langle \hat{f}, \hat{u}\rangle\langle \hat{g}, \hat{u}\rangle}, \tag{18}$$

where, in general, $\hat{h}$ is a nonzero vector in h.

Moreover, there exists a connection form β on L whose curvature is symplectic and equal to Ω, and such that, for any u non zero vector in a line D, the section

$$\xi_u : P \mapsto (u, f) \text{ such that } \langle u, f\rangle = 1, \text{ and } f \in P,$$

is parallel for β above $\{D\} \times (\mathbb{P}(E^*) \setminus \{P \mid D \subset P^\perp\})$.

Finally, let A be an element of $\mathrm{PSL}(n, \mathbb{R})$. Let D (resp $\bar{D}$) be an A-eigenspace of dimension one for the eigenvalue λ (resp. μ). Then $(D, \bar{D}^\perp)$ is a fixed point of A in $\mathbb{P}^{2*}$ and he action of A on $L_{(D,\bar{D}^\perp)}$ is the translation by

$$\log|\lambda/\mu|.$$

PROOF: We consider the standard symplectic form Ω^0 on $E \times E^*/\{+1, -1\}$. We observe that $\Omega^0 = d\beta^0$ where $\beta^0_{(u,f)}(v, g) = \langle u, g\rangle$. A symplectic action of $\mathbb{R}$ is given by

$$\lambda.(u, f) = (e^{-\lambda}u, e^{\lambda}f),$$

with moment map

$$\mu\big((u, f)\big) = \langle f, u\rangle.$$

Then $L = \mu^{-1}(1)$. Therefore, we obtain that $\beta = \beta_0|_L$ is a connection form for the $\mathbb{R}$-action, whose curvature Ω is the symplectic form obtained by reduction of the Hamiltonian action of $\mathbb{R}$.

We now compute Ω explicitly. Let (D, P) be an element of $\mathbb{P}^{2*}$. Let π be the projection onto P parallel to D. We identify $T_{(D,P)}\mathbb{P}^{2*}$ with $\mathrm{Hom}(D, P^\perp) \oplus \mathrm{Hom}(P^\perp, D)$. Let $(f, \hat{g})$ be an element of $T_{(D,P)}\mathbb{P}^{2*}$, Let $(u, \alpha) \in D \times P$be an element of L. Now, $(f(u), \hat{g}(\alpha))$ is an element of $T_{(u,\alpha)}L$ which projects to $(f, \hat{g})$. By definition of the symplectic reduction, if $(f, \hat{g})$ and $(h, \hat{l})$ are elements of $T_{(D,P)}\mathbb{P}^{2*}$, then

$$\Omega((f, \hat{g}), (h, \hat{l})) = \langle \hat{l}(\alpha), f(u)\rangle - \langle \hat{g}(\alpha), h(u)\rangle.$$

Finally, let π be the projection onto $P^\perp$ in the D direction. We consider the map

$$\begin{cases} \mathrm{Hom}(P^\perp, D) & \rightarrow & \mathrm{Hom}(P^\perp, D) \\ f & \mapsto & \hat{f} = (f \circ \pi)^* \end{cases}.$$

In particular

$$\langle \hat{l}(\alpha), f(u)\rangle = \mathrm{tr}(l \circ f)\langle \alpha, u\rangle.$$

The description of the parallel sections follows from the explicit formula for β^0. This description allows to compute the holonomy of this connection along squares. This prove Formula (18).

The last point is obvious. Q.E.D.

4.4.4 Appendix: Periods, action difference and triple ratios

The results of this paragraph are not used in the sequel. We come back to the general setting of the beginning of the Section and make further definitions and remarks

Definition 4.8 [Action difference] *The* action difference *is*

$$\Delta_\psi(\alpha,\zeta,c) = \exp\left(\int_D \omega\right).$$

We have

Proposition 4.9 *The quantity* $\Delta := \Delta_\psi(\alpha,\zeta,c)$ *just depends on the homotopy class of* c*.*

Proof: This follows from the invariance of ω under ψ. The reader should observe the analogy with the action difference for Hamiltonian diffeomorphisms. Q.e.d.

In our context, we have a preferred class of curves joining two point. Let $\alpha = (a,b)$ and $\zeta = (\bar{a},\bar{b})$ be two points of O. Since squares are homotopic to zero, we define the homotopy class $c_{a,b,\bar{a},\bar{b}}$ of curves from (a,b) to $(\bar{a},\bar{b})$ to be curves homotopic to $c^+ \cup c^- \cup \bar{c}^+$, where c^+ is a curve along $\mathcal{F}^+$ going from (a,b) to (y,b), c^- a curve along $\mathcal{F}^-$ going from (y,b) to $(y,\bar{b})$, and c^- a curve along $\mathcal{F}^+$ going from $(y,\bar{b})$ to $(\bar{a},\bar{b})$. By convention we set

$$\Delta_\gamma(\alpha,\zeta) = \Delta_\gamma(\alpha,\zeta,c_{a,b,\bar{a},\bar{b}}).$$

Periods and action difference Using the notations of the previous paragraph, we have

Proposition 4.10 *Let* γ *be an element of* $\pi_1(\Sigma)$ *then,*

$$\mathbb{B}_{\xi,\xi^*}(\gamma^+,y,\gamma^-,\gamma y)^2 = \Delta_{\phi(\gamma)}\big((\xi(\gamma^+),\xi^*(\gamma^-)),(\xi(\gamma^-),\xi^*(\gamma^+))\big).$$

In particular,

$$\ell_{\mathbb{B}_{\xi,\xi^*}}(\gamma) = \frac{1}{2}\log\left|\Delta_{\rho(\gamma)}\big((\xi(\gamma^+),\xi^*(\gamma^-)),(\xi(\gamma^-),\xi^*(\gamma^+))\big)\right|.$$

Proof: Let $f = (g,\bar{g})$ be a diffeomorphism of O preserving ω, restriction of an element of $\mathrm{Diff}(V)\times\mathrm{Diff}(W)$. Let (a,b) and $(\bar{a},\bar{b})$ be two fixed points of f. Let as before $c = c^+ \cup c^- \cup \bar{c}^+$ composition of

- c^+ a curve along $\mathcal{F}^+$ from (a,b) to (y,b),
- c^- a curve along $\mathcal{F}^-$ from (y,b) to $(y,\bar{b})$,

- and $\overline{c}^+$ a curve along $\mathcal{F}^+$ from $(y,\bar{b})$ to $(\bar{a},\bar{b})$.

Assume (a,b) and $(\bar{a},\bar{b})$ be fixed points of f. Then $c \cup \gamma(c)$ is a square. Let D be a disk whose boundary is this square. By definition

$$\Delta_f\big((a,b),(\bar{a},\bar{b})\big) = \int_D \omega.$$

The proposition follows from the definition of the weak cross ratio associated to (ξ,ξ^*), when we take $f = (\rho(\gamma),\rho^*(\gamma))$ and

$$(a,b,\bar{a},\bar{b}) = (\xi(\gamma^+),\xi^*(\gamma^-),\xi(\gamma^+),\xi^*(\gamma^-)).$$

Q.e.d.

Symplectic interpretation of triple ratio We explain quickly a similar construction for triple ratio. We consider a sextuplet (e,u,f,v,g,w) in $V\times W\times V\times W\times V\times W$. Let now ϕ be a map from the interior of the regular hexagon H in $V\times W$ such that the image of the edges lies in $\mathcal{F}^+$ or $\mathcal{F}^-$, and the (ordered) image of the vertexes are (e,u), (f,u), (f,v), (g,v), (g,w), (e,w) (Figure 1). Then, the following quantity does not depend on the choice of ϕ:

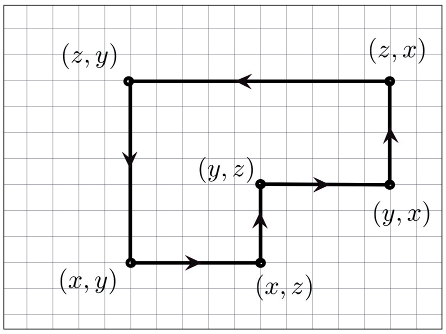

Figure 1: Triple ratio

$$T(e,u,f,v,g,w) = e^{\frac{1}{2}\int_H \phi^*\omega}.$$

Finally using the same notations as above, we verify that

$$t(x,y,z) = T(\xi(x),\xi^*(z),\xi(y),\xi^*(x),\xi(z),\xi^*(y)),$$

is the triple ratio as defined in Paragraph 4.

5 Hitchin representations and cross ratios

We prove in this section Theorem 1.1. This result is as a consequence of Theorem 5.3 that generalises Proposition 4.1 in higher dimensions.

5.1 Rank-n weak cross ratios

We extend a definition given in the introduction. Let S_*^p be the set of pairs

$$(e,u) = ((e_0, e_1, \ldots, e_p), (u_0, u_1, \ldots, u_p)),$$

of $p+1$-tuples of points of a set S such that

$$e_j \neq e_i \neq u_0, \quad u_j \neq u_i \neq e_0,$$

whenever $j > i > 0$. Let $\mathbb{B}$ be a weak cross ratio on S and let $\chi_{\mathbb{B}}^p$ be the map from S_*^p to $\mathbb{R}$ defined by

$$\chi_{\mathbb{B}}^p(e,u) = \det_{i,j>0}((\mathbb{B}(e_i, u_j, e_0, u_0)).$$

Definition 5.1 [Rank-n weak cross ratio] *A weak cross ratio $\mathbb{B}$ has* rank-n *if*

- $\forall (e,u) \in S_*^n, \quad \chi_{\mathbb{B}}^n(e,u) \neq 0,$
- $\forall (e,u) \in S_*^{n+1}, \quad \chi_{\mathbb{B}}^{n+1}(e,u) = 0.$

When the context makes it obvious, we omit the subscript $\mathbb{B}$.

Remarks:

- We prove in Paragraph 5.1.1 that for all weak cross ratios the nullity of $\chi^p(e,u)$ for a given e and u does not depend on e_0 and u_0.
- *The function $\chi_{\mathbb{B}}^2$ never vanishes for a strict weak cross ratio $\mathbb{B}$.* Indeed, we have

$$\begin{aligned}\chi^2((e,e,f),(u,u,v)) &= \begin{vmatrix} 1 & 1 \\ 1 & \mathbb{B}(f,v,e,u) \end{vmatrix} \\ &= \mathbb{B}(f,v,e,u) - 1.\end{aligned}$$

  The remark now follows from the previous one.

- *A cross ratio has rank-2 if if and only if it satisfies Equation (12), or the equivalent Equation (11).* Indeed

$$\begin{aligned}\chi^3((e,e,f,g),(u,u,v,w)) &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \mathbb{B}(f,v,e,u) & \mathbb{B}(g,v,e,u) \\ 1 & \mathbb{B}(f,w,e,u) & \mathbb{B}(g,w,e,u) \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 & 0 \\ 1 & \mathbb{B}(f,v,e,u)-1 & \mathbb{B}(g,v,e,u)-1 \\ 1 & \mathbb{B}(f,w,e,u)-1 & \mathbb{B}(g,w,e,u)-1 \end{vmatrix} \\ &= (\mathbb{B}(f,v,e,u)-1)(\mathbb{B}(g,w,e,u)-1) \\ &\quad -(\mathbb{B}(f,w,e,u)-1)(\mathbb{B}(g,v,e,u)-1).\end{aligned}$$

  Therefore by the previous remarks, a cross ratio has rank 2 if and only if it satisfies Equation (12).

5.1.1 Nullity of χ.

We first prove the following result of independent interest.

Proposition 5.2 *For a weak cross ratio, the nullity of*

$$\chi^n((e_0, e_1, \ldots, e_n), (u_0, u_1, \ldots, u_n)),$$

is independent of the choice of e_0 and u_0.

Proof: Let f_0 and v_0 be arbitrary points of $\mathbb{T}$ such that

$$f_0 \neq v_0 \neq e_i, u_j \neq e_0 \neq f_0.$$

By the Cocycle Identities (8) and (9), we have

$$\begin{aligned} \mathbb{B}(e_i, u_j, e_0, u_0)\mathbb{B}(e_i, u_0, e_0, v_0) &= \mathbb{B}(e_i, u_j, e_0, v_0) \\ &= \mathbb{B}(e_i, u_j, f_0, v_0)\mathbb{B}(f_0, u_j, e_0, v_0). \end{aligned}$$

Therefore

$$\begin{aligned} & \chi^n((e_0, e_1, \ldots, e_n), (u_0, u_1, \ldots, u_n)) \\ = & \Big(\prod_{i,j} \frac{\mathbb{B}(f_0, u_j, e_0, v_0)}{\mathbb{B}(e_i, v_0, e_0, u_0)}\Big)\chi^n((f_0, e_1, \ldots, e_n), (v_0, u_1, \ldots, u_n)). \end{aligned}$$

The proposition immediately follows. Q.E.D.

5.2 Hyperconvex curves and rank-n cross ratios

The main result in this section is

Theorem 5.3 *Let ξ and ξ^* be two hyperconvex curves from $\mathbb{T}$ to $\mathbb{P}(E)$ and $\mathbb{P}(E^*)$. Assume that $\xi(y) \in \ker \xi^*(x)$ if and only if $x = y$. Then the associated weak cross ratio B_{ξ,ξ^*} has rank n.*

Moreover, if ξ and ξ^* are Frenet and if ξ^* is the osculating hyperplane of ξ, then $\mathbb{B}_{\xi,\xi^*}$ strict.

Conversely, let $\mathbb{B}$ be a rank-n cross ratio on $\mathbb{T}$. Then, there exist two hyperconvex curves ξ and ξ^* with values in $\mathbb{P}(E)$ and $\mathbb{P}(E^*)$ respectively, unique up to projective transformations, such that $\mathbb{B} = \mathbb{B}_{\xi,\xi^*}$. Moreover $\xi(y) \in \ker \xi^*(x)$ if and only if $x = y$.

We prove this theorem in Paragraph 5.3.

5.3 Proof of Theorem 5.3

5.3.1 Cross ratio associated to curves

Let ξ and ξ^* be two hyperconvex curves with values in $\mathbb{P}(E)$ and $\mathbb{P}(E^*)$ such that

$$x = y \Leftrightarrow \xi(y) \in \ker \xi^*(x).$$

Let e_0 and u_0 be two distinct points of $\mathbb{T}$. Let E_0 be a nonzero vector in $\xi(e_0)$. Let U_0 be a covector in $\xi^*(u_0)$ such that $\langle U_0, E_0\rangle = 1$. We now lift the curves ξ and ξ^* with values in $\mathbb{P}(E)$ and $\mathbb{P}(E^*)$ to continuous curves $\hat{\xi}$ and $\hat{\xi}^*$ from $\mathbb{T}\setminus\{e_0, u_0\}$ to E and E^*, such that

$$\langle\hat{\xi}^*(v), E_0\rangle = 1 = \langle U_0, \hat{\xi}(e)\rangle.$$

Then the associated cross ratio is

$$\mathbb{B}(f, v, e_0, u_0) = \langle\hat{\xi}^*(v), \hat{\xi}(f)\rangle.$$

From this expression it follows that $\chi^{n+1} = 0$ and that by hyperconvexity $\chi^n \neq 0$.

5.3.2 Strict cross ratio and Frenet curves

We now prove that $\mathbb{B}_{\xi,\xi^*}$ is strict if ξ and ξ^* are Frenet and if ξ^* is the osculating hyperplane of ξ.

Since $\xi(\mathbb{T})$ and $\xi^*(\mathbb{T})$ are both C^1 submanifolds, there exist homeomorphisms σ and σ^* of $\mathbb{T}$ so that

$$\eta = (\xi\circ\sigma, \xi\circ\sigma^*),$$

is a C^1 map.

The following preliminary result is of independent interest

Proposition 5.4 *The two-form $\omega = \eta^*\Omega$ is symplectic.*

PROOF: The regularity of η implies that $\eta^*\Omega$ is continuous. We begin by an observation. Let D be a line in $\mathbb{R}^n$, P a line in $\mathbb{R}^{n*}$, such that $D\oplus P^\perp = \mathbb{R}^n$. Let W be a two-plane containing D. Let

$$\hat{W} = T_D\mathbb{P}(W) \subset T_D\mathbb{P}(\mathbb{R}^n).$$

Let V an $n-2$-plane contained in $P^\perp$. Let

$$\hat{V} = T_P\mathbb{P}(W^\perp) \subset T_P\mathbb{P}(\mathbb{R}^{*n}).$$

By the definition of Ω in Section 4.4.3, if

$$V\oplus W = \mathbb{R}^n,$$

then

$$\Omega|_{\hat{V}\oplus\hat{W}} \neq 0.$$

In the case of hyperconvex curves,

$$T_{\xi^1(x)}\big(\xi^1(\partial_\infty\pi_1(\Sigma))\big) = \widehat{\xi^2(x)}, \qquad T_{\xi^{n-1}(x)}\big(\xi^{n-1}(\partial_\infty\pi_1(\Sigma))\big) = \widehat{\xi^{n-2}(x)}.$$

Since by hyperconvexity $\xi^2(x)\oplus\xi^{n-2}(y) = \mathbb{R}^n$ for $x\neq y$, we conclude that $\eta^*\omega$ is symplectic. Q.E.D.

The following regularity result will be used in the sequel

Proposition 5.5 *If moreover the osculating flags of ξ and ξ^* are Hölder, so is ω.*

PROOF: If we furthermore assume that the osculating flags of ξ and ξ^* are Hölder, then we can choose the homeomorphisms σ to be Hölder. Indeed, σ and σ^* are the inverse of the arc parametrisation of the submanifolds $\xi(\mathbb{T})$ and $\xi^*(\mathbb{T})$. Therefore, σ and σ^* are Hölder. Since the tangent spaces to these submanifolds are Hölder, η is C^1 with Hölder derivatives. It follows that ω is Hölder. Q.E.D.

As a corollary, we obtain

Proposition 5.6 *Let ξ and ξ^* be Frenet curves such that ξ^* is the osculating hyperplane of ξ. Then $\mathbb{B}_\rho$ is strict.*

PROOF: Let (x, y, z, t) be a quadruple of pairwise distinct points. Let $\dot{\eta} = \eta \circ (\sigma, \sigma^*)^{-1}$. Let Q be the square in $\partial_\infty \pi_1(\Sigma)^{2*}$ whose vertexes are

$$(x, y), (z, y), (x, t), (z, t).$$

By Formula (18), we know that

$$|\mathbb{B}_\rho(x, y, z, t)| = e^{\frac{1}{2}\int_{\dot{\eta}(Q)} \omega}.$$

Since ω is symplectic, and Q has a nonempty interior, we have that $\int_{\dot{\eta}(Q)} \omega \neq 0$. Hence $\mathbb{B}_\rho(x, y, z, t) \neq 1$. Q.E.D.

5.3.3 Curves associated to cross ratios

We prove in this paragraph

Proposition 5.7 *Let $\mathbb{B}$ be a rank-n cross ratio on $\mathbb{T}$. Then there exists hyperconvex curves (ξ, ξ^*) with values in $\mathbb{P}(\mathbb{R}^n) \times \mathbb{P}(\mathbb{R}^{*n})$, unique up to projective equivalence, so that*

$$\mathbb{B} = \mathbb{B}_{\xi,\xi^*}$$

PROOF: Let $\mathbb{B}$ be a rank-n cross ratio. Let us fix $(e, u) = ((e_0, e_1, \ldots, e_n), (u_0, u_1, \ldots, u_n))$ in $\mathbb{T}^{n*}$. We consider

$$\hat{\xi} : \begin{cases} \mathbb{T} \setminus \{u_0\} & \to \quad \mathbb{R}^n, \\ \qquad f & \mapsto \quad (\mathbb{B}(f, u_1, e_0, u_0), \ldots, \mathbb{B}(f, u_n, e_0, u_0)). \end{cases}$$

Since

$$\chi^n_{\mathbb{B}}(e, u) \neq 0, \tag{19}$$

the set $(\hat{\xi}(e_1), \ldots, \hat{\xi}(e_n))$ is a basis of $\mathbb{R}^n$. Let $(E^*_1, \ldots, E^*_n)$ be its dual basis. We now consider the map

$$\hat{\xi}^* : \begin{cases} \mathbb{T} \setminus \{e_0\} & \to \quad \mathbb{R}^{*n}, \\ \qquad v & \mapsto \quad \sum_{i=1}^{i=n} \mathbb{B}(v, e_i, e_0, u_0) E^*_i. \end{cases}$$

We now prove that

$$\mathbb{B}(f, v, e_0, u_0) = \langle \hat{\xi}(f), \hat{\xi}^*(v) \rangle. \tag{20}$$

We first observe that

$$\langle \hat{\xi}(e_i), \hat{\xi}^*(v) \rangle = \mathbb{B}(e_i, v, e_0, u_0). \tag{21}$$

In particular

$$\langle \hat{\xi}(e_i), \hat{\xi}^*(u_j) \rangle = \mathbb{B}(e_i, u_j, e_0, u_0), \tag{22}$$

and thus

$$(\hat{\xi}^*(u_1), \ldots, \hat{\xi}^*(u_n)),$$

is the canonical basis of $\mathbb{R}^{*n}$. As a consequence

$$\langle \hat{\xi}(f), \hat{\xi}^*(u_j) \rangle = \mathbb{B}(f, u_j, e_0, u_0). \tag{23}$$

Let M^n be the $n \times n$-matrix whose coefficients are $\langle \hat{\xi}(e_i), \hat{\xi}^*(u_j) \rangle$. For any any f anf v, let $\hat{e} = (e_0, \ldots, e_n, f)$ and $\hat{u} = (u_0, \ldots, u_n, v)$. Let also M^{n+1} be the degenerate $(n+1) \times (n+1)$-matrix whose coefficients are $\langle \hat{\xi}(e_i), \hat{\xi}^*(u_j) \rangle$ with the convention that $e_{n+1} = f$ and $u_{n+1} = v$. By Equation (22),

$$\det(M^n) = \chi^n_{\mathbb{B}}(e, u) \neq 0. \tag{24}$$

The equation

$$\chi^{n+1}_{\mathbb{B}}(\hat{e}, \hat{u}) = 0,$$

yields, after developing the determinant along the last line,

$$\mathbb{B}(f, v, e_0, v_0)\chi^n_{\mathbb{B}}(e, u) = F(\ldots, \mathbb{B}(f, u_i, e_0, u_0), \ldots, \mathbb{B}(e_i, v, e_0, u_0), \ldots),$$

where the right hand term is polynomial in $\mathbb{B}(f, u_i, e_0, u_0)$ and $\mathbb{B}(e_i, v_i, e_0, u_0)$. The same argument applied to the determinant of M^{n+1} yields

$$\langle \hat{\xi}(f), \hat{\xi}^*(v) \rangle \det(M^n) = F(\ldots, \langle \hat{\xi}(f), \hat{\xi}^*(u_j) \rangle, \ldots, \langle \hat{\xi}(e_i), \hat{\xi}^*(v) \rangle, \ldots).$$

Therefore, we complete the proof of Equation (20) using Equations (21), (23) and (24).

Let now $\hat{\xi}$ and $\hat{\xi}^*$ be defined from $S = \mathbb{T} \setminus \{e_0, u_0\}$ to $\mathbb{P}(\mathbb{R}^n)$ and $\mathbb{P}(\mathbb{R}^{*n})$ as the projections of the map ξ and ξ^*.

By the Normalisation Relation (6) and Equation (20)

$$x = y \Leftrightarrow \xi(y) \in \ker \xi^*(x).$$

Equation (20) applied four times yields that $\mathbb{B} = \mathbb{B}_{\xi,\xi^*}$. Finally since χ^n never vanishes, ξ is hyperconvex as well as ξ^*.

By Lemma 4.3, this shows that the curves ξ and ξ^* are unique up to projective transformations, and in particular do not depend on the choice of e and u. We can therefore extend ξ and ξ^* to $\mathbb{T}$ by a simple gluing argument and the proposition follows. Q.E.D.

5.4 Hyperconvex representations and cross ratios

Using Theorem 2.6, the following proposition relates hyperconvex representations and cross ratios.

Proposition 5.8 *Let ρ be a a n-hyperconvex representation with limit curve ξ and osculating hyperplane ξ^*. Then $\mathbb{B}_\rho = \mathbb{B}_{\xi,\xi^*}$ is a cross ratio defined on $\partial_\infty \pi_1(\Sigma)$ and its periods are*

$$\ell_{\mathbb{B}}(\gamma) = w_\rho(\gamma) := \log\left(\left|\frac{\lambda_{\max}(\rho(\gamma))}{\lambda_{\min}(\rho(\gamma))}\right|\right),$$

where $\lambda_{\max}(\rho(\gamma))$ and $\lambda_{\min}(\rho(\gamma))$ are the eigenvalues of the real split element $\rho(\gamma)$ having maximal and minimal absolute values respectively.

Proof: By Theorem 5.3, $\mathbb{B}_\rho$ is rank-n weak cross ratio. Furthermore, since ξ and ξ^* are Hölder, $\mathbb{B}_\rho$ is Hölder. By Proposition 5.6, $\mathbb{B}_\rho$ is strict. It remains to compute the periods. By Theorem 2.6, if γ^+ is the attracting fixed point of γ in $\partial_\infty\pi_1(\Sigma)$, then $\xi(\gamma^+)$, (resp. $\xi^*(\gamma^-)$) is the unique attracting (respectively repelling) fixed point of $\rho(\gamma)$ in $\mathbb{P}(E)$. In particular

$$\begin{aligned}
\rho(\gamma)\hat{\xi}(\gamma^+) &= \lambda_{\max}\hat{\xi}(\gamma^+),\\
\rho(\gamma)\hat{\xi}(\gamma^-) &= \lambda_{\min}\hat{\xi}(\gamma^-).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\ell_{\mathbb{B}}(\gamma) &= \log|\mathbb{B}(\gamma^-, y, \gamma^+, \gamma^{-1}y)|\\
&= \log\left|\frac{\langle\hat{\xi}(\gamma^-), \hat{\xi}^*(y)\rangle\langle\hat{\xi}(\gamma^+), \hat{\xi}^*(\gamma^{-1}y)\rangle}{\langle\hat{\xi}(\gamma^-), \hat{\xi}^*(\gamma^{-1}y)\rangle\langle\hat{\xi}(\gamma^+), \hat{\xi}^*(y)\rangle}\right|\\
&= \log\left|\frac{\langle\hat{\xi}(\gamma^-), \hat{\xi}^*(y)\rangle\langle\hat{\xi}(\gamma^+), \rho(\gamma)^*\hat{\xi}^*(y)\rangle}{\langle\hat{\xi}(\gamma^-), \rho(\gamma)^*\hat{\xi}^*(y)\rangle\langle\hat{\xi}(\gamma^+), \hat{\xi}^*(y)\rangle}\right|\\
&= \log\left|\frac{\langle\hat{\xi}(\gamma^-), \hat{\xi}^*(y)\rangle\langle\rho(\gamma)\hat{\xi}(\gamma^+), \hat{\xi}^*(y)\rangle}{\langle\rho(\gamma)\hat{\xi}(\gamma^-), \hat{\xi}^*(y)\rangle\langle\hat{\xi}(\gamma^+), \hat{\xi}^*(y)\rangle}\right|\\
&= \log\left|\frac{\lambda_{\max}}{\lambda_{\min}}\right|.
\end{aligned}$$

Q.e.d.

Remark: Let ρ^* be the contragredient representation of ρ defined by

$$\rho^*(\gamma) = (\rho(\gamma^{-1}))^*.$$

Let $\mathbb{B}$ be a cross ratio. We define $\mathbb{B}^*$ by $\mathbb{B}^*(x, y, z, t) = \mathbb{B}(y, x, t, z)$. Then $(\mathbb{B}_\rho)^* = \mathbb{B}_{\rho^*}$. By the above Proposition, the cross ratios $\mathbb{B}_\rho$ and $\mathbb{B}_{\rho^*}$ have the same periods, however $\mathbb{B}_\rho$ and $\mathbb{B}_{\rho^*}$ are not necessarily conjugated.

5.5 Proof of the main Theorem 1.1

Theorem 1.1 stated in the introduction is a consequence of Proposition 5.8 for one side of the equivalence, and of Theorem 5.3, Lemma 4.3 and Guichard's Theorem 2.4 [18] for the other side of the equivalence.

6 The jet space $J^1(\mathbb{T},\mathbb{R})$

Let $J = J^1(\mathbb{T},\mathbb{R})$ be the space of one-jets of real-valued functions on the circle $\mathbb{T}$. The structure of this section is as follows:

- In Paragraph 6.1, we describe the action of the group $C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ on J as well as the geometry of this latter space.
- In Paragraph 6.2, we characterise geometrically the action of $C^\infty(\mathbb{T}) \rtimes \mathrm{Diff}^\infty(\mathbb{T})$ on J.
- In Paragraph 6.3, we describe a homomorphism from $\mathrm{PSL}(2,\mathbb{R})$ to $C^\infty(\mathbb{T})\rtimes \mathrm{Diff}^\infty(\mathbb{T})$ whose image acts faithfully and transitively on J.

6.1 Description of the jet space

We describe in this section the geometric features of J that will be useful in the sequel, namely:

- A structure of a principal $\mathbb{R}$-bundle with connection given by
  - a projection δ onto $T^*\mathbb{T}$,
  - an $\mathbb{R}$-action given by a flow $\{\varphi_t\}$,
  - a connection form β which is a contact form.
- A foliation of J by affine leaves $\mathcal{F}$.
- A projection π onto $\mathbb{T}$.
- An action of $C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ on J.

In Lemma 10.1, we shall characterise the action of $C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ using the principal bundle structure and the foliation $\mathcal{F}$. We finally give in Paragraph 6.3.2 two alternate descriptions of J

6.1.1 Projections

We denote by $j^1_x(f)$ the one-jet of the function f at the point x of $\mathbb{T}$. We define the projections

$$\pi \;:\; \left\{ \begin{array}{rcl} J & \rightarrow & \mathbb{T}, \\ j^1_x(f) & \mapsto & x, \end{array} \right.$$

$$\delta \;:\; \begin{cases} J & \to & T^*\mathbb{T}, \\ j^1_x(f) & \mapsto & d_x f. \end{cases}$$

We observe that each fibre of π carries an affine structure. In order to help the readers remember the heavy notation, we remark that δ is the projection that takes in account the "derivative" part.

6.1.2 Action of $C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$

The group $H(\mathbb{T}) = C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ acts by Hölder homeomorphisms on J in the following way. Let (ϕ, h) be an element of $C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$, where ϕ a C^1-diffeomorphism with Hölder derivatives of $\mathbb{T}$ and h is a C^1-function on $\mathbb{T}$ with Hölder derivatives. Let $F = F(h, \phi)$ be the homeomorphism of J given by

$$F(h,\phi) \;:\; j^1_x(f) \mapsto j^1_{\phi(x)}((h+f)\circ\phi^{-1}).$$

The homeomorphism $F = F(h, \phi)$ has the following properties:

- F preserves the fibres of π, that is: $\pi \circ F = \phi \circ \pi$.
- For every x in $\mathbb{T}$, F restricted to $\pi^{-1}(x)$ is affine – in particular C^∞ – and the derivatives of $F|_{\pi^{-1}(x)}$ vary continuously on J.

Finally, the map $(h, \phi) \to F(h\phi)$ defines an action of $C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ on J by Hölder homeomorphisms.

Alternatively, if we choose a coordinate θ on $\mathbb{T}$ and consider the identification

$$\begin{cases} J & \to & \mathbb{T}\times\mathbb{R}\times\mathbb{R}, \\ j^1_\theta(f) & \mapsto & (\theta, r, f) = (\theta, \frac{\partial f}{\partial\theta}, f(\theta)), \end{cases}$$

then $F(h, \phi)$ is given by

$$(\eta, r, f) \mapsto (\phi(\eta), \frac{\partial\phi}{\partial\theta}^{-1}(\eta)\big(\frac{\partial h}{\partial\theta}(\eta) + r\big), f + h(\eta)). \tag{25}$$

6.1.3 Foliation

Let $\mathcal{F}$ be the 1-dimensional foliation of J given by the fibres of the projection

$$\begin{cases} J & \to & \mathbb{T}\times\mathbb{R}, \\ j^1_x(f) & \mapsto & (x, f(x)). \end{cases}$$

Each leaf of $\mathcal{F}$ is included in a fibre of π and is an affine line with respect to the affine structure on the fibres of π. We observe that the leaf through z is identified (as an affine line) with $T^*_x\mathbb{T}$ for $x = \pi(z)$.

This affine structure is invariant by the action of $C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ described above.

6.1.4 Canonical flow

We define the *canonical flow* of J to be the flow

$$\varphi_t(j^1_x(f)) = j^1_x(f+t),$$

where we identify the real number t with the constant function that takes value t. The canonical flow commutes with the action of $C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ on J. Notice also that

$$J/Z(H(\mathbb{T})) = T^*\mathbb{T},$$

and that this identification turns $\delta : J \mapsto T^*\mathbb{T}$ into a principal $\mathbb{R}$-bundle.

6.1.5 Contact form

We finally recall that J admits a contact form β. If we choose a coordinate θ on $\mathbb{T}$ and consider the identification

$$\left\{ \begin{array}{ccc} J & \rightarrow & \mathbb{T}\times\mathbb{R}\times\mathbb{R}, \\ j^1_\theta(f) & \mapsto & (\theta, r, f) = (\theta, \frac{\partial f}{\partial \theta}, f(\theta)), \end{array} \right.$$

then

$$\beta = df - rd\theta.$$

REMARKS:

1. Note that a Legendrian curve for β which is locally a graph above $\mathbb{T}$ is the graph of one-jet of a function.

2. Moreover, the canonical flow φ_t preserves the 1-form β.

3. Observe that β is a connection form for the principal $\mathbb{R}$-bundle defined by δ, and that its curvature form is the canonical symplectic form of $T^*\mathbb{T}$.

4. Here is another description of β. Recall that the *Liouville form* λ on the cotangent bundle $p : T^*\mathbb{M} \rightarrow M$ is given by

   $$\lambda_q(u) = \langle q | T_q p(u) \rangle.$$

   In coordinates for $T^*\mathbb{T}$, we have, $\lambda = rd\theta$. If f is the function on J given by $f(j^1_x(g)) = g(x)$, then $\beta = df - \delta^*\lambda$.

6.2 A Geometric Characterisation of $C^\infty(\mathbb{T}) \rtimes \mathrm{Diff}^\infty(\mathbb{T})$

The following proposition shows that $\mathcal{F}$, β and φ_t characterise the action of $C^\infty(\mathbb{T}) \rtimes \mathrm{Diff}^\infty(\mathbb{T})$ on J. Later, in Lemma 10.1, we characterise $C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$.

Proposition 6.1 *Let ψ be a C^∞-diffeomorphism of J. Assume that*

1. *ψ commutes with the flow φ_t,*

2. *ψ preserves the 1-form β,*

3. *ψ preserves the foliation $\mathcal{F}$.*

Then ψ belongs to $C^\infty(\mathbb{T}) \rtimes \mathrm{Diff}^\infty(\mathbb{T})$.

6.2.1 A preliminary proposition

We first prove an elementary remark

Proposition 6.2 *Let α be a connection one-form on the $\mathbb{R}$-bundle $\delta : J \to T^*\mathbb{T}$. Assume that the curvature of α is the canonical symplectic form on $T^*\mathbb{T}$. Then there exists a diffeomorphism ξ of J which*

- *commutes with the action of the canonical flow,*
- *preserves the fibres of π and δ (that is send fibres to fibres),*
- *is above a symplectic diffeomorphism of $T^*\mathbb{T}$,*
- *satisfies $\xi^*\beta = \alpha$.*

Proof: We choose a coordinate θ of $\mathbb{T}$. Thus, $T^*\mathbb{T}$ is identified with $\mathbb{T} \times \mathbb{R}$ with coordinates (θ, r), and J with $\mathbb{T} \times \mathbb{R} \times \mathbb{R}$ with coordinates (θ, r, f). Since α is a connection form, there exist functions α_r and α_θ such that

$$\alpha = df + \alpha_r(r, \theta)dr + \alpha_\theta(r, \theta)d\theta.$$

Let

$$\gamma = \alpha - \beta = \alpha_r(r, \theta)dr + (\alpha_\theta(r, \theta) + r)d\theta.$$

Since the curvature of α is $d\theta \wedge dr$, γ is closed. Therefore, there exist a function h on $T^*\mathbb{T}$ and a constant λ such that

$$\gamma = dh + \lambda d\theta.$$

A straightforward check now shows that the diffeomorphism ξ of J given by

$$\xi(\theta, r, f) = (\theta, r - \lambda, f + h(\theta, r)).$$

satisfies the condition of our proposition. Q.e.d.

6.2.2 Proof of Proposition 6.1

Proof: We use the notations and assumptions of Proposition 6.1. By Assumptions (1) and (3), ψ preserves the fibres of π. There exists thus a C^∞-diffeomorphism ϕ of $T^*\mathbb{T}$ such that

$$\pi \circ \psi = \phi \circ \pi.$$

Replacing ψ by $\psi \circ (0, \phi^{-1})$, we may as well assume that $\phi = id$. By Assumption (1), ψ also preserves the fibres of δ.

We choose a coordinate on $\mathbb{T}$ and use the identification $J = \mathbb{T} \times \mathbb{R} \times \mathbb{R}$ given in Paragraph 6.1.5. The previous discussion shows that

$$\psi(\theta, r, f) = (\theta, F(\theta, r), H(\theta, r, f)).$$

Since $\psi^* \beta = \beta$, we obtain that

$$dH - F d\theta = df - r d\theta.$$

Hence

$$\frac{\partial H}{\partial f} = 1,\ F - \frac{\partial H}{\partial \theta} = r,\ \frac{\partial H}{\partial r} = 0.$$

Therefore, there exists a C^∞-function g such that $H = f + g(\theta)$. It follows that

$$\psi(\theta, r, f) = (\theta, r + \frac{\partial g}{\partial \theta}, f + g(\theta)).$$

This exactly means that

$$\psi(\theta, r, f) = (g, id).j^1(f).$$

In other words, ψ belongs to $C^\infty(\mathbb{T}) \rtimes \mathrm{Diff}^\infty(\mathbb{T})$. Q.E.D.

6.3 $\mathrm{PSL}(2,\mathbb{R})$ and $C^\infty(\mathbb{T}) \rtimes \mathrm{Diff}^\infty(\mathbb{T})$

We consider the 3-manifold $\overline{J} = \mathrm{PSL}(2,\mathbb{R})$. We denote by g the Killing form, which we consider as a biinvariant – with respect to both left and right actions of $\mathrm{PSL}(2,\mathbb{R})$ – Lorentz metric on $\overline{J}$. Then:

- Let $\overline{\varphi_t} = \Delta$ be the one-parameter group of diagonal matrices acting on the right on $\overline{J}$.
- Let $\overline{\mathcal{F}}$ the orbit foliation by the right action of the one-parameter group of strictly upper triangular matrices.
- Let $\overline{\beta} = i_X g$, where X is the vector field generating $\overline{\varphi}_t$.

Alternatively, describing $\mathrm{PSL}(2,\mathbb{R})$ as the unit tangent bundle of the hyperbolic plane, we can identify $\overline{\varphi_t}$ with the geodesic flow, $\overline{\mathcal{F}}$ with the horospherical foliation and β with the Liouville form.

Then

Proposition 6.3 *The one-form $\overline{\beta}$ is a contact form. Furthermore, there exists a C^∞-diffeomorphism Ψ from J to $\overline{J}$, that sends $(\varphi_t, \mathcal{F}, \beta)$ to $(\overline{\varphi_t}, \overline{\mathcal{F}}, \overline{\beta})$ respectively. Finally, this diffeomorphism Ψ is unique up to left composition by an element of $C^\infty(\mathbb{T}) \rtimes \mathrm{Diff}^\infty(\mathbb{T})$.*

As an immediate application, we have

Definition 6.4 [Standard representation] *Since the left action of* $\mathrm{PSL}(2,\mathbb{R})$ *preserves the flow* $\overline{\varphi_t}$*, the foliation* $\overline{\mathcal{F}}$ *and the 1-form* $\overline{\beta}$*, combining Propositions 6.1 and 6.3, we obtain a group homomorphism*

$$\iota : \left\{ \begin{array}{rcl} \mathrm{PSL}(2,\mathbb{R}) & \rightarrow & C^\infty(\mathbb{T}) \rtimes \mathrm{Diff}^\infty(\mathbb{T}), \\ g & \mapsto & \Psi \circ g \circ \Psi^{-1}, \end{array} \right.$$

well defined up to conjugation by an element of $C^\infty(\mathbb{T}) \rtimes \mathrm{Diff}^\infty(\mathbb{T})$*. The corresponding representation from* $\mathrm{PSL}(2,\mathbb{R})$ *to* $C^\infty(\mathbb{T}) \rtimes \mathrm{Diff}^\infty(\mathbb{T})$ *is called* standard.

Remark : The action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{RP}^1 \simeq \mathbb{T}$ gives rise to an embedding of $\mathrm{PSL}(2,\mathbb{R})$ in $\mathrm{Diff}^\infty(\mathbb{T})$. The above standard representation is a nontrivial extension of this representation. Indeed, the natural lift of the action of $\mathrm{Diff}^\infty(\mathbb{T})$ on $T^*\mathbb{T}$ does not act transitively since it preserves the zero section. On the contrary, $\mathrm{PSL}(2,\mathbb{R})$ does act transitively through the standard representation. In Paragraph 6.3.2, we consider an alternate description of J which make the action of $\mathrm{PSL}(2,\mathbb{R})$ more canonical.

6.3.1 Proof of Proposition 6.3

Proof: It is immediate to check that

$$\mathrm{PSL}(2,\mathbb{R}) \rightarrow \mathrm{PSL}(2,\mathbb{R})/\Delta = W,$$

is a principal $\mathbb{R}$-bundle, whose connexion form is $\overline{\beta}$ and whose curvature is symplectic. It follows that $\overline{\beta}$ is a contact form.

Let $\overline{\pi}$ be the projection

$$W \rightarrow \mathbb{RP}^1 = \mathrm{PSL}(2,\mathbb{R})/B,$$

where B be the 2-dimensional group of upper triangular matrices.

Let ψ_0 be a symplectic diffeomorphism from W to $T^*\mathbb{T}$ over a diffeomorphism ϕ_0 from $\mathbb{RP}^1$ to $\mathbb{T}$, that is so that $\pi \circ \psi_0 = \phi_0 \overline{\pi}$. Let $\Psi_0 : \overline{J} \rightarrow J$ be a principal $\mathbb{R}$-bundle equivalence over ψ_0. Let $\alpha = (\psi_0^{-1})^* \overline{\beta}$. Let ξ be the symplectic diffeomorphism of J obtained by Proposition 6.2 applied to $\alpha = (\psi_0^{-1})^* \overline{\beta}$. It follows $\Psi = \xi \circ \Psi_0$ has all the properties required. Finally, by Proposition 6.1, Ψ is well defined up to multiplication by an element of $C^\infty(\mathbb{T}) \rtimes \mathrm{Diff}^\infty(\mathbb{T})$.

Since the right action of $\mathrm{PSL}(2,\mathbb{R})$ preserves $\overline{\mathcal{F}}$, $\overline{\beta}$, and $\overline{\varphi}_t$, we obtain a representation of $\mathrm{PSL}(2,\mathbb{R})$ well defined up to conjugation

$$\left\{ \begin{array}{rcl} \mathrm{PSL}(2,\mathbb{R}) & \rightarrow & C^\infty(\mathbb{T}) \rtimes \mathrm{Diff}^\infty(\mathbb{T}), \\ g & \mapsto & \Psi \circ g \circ \Psi^{-1}. \end{array} \right.$$

Q.e.d.

6.3.2 Alternate description of J

For the sake of completeness, we introduce two alternate descriptions of J. The second one – which we shall not use – has been suggested by the referee:

- We have just seen that $\mathrm{PSL}(2,\mathbb{R})$ is an alternate description of J.
- We also may consider the space $E_{1/2}$ of half densities on $\mathbb{T}$. Then J will be the space of 1-jets of sections. The reader can work out the details and see that the natural action of $\mathrm{PSL}(2,\mathbb{R})$ which comes from its injection into $\mathrm{Diff}^{\infty}(\mathbb{T})$ coincides with the above action.

7 Anosov and ∞-Hitchin homomorphisms

The purpose of this section is to define various types of homomorphisms from $\pi_1(\Sigma)$ to $H(\mathbb{T}) = C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$.

7.1 ∞-Fuchsian homomorphism and H-Fuchsian actions

Definition 7.1 [Fuchsian homomorphism] *Let ρ be a Fuchsian homomorphism from $\pi_1(\Sigma)$ to $\mathrm{PSL}(2,\mathbb{R})$. We say that the composition of ρ by the standard representation ι from $\mathrm{PSL}(2,\mathbb{R})$ to $H(\mathbb{T})$ (*cf *Definition 6.4) is* ∞-Fuchsian*, or in short* Fuchsian *when there is no ambiguity.*

Definition 7.2 [H-Fuchsian action on $\mathbb{T}$] *We say an action of $\pi_1(\Sigma)$ on $\mathbb{T}$ is* H-Fuchsian *if it is an action by C^1-diffeomorphisms with Hölder derivatives, which is Hölder conjugate to the action of a cocompact group of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{RP}^1$.*

7.2 Anosov homomorphisms

We begin with a general definition. Let $\mathcal{F}$ be a foliation on a compact space, and $d_{\mathcal{F}}$ be a Riemannian distance along the leaves of $\mathcal{F}$ which comes from a leafwise continuous metric. In particular, $d_{\mathcal{F}}(x,y) < \infty$ if and only if x and y are in the same leaf.

Definition 7.3 [Contracting the leaves] *A*flow φ_t contracts uniformly the leaves of a foliation $\mathcal{F}$*, if φ_t preserves $\mathcal{F}$ and if moreover*

$$\forall \epsilon > 0,\ \forall \alpha > 0,\ \exists t_0 :\ t \geq t_0,\ d_{\mathcal{F}}(x,y) \leq \alpha \implies d_{\mathcal{F}}(\varphi_t(x), \varphi_t(y)) \leq \epsilon$$

This definition does not depend on the choice of $d_{\mathcal{F}}$.

We now use the notations of Section 6.1.

Definition 7.4 [Anosov homomorphisms] *A homomorphism ρ from $\pi_1(\Sigma)$ to $H(\mathbb{T})$ is* Anosov *if:*

- *the group $\rho(\pi_1(\Sigma))$ acts with a compact quotient on $J = J^1(\mathbb{T},\mathbb{R})$,*
- *the flow induced by the canonical flow φ_t on $\rho(\pi_1(\Sigma))\backslash J$ contracts uniformly the leaves of $\mathcal{F}$ (cf. definition above),*

- *the induced action on $\mathbb{T}$ is H-Fuchsian.*

We denote the space of Anosov homomorphisms by Hom^*.

Remarks:

1. If ρ is Anosov, $\rho(\pi_1(\Sigma))$ usually only acts by homeomorphisms on J. However, if we consider J as a C^∞-filtered space (See the definitions in Paragraph 8.1.1), whose nested foliated structures are given by the fibres of the two projections δ and π, then $\rho(\pi_1(\Sigma))$ acts by C^∞-filtered maps (*i.e* smoothly along the leaves with continuous derivatives). Therefore, $\rho(\pi_1(\Sigma))\backslash J$ has the structure of a C^∞-lamination such that the flow induced by φ_t is C^∞ leafwise.

2. Our first examples of Anosov homomorphisms are ∞-Fuchsian homomorphisms from $\pi_1(\Sigma)$ in $H(\mathbb{T})$. Indeed in this case, the canonical flow on $\rho(\pi_1(\Sigma))\backslash J$ is conjugate to the geodesic flow for the corresponding hyperbolic surface.

Finally we define

Definition 7.5 [∞-Hitchin homomorphisms] *An* ∞-Hitchin homomorphism *is a homomorphism from $\pi_1(\Sigma)$ to $H(\mathbb{T})$ which may be deformed into an ∞-Fuchsian homomorphism, through Anosov homomorphisms. In other words, the set* Hom_H *of Hitchin representations is the connected component of the set* Hom^* *of Anosov homomorphisms, containing the Fuchsian homomorphisms.*

8 Stability of ∞-Hitchin homomorphisms

The set of homomorphisms from $\pi_1(\Sigma)$ to a semi-simple Lie group G with Zariski dense images satisfies the following properties:

- It is open in the set of all homomorphisms.
- The group $G/Z(G)$ acts properly on it, where $Z(G)$ is the centre of G.

We aim to prove that the set Hom_H of ∞-Hitchin homomorphisms from $\pi_1(\Sigma)$ to $H(\mathbb{T})$ enjoys the same properties.

The canonical flow φ_t – considered as a subgroup of $H(\mathbb{T})$ – is in the centre $Z(H((T))$ of $H(\mathbb{T})$ and hence acts trivially on Hom_H. It follows that $H(\mathbb{T})/Z(H(\mathbb{T}))$ acts on Hom_H.

The main result of this section is the following:

Theorem 8.1 [Openness Theorem] *The set of Hitchin homomorphisms is open in the space of all homomorphisms.*

We shall also see in the next section that the action by conjugation of $H(\mathbb{T})/Z(H(\mathbb{T}))$ on Hom_H is proper and that the quotient is Hausdorff.

In the proof of Theorem 8.1, we use the definitions and results obtained in the Paragraph (8.1) which is independent of the rest of the article. It can be skipped at a first reading and is summarised as follows. Roughly speaking, a filtered space is equipped with a family of nested foliations which are transverse continuously, but posses a smooth structure along the leaves. A C^∞-filtered function is continuous, smooth along the leaves and with continuous leafwise derivatives, and a C^∞-filtered map between filtered spaces sends leaves to leaves and C^∞-filtered functions to C^∞-filtered functions.

The theorem follows from the Stability Lemma 8.10 which has its own interest and applications.

8.1 Filtered Spaces and Holonomy.

In this paragraph, logically independent from the rest of the article, we describe a notion of a topological space with "nested" laminations. This requires some definitions and notations. Let $Z_1, \ldots, Z_p$ be a family of sets. Let

$$Z = Z_1 \times \ldots \times Z_p.$$

For $k < p$, we note p_k the projection

$$Z \to Z_1 \times \ldots \times Z_k.$$

Let O be a subset of Z and x a point of O. The k^{th}*-leave* through x in O is

$$O_x^{(k)} = O \cap p_k^{-1}\{p_k(x)\}.$$

We note that $O_x^{(k+1)} \subset O_x^{(k)}$. The *higher dimensional leaf* is $O_x^{(1)}$. If ϕ is a map from O to to a set V, we write

$$\phi_x^{(k)} = \phi|_{O_x^{(k)}}.$$

If $V \subset W = W_1 \times \ldots \times W_k$, we say ϕ is a *filtered map* if

$$\forall k,\ \phi(O_x^{(k)}) \subset V_{\phi(x)}^{(k)}.$$

8.1.1 Laminations and Filtered Space

In this section, we define a notion of filtered space for which it does make sense to say some maps are "smooth along leaves with derivatives varying continuously".

Definition 8.2 [C^∞-FILTERED SPACE] *A metric space P is C^∞-*filtered *if the following conditions are satisfied:*

- *There exists a covering of P by open sets U_i, called* charts.

- *There exist Hölder homeomorphisms φ_i, called* coordinates, *from U_i to $V_1^i \times V_2^i \times \ldots \times V_p^i$ where for $k > 1$, V_i^k is an open set in a finite dimensional affine space.*

- *Let*
$$\varphi_{ij} = \varphi_i \circ (\varphi_j)^{-1},$$
defined from $W_{ij} = \varphi_j^{-1}(U_i \cap U_j)$ to W_{ji} be the coordinate changes. *Then*
  - *the coordinates changes are filtered map,*
  - *the restriction $(\varphi_{ij}^{(1)})_x$ of the coordinate changes to the maximal leaf through x is a C^∞-map whose derivatives depends continuously on x.*

Remarks:

1. Due to the last assumption, for all k, $(\varphi_{ij}^{(k)})_x$ is a C^∞-map whose derivatives depends continuously on x.
2. When $p = 2$, we speak of a *laminated space*.
3. We may want to specify the number of nested laminations, in which case we talk of p-filtered objects.
4. The Hölder hypothesis is somewhat irrelevant and could be replaced by other regularity assumptions, but it is needed in the applications.

We now extend the definitions of the previous paragraph. Let P be a C^∞-p-filtered space. Let $k < p$.

Definition 8.3 [Leaves] *The* k^{th}-leaves *of P are the equivalence classes of the equivalence relation generated by*
$$y \; \mathcal{R} \; x \iff p_k(\varphi_i(y)) = p_k(\varphi_i(x)).$$

Let P and $\overline{P}$ be C^∞-p-filtered spaces. Let φ_i be coordinates on P and $\overline{\varphi}_j$ be coordinates on $\overline{P}$.

Definition 8.4 [Filtered maps and immersions] *A map ψ from $\overline{P}$ to P is* C^∞-filtered *if*

- *ψ is Hölder,*
- *ψ send k^{th}-leaves to k^{th}-leaves,*
- *$(\varphi_i \circ \psi \circ \overline{\varphi}_j)_x^{(k)}$ are C^∞ and their derivatives vary continuously with x.*

A filtered immersion *is a filtered map ψ whose leafwise tangent map is injective: in other words, $(\varphi_i \circ \psi \circ \overline{\varphi}_j^{-1})_x^1$ is an immersion.*

Observe that a filtered immersion is only an immersion along the leaves, there is no requirement about what happens transversely to the leaves.

Definition 8.5 [CONVERGENCE OF FILTERED MAPS] *Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a sequence of C^∞-filtered maps from $\overline{P}$ to P. The sequence C^∞* converges on every compact set *if*

- *it converges uniformly on every compact set,*
- *all the derivatives of $(\varphi_i \circ \psi_n \circ \overline{\varphi}_j)^{(k)}_x$ converges uniformly on every compact set (as functions of x).*

Definition 8.6 [AFFINE LEAVES] *A P is C^∞-filtered by* affine leaves *or carries a* leafwise affine structure *if furthermore $\varphi_{ij}|_{U_x}$ is an affine map.*

8.1.2 Holonomy Theorem

We now prove the following result which generalises Ehresmann-Thurston Holonomy Theorem [10]. For P a p-filtered space, we denote by $\mathrm{Fil}(P)$ be the group of all bijections of P to itself which are all C^∞-filtered leafwise immersions of P, equipped with the topology of C^∞-convergence on compact sets.

Theorem 8.7 *Let P be a p-filtered space. Let $\tilde{V}$ be a connected p-filtered space. Let Γ be a discrete subgroup of $\mathrm{Fil}(\tilde{V})$, such that $\tilde{V}/\Gamma$ is a compact filtered space.*

Let U be the set of homomorphisms ρ from Γ to $\mathrm{Fil}(P)$ such that there exists a ρ-equivariant filtered immersion from $\tilde{V}$ to P. Then U is open.

Moreover, suppose ρ_0 belongs to U. Let f_0 be a ρ_0-equivariant filtered immersion from $\tilde{V}$ to P. If ρ is close enough to ρ_0, we may choose a ρ-equivariant filtered immersion f arbitrarily close to f_0 on compact sets.

We note again that we could have replaced the word "Hölder" by "continuous" from all the definitions and still have a valid theorem.

PROOF: Let $\{U^i\}$ be a finite covering of $\tilde{V}/\Gamma$ such that:

- The open sets U^i are charts on $\tilde{V}/\Gamma$.
- The open sets U^i are trivialising open sets for the covering $\pi : \tilde{V} \to \tilde{V}/\Gamma$. In other words, $\pi^{-1}(U^i)$ is a disjoint union of open sets which are mapped homeomorphically to U^i by π.

We choose an open set $\tilde{U}^1 \subset \tilde{V}$, such that π is a homeomorphism from $\tilde{U}^1$ to U^1. We make the following temporary definitions.

A *loop* is a sequence of indexes $i_1, \dots i_l$ such that $i_l = i_1 = 1$ and

$$U^{i_j} \cap U^{i_{j+1}} \neq \emptyset.$$

A loop defines uniquely a sequence of open sets $\tilde{U}^{i_j}$ such that

- π is a homeomorphism from $\tilde{U}^{i_j}$ to U^{i_j},
- $\tilde{U}^{i_j} \cap \tilde{U}^{i_{j+1}} \neq \emptyset$.

A loop $i_1, \dots i_l$ is *trivialising* if $\tilde{U}^1 = \tilde{U}^{i_l}$. In general, we associate to a loop the element γ of Γ such that $\gamma(\tilde{U}^1) = \tilde{U}^{i_l}$. The group Γ is the group of loops (with the product structure given by concatenation) modulo trivialising loops. This is just a way to choose a presentation of Γ adapted to the charts U^i.

A *cocycle* is a finite sequence of $g = \{g^{ij}\}$ of elements of G such that for every trivialising loop $i_1, \dots, i_p$ we have

$$g^{i_1 i_2} \dots g^{i_{p-1} i_p} = 1.$$

Every cocycle g defines uniquely a homomorphism ρ_g from Γ to $G = \mathrm{Fil}(P)$, and furthermore the map $g \mapsto \rho_g$ is open.

Let g be a cocycle. A *g-equivariant map* f is a finite collection $\{f_i\}$ such that

- f_i is a filtered map from U^i to P,
- $f_j = g^{ij} f_i$ on $U^i \cap U^j$.

It is easy to check that there is a one to one correspondence between ρ_g-equivariant filtered maps and g-equivariant maps.

The following fact follows from the existence of partition of unity:

Let W_0, W_1, W_2 be three open set in $\tilde{V}/\Gamma$, such that

$$\overline{W_0} \subset W_1 \subset \overline{W_1} \subset W_2.$$

Let h be a filtered map defined on W_2. Then there exists ϵ_0 such that for any positive ϵ smaller ϵ_0, for any filtered map h_1 defined on W_1 and ϵ-close to h on W_1 then there exists h_0, 2ϵ-close to h on W_2, which coincides with h_1 on W_0.

Let us now begin the proof. Let g be a cocycle associated to a covering $\mathcal{U} = (U_1, \dots, U_m)$. Let f be a g-equivariant immersion. Let now $\overline{g}$ be a cocycle arbitrarily close to g.

We proceed by induction to build a $\overline{g}$-equivariant map $\overline{f}$ defined on a smaller covering $\mathcal{V} = (V_1, \dots, V_m)$ and close to f. Our induction hypothesis is the following

- Let $\mathcal{V}^i$ be a collection of opens sets : $\mathcal{V}^i = (V_1^i, \dots, V_{i-1}^i)$, with $V_k^i \subset U_k$ and $(V_1^i, \dots, V_{i-1}^i, U_i, \dots, U_m)$ is a covering.
- Let
$$\overline{f}^i = \{\overline{f}_1^i, \dots, \overline{f}_{i-1}^i\},$$
be a $\overline{g}$-equivariant map defined on $\mathcal{V}^i = (V_1^i, \dots, V_{i-1}^i)$.
- Assume that $\overline{f}$ is close to f on V_l^i, for all $l < i$.

Let

$$W_i = V_1^i \cup \ldots \cup V_{i-1}^i.$$

By the definition of a $\overline{g}$-equivariant map, there exists a map h_i defined on $W_i \cap U_i$ such that h_i restricted to $V_l^i \cap U_i$ is equal to $\overline{g}^{il}\overline{f}_l^i$. The map h_i is close to f_i. Let Z_i be a strictly smaller subset of W_i such that $(Z_i, U_i, \ldots, U_m)$ is still a covering. We use our preliminary observation to build $\overline{f}_i^i$ close to f_i on U_i and coinciding with h_i on $W_{i+1} \cap U_i$. We finally define

- $V_l^{i+1} = V_l^i \cap Z_i$,
- $V_i^{i+1} = U_i$,
- $\overline{f}_l^{i+1} = f_l^i$ for $l < i$.

This completes the induction.

In the end, we obtain a $\overline{g}$-equivariant map $\overline{f}$ defined on slightly smaller open subsets of U_i, and close to f. Therefore, it follows $\overline{f}$ is an immersion.

The construction above proves also the last part of the statement about continuity. Q.E.D.

8.1.3 Completeness of affine structure along leaves

Definition 8.8 [Leafwise complete] *The space V, C^∞-filtered space by affine leaves, is* leafwise complete *if the universal cover of every leaf is isomorphic, in the affine category, to the affine space.*

We prove the following

Lemma 8.9 *Let V be a compact space C^∞-filtered by affine leaves. Let E be the vector bundle over V whose fibre at x is the tangent space at x of the leaf $\mathcal{L}_x$. Assume that there exists a one-parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ of homeomorphisms of V such that:*

- *For every leaf $\mathcal{L}_x$, φ_t preserves the leaf $\mathcal{L}_x$ and acts as a one parameter group of translation on $\mathcal{L}_x$, generated by the vector field X.*
- *The induced action of φ_t on the vector bundle $E/\mathbb{R}.X$ is uniformly contracting.*

Then V is leafwise complete.

Proof: For every x in V, let

$$\begin{aligned} O_x &= \{u \in E_x \mid x + u \in \mathcal{L}_x\}, \\ O &= \cup_{x \in V} O_x. \end{aligned}$$

We observe that O is an open subset of E invariant by φ_t. By hypothesis, we have

$$L \subset O.$$

Let L be the line bundle $\mathbb{R}.X$. Since φ_t is contracting on $F = E/L$ and V is compact, L admits a $\phi = \varphi_1$ invariant supplementary F_0. Let us recall the classical proof of this fact. We choose a supplementary F_1 to L. Then $\phi_* F_1$ is the graph of an element ω in $K = F_1^* \otimes L$. We now identify F_1 with F using the projection. Since the action of φ_t is uniformly contracting on F, the action of φ_{-t} is uniformly contracting on $K = F_1^* \otimes L$, hence exponentially contracting by compactness. It follows that the following element of $K = F_1^* \otimes L$ is well defined:

$$\alpha = \sum_{p=-1}^{p=-\infty} (\phi^p)^* \omega.$$

This section α satisfies the *cohomological equation*

$$\phi^* \alpha - \alpha = \omega.$$

This last equation exactly means that the graph F_0 of α is ϕ invariant.

Let $u \in E$. Write $u = v + \lambda X$, with $v \in F_0$. For n large enough, $\phi^n(u)$ belongs to O. Since O is invariant by ϕ, we deduce that $u \in O$. Hence $O = E$ and V is leafwise complete. Q.E.D.

8.2 A Stability Lemma for Anosov Homomorphisms

Let ρ be a homomorphism from $\pi_1(\Sigma)$ to $H(\mathbb{T}) = C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$. We denote by $\overline{\rho}$ the associated representation (by projection) to $\mathrm{Diff}^h(\mathbb{T})$.

The openness of the space of Anosov homomorphisms Hom^* is an immediate consequence of the following Stability Lemma. Moreover, corollaries of this Lemma will enable us to associate to every ∞-Hitchin representation a cross ratio and a spectrum – see Paragraph 9.2.1 and 9.2.2.

Lemma 8.10 [Stability Lemma] *Let ρ_0 be an Anosov homomorphism from $\pi_1(\Sigma)$ to $H(\mathbb{T})$. Then for ρ close enough to ρ_0, there exists a Hölder homeomorphism Φ of J close to the identity which is a C^∞-filtered immersion as well as its inverse intertwining ρ_0 and ρ, that is*

$$\forall \gamma \in \pi_1(\Sigma), \quad \rho_0(\gamma) = \Phi^{-1} \circ \rho(\gamma) \circ \Phi.$$

This Lemma is proved in Paragraph 8.2.2.

8.2.1 Minimal action on the circle

The following Lemma is independent of the rest of the article.

Lemma 8.11 *Let ρ_0 and ρ_1 be two homomorphisms from a group Γ to the group of homeomorphisms of $\mathbb{T}$. Suppose that every element of $\rho_0(\Gamma)$ different from the identity has exactly two fixed points in $\mathbb{T}$, one attractive and one repulsive. Suppose moreover that the fixed points of the nontrivial elements $\rho_0(\Gamma)$ are dense in $\mathbb{T} \times \mathbb{T}$.*

Let f be a continuous map of nonzero degree from $\mathbb{T}$ to $\mathbb{T}$ intertwining ρ_0 and ρ_1. Then f is a homeomorphism.

PROOF: We first prove that f is injective. Let a and b be two distinct points of $\mathbb{T}$ such that $f(a) = f(b) = c$. Let I and J be the connected components of $\mathbb{T} \setminus \{a, b\}$.

Since f has a nonzero degree, either $\mathbb{T}\setminus\{c\} \subset f(I)$ or $\mathbb{T}\setminus\{c\} \subset f(J)$. Assume $\mathbb{T} \setminus \{c\} \subset f(I)$. By the density of fixed point in $\mathbb{T} \times \mathbb{T}$, we can find an element γ in Γ such that the attractive fixed point γ^+ of $\rho_0(\gamma)$ belongs to I and the repulsive point γ^- belongs to J.

We observe that for all n,

$$\mathbb{T} \setminus \{\rho_1(\gamma^n)(c)\} \subset f(\rho_0(\gamma)^n(I)).$$

Therefore, there exist a positive ϵ and two sequences, $\{x_n\}_{n\in\mathbb{N}}$ and$\{y_n\}_{n\in\mathbb{N}}$, such that

- x_n belongs to $\rho_0(\gamma)^n(I)$, y_n belongs to $\rho_0(\gamma)^n(I)$,
- $d(f(x_n), f(y_n)) > \epsilon$.

Since γ^- belongs to J, the sequence $\{\rho_0(\gamma)^n(I)\}$ converges uniformly to γ^+. Hence,

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

The contradiction follows. We have proved that f is injective. Since f has nonzero degree, it is onto. Hence f is a homeomorphism. Q.E.D.

8.2.2 Proof of Lemma 8.10

PROOF: Let ρ_0 be an ∞-Hitchin homomorphism from $\pi_1(\Sigma)$ to $H(\mathbb{T})$. By definition, $P = \rho_0(\pi_1(\Sigma))\backslash J$ is compact. The topological space J is a C^0-manifold which is C^∞-filtered by the fibres of the projection $\pi : J \to \mathbb{T}$ and $\delta : J \to T^*\mathbb{T}$. Observe that $H(\mathbb{T})$ acts by C^∞-filtered maps on J. It follows that the space P is also C^0-manifold which is C^∞-filtered by the images of the leaves of J.

Let now ρ be a homomorphism of $\pi_1(\Sigma)$ close enough to ρ_0. Let $\Gamma = \rho_0(\pi_1(\Sigma))$. According to Theorem 8.7, we obtain a C^∞-filtered immersion Φ from J to J, intertwining ρ and ρ_0, and arbitrarily close to the identity on compact sets provided ρ is close enough to ρ_0. It remains to show that Φ is a homeomorphism.

As a first step, we prove that $\overline{\rho}$ is H-fuchsian. Since Φ is a filtered map, there exists a Hölder map f from $\mathbb{T}$ to $\mathbb{T}$ close to the identity such that

$$\pi \circ \Phi = f \circ \pi.$$

In particular, f intertwines $\overline{\rho}_0$ and $\overline{\rho}$. Since ρ_0 is Anosov, by definition $\overline{\rho}_0$ is an H-Fuchsian representation, and thus satisfies the hypothesis of Lemma 8.11. The degree of f is nonzero since f is close to the identity. By Lemma 8.11, we obtain that f is a homeomorphism. Thus, $\overline{\rho}$ is also H-Fuchsian as we claimed.

The filtered equivariant immersion Φ induces an affine structure on the leaves of P, according to the definition of Paragraph 8.1.1. Due to the compactness of P, we deduce that there exists an action of a one-parameter group ψ_t on J such that,

$$\Phi \circ \psi_t = \varphi_t \circ \Phi,$$

where φ_t is the canonical flow. We observe that ψ_t preserves any given leaf and acts as a one-parameter group of translation on it.

We recall that by definition the canonical flow contracts uniformly the leaves of $\mathcal{F}$ on the quotient $\rho_0(\pi_1(\Sigma))\backslash J$. It follows that the same holds for ψ_t, for Φ close enough to the identity.

By Lemma (8.9), the affine structure induced by ϕ is leafwise complete. In particular, for every x in $\mathbb{T}$, Φ is an affine bijection from $\pi^{-1}\{x\}$ to $\pi^{-1}\{f(x)\}$. Since f is a homeomorphism, we deduce that Φ itself is a homeomorphism, its inverse being also a filtered immersion.

The result now follows. Q.E.D.

9 Cross ratios and properness of the action

We prove in this section

Theorem 9.1 *The action by conjugation of $H(\mathbb{T})/Z(H(\mathbb{T}))$ on Hom_H is proper and the quotient is Hausdorff.*

This is a consequence of the more precise Theorem 9.10. In order to state and prove this last Theorem, we associate to every ∞-Hitchin representation a cross ratio. This is done in the first two paragraphs of this section.

9.1 Corollaries of the Stability Lemma 8.10

Since the space Hom_H is connected, we have

Corollary 9.2 *Let ρ_0 and ρ_1 be two ∞-Hitchin homomorphisms. Then, there exists a Hölder homeomorphism Φ of J which is a filtered immersion as well as its inverse which intertwines ρ_0 and ρ_1, that is, for all γ in $\pi_1(\Sigma)$ $\rho_0(\gamma) = \Phi^{-1} \circ \rho_1(\gamma) \circ \Phi$.*

Let $\partial_\infty \pi_1(\Sigma)^{2*} = \{(x, y) \in \partial_\infty \pi_1(\Sigma)^2 \mid x \neq y\}$. Recall that $H(\mathbb{T})$ acts on $T^*\mathbb{T} = J/Z(H(\mathbb{T}))$. We also prove

Proposition 9.3 *Let ρ be an ∞-Hitchin homomorphism. Then, there exists a unique Hölder homeomorphism*

$$\Theta_\rho : T^*\mathbb{T} \to \partial_\infty \pi_1(\Sigma)^{2*},$$

that intertwines

- *the action of $\rho(\pi_1(\Sigma))$ and the action of $\pi_1(\Sigma)$,*

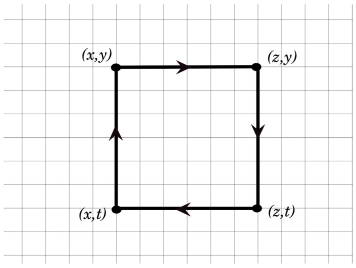

Figure 2: *The curve* $\widehat{q}$

- *the projection from* $T^*\mathbb{T}$ *onto* $\mathbb{T}$ *and the projection on the first factor from* $\partial_\infty\pi_1(\Sigma)^{2*}$ *onto* $\partial_\infty\pi_1(\Sigma)$.

In other words:

- *For all* γ *in* $\pi_1(\Sigma)$, *we have* $\gamma \circ \Theta_\rho = \Theta_\rho \circ \rho(\gamma)$.
- *There exists a Hölder map* f *from* $\mathbb{T}$ *to* $\partial_\infty\pi_1(\Sigma)$ *such that* $\{f(x)\} \times \partial_\infty\pi_1(\Sigma)$ *contains* $\Theta_\rho(T^*_x\mathbb{T})$.

Proof: By Corollary 9.2, it suffices to show this result for the action of a Fuchsian representation. In this case the statement is obvious. Q.e.d.

9.2 Cross ratio, Spectrum and Ghys deformations

In this section, we associate a cross ratio to every ∞-Hitchin representation. We also define a natural class of deformations of these representations. Similar deformations were introduced by E. Ghys in the context of the geodesic flow of hyperbolic surfaces in [13].

9.2.1 Cross ratio associated to ∞-Hitchin representations

Let (a, b, c) be a triple of distinct elements of $\partial_\infty\pi_1(\Sigma)$. We define $[b, c]_a$ to be the closure of the connected component of $\partial_\infty\pi_1(\Sigma) \setminus \{b, c\}$ not containing a. Let $q = (x, y, z, t)$ be a quadruple of elements of $\partial_\infty\pi_1(\Sigma)$. We define $\widehat{q}$ to be following closed curve, embedded in $\partial_\infty\pi_1(\Sigma)^{2*}$ (Figure (2))

$$\widehat{q} = (\{x\} \times [y, t]_x) \cup ([x, z]_t \times \{t\}) \cup (\{z\} \times [y, t]_z) \cup ([x, z]_y \times \{y\}).$$

We choose the orientation on $\widehat{q}$ such that $(x, y), (x, t), (z, t), (z, y)$ are cyclically ordered.

We finally define $\varepsilon_q \in \{-1, 1\}$ so that

$$\begin{cases} \varepsilon_q &= -1, \quad \text{if } (x, y, z, t) \text{ are cyclically ordered,} \\ \varepsilon_{(\sigma(x),\sigma(y),\sigma(z),\sigma(t))} &= \varepsilon(\sigma)\varepsilon_{(x,y,z,t)}. \end{cases}$$

Let ρ be an ∞-Hitchin representation. Let Θ_ρ be the homeomorphism from $T^*\mathbb{T}$ to $\partial_\infty\pi_1(\Sigma)^{2*}$ obtained in Proposition 9.3. Let $\lambda = rd\theta$ be the Liouville form on $T^*\mathbb{T}$ (see Paragraph 6.1.5 .(4) for a definition).

Definition 9.4 [Associated cross ratio] *The* associated cross ratio *to the Hitchin representation ρ is*

$$\mathbb{B}_\rho(x,y,z,t) = \varepsilon_q . \exp\Big(\frac{1}{2}\int_{\Theta_\rho^{-1}(\widehat{q})} \lambda\Big).$$

Remarks:

1. *The cross ratio is well defined and finite.* Indeed $\Theta_\rho^{-1}(\widehat{q})$ is the union of four arcs in $T^*\mathbb{T}$. Two of these arcs project injectively on $\mathbb{T}$, the other two arcs project to a constant. In both cases, the integral of $rd\theta = \lambda$ is well defined and finite on such arcs.

2. *The cross ratio just depends on the action of $\pi_1(\Sigma)$ on $T^*\mathbb{T}$.* Here is another formulation of this observation. Let $\Omega^h(\mathbb{T})$ be the space of Hölder one-forms on $\mathbb{T}$. Note that $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ naturally acts on $T^*\mathbb{T}$. We also have a natural homomorphism

$$d : \left\{ \begin{array}{rcl} C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T}) & \to & \Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T}), \\ (f,\phi) & \mapsto & (df,\phi), \end{array} \right.$$

whose kernel is the canonical flow. Therefore, two homomorphisms ρ_1 and ρ_2 such that $d\circ\rho_1 = d\circ\rho_2$ have the same associated cross ratio. In other words, the cross ratio only depends on the representation as with values in $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$.

9.2.2 Spectrum

Definition 9.5 [ρ-length] *Let ρ be an ∞-Hitchin representation. Let γ be an element in $\pi_1(\Sigma)$. The* ρ-length *of γ – denoted by $\ell_\rho(\gamma)$ – is the positive number t such that*

$$\exists u \in J, \quad \varphi_t(u) = \rho(\gamma)u.$$

Remarks:

1. In other words, $\ell_\rho(\gamma)$ is the length of the periodic orbit of φ_t in $\rho(\pi_1(\Sigma))\backslash J$ freely homotopic to γ. It is clear that $\ell_\rho(\gamma)$ just depends on the conjugacy class of γ.

2. The existence and uniqueness of such a positive number t – or of the previously discussed closed orbit – follow from Corollary 9.2 and the description of standard representations.

Definition 9.6 [Spectrum] *The* marked spectrum *of an ∞-Hitchin representation ρ is the map*

$$\ell_\rho : \gamma \mapsto \ell_\rho(\gamma).$$

Definition 9.7 [Symmetric representation] *A representation is* symmetric *if $\ell_\rho(\gamma) = \ell_\rho(\gamma^{-1})$.*

9.2.3 Coherent representations and Ghys deformations

Definition 9.8 [Coherent representation] *An ∞-Hitchin representation ρ is* coherent *if its marked spectrum coincides with the periods of the associated cross ratio.*

Remarks:

1. Every coherent representation is symmetric.
2. Conversely, we prove in Proposition 10.12 that if $\ell_{\mathbb{B}_\rho}$ is the period of the cross ratio associated to the representation ρ, then

   $$\ell_{\mathbb{B}_\rho}(\gamma) = \frac{1}{2}(\ell_\rho(\gamma) + \ell_\rho(\gamma^{-1})).$$

   Consequently, every symmetric representation is coherent.
3. Theorems 13.1 and 12.1 provide numerous examples of coherent representations.

Definition 9.9 [Ghys deformation] *Let ρ be a Hitchin representation. Let ω be a nontrivial element in $H^1(\Gamma, \mathbb{R})$. We identify the centre of $H(\mathbb{T})$ with the canonical flow φ_t. Let ρ^ω be defined by*

$$\rho^\omega(\gamma) = \varphi_{\omega(\gamma)}.\rho(\gamma).$$

We say that ρ^ω is a Ghys deformation *of ρ if the representation ρ^ω is Hitchin.*

Remarks:

1. For ω small enough, ρ^ω is a Ghys deformation. Indeed, for ω small enough, ρ^ω is Hitchin by the Openness Theorem 8.1.
2. We observe that ρ^ω and ρ have the same associated cross ratio since they have the same action on $T^*\mathbb{T}$. However, one immediately checks that

   $$\ell_{\rho^\omega}(\gamma) = \ell_\rho(\gamma) + \omega(\gamma). \qquad (26)$$

3. The previous remarks provide many examples of representations with the same cross ratio but different marked spectra.

9.3 Action of $H(\mathbb{T})$ on Hom_H

We prove in this section that $H(\mathbb{T})/Z(H(\mathbb{T}))$ acts properly on Hom_H. This is a consequence of the following result.

Theorem 9.10 *The group $H(\mathbb{T})/Z(H(\mathbb{T}))$ acts properly with a Hausdorff quotient on the set of ∞-Hitchin homomorphisms.*

Moreover, if two ∞-Hitchin homomorphisms ρ_0 and ρ_1 have the same associated cross ratio, then there exists $\omega \in H^1(\pi_1(\Sigma))$ such that ρ_0 and ρ_1^ω are conjugated by some element in $H(\mathbb{T})$.

Consequently two representations with the same cross ratio and the same spectrum are equal.

Remark: Two representations with the same spectrum are not necessarily identical (see remark after Theorem 13.1).

9.3.1 Characterisation of $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$

Let $\Omega^h(\mathbb{T})$ be the space of Hölder one-form on $\mathbb{T}$. We observe first that $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ acts naturally on $T^*\mathbb{T}$ and preserves the area. Conversely

Lemma 9.11 *Let G be an area preserving Hölder homeomorphism of $T^*\mathbb{T}$. Assume G is above a homeomorphism f of $\mathbb{T}$. Then G belongs to $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$.*

Proof: This Lemma is classical for all cotangent spaces. We give the proof for completeness. We use the coordinates (r,θ) on $T^*\mathbb{T}$. By hypothesis,

$$G(r,\theta) = (g(r,\theta), f(\theta)).$$

Since G preserves the area, we obtain

$$\begin{aligned}(r_0 - r_1)(\theta_0 - \theta_1) &= \int_{f(\theta_0)}^{f(\theta_1)} \left(\int_{g(r_0,\theta)}^{g(r_1,\theta)} dr\right) d\theta \\ &= \int_{f(\theta_0)}^{f(\theta_1)} (g(r_0,\theta) - g(r_1,\theta)) d\theta. \end{aligned} \tag{27}$$

Hence $g(r,\theta)$ is affine in r:

$$g(r,\theta) = \omega(\theta) + r\beta(\theta),$$

where $\omega(\theta)$ and β are Hölder. Since G is a homeomorphism, we observe that for all θ, $\beta(\theta) \neq 0$ and f is an homeomorphism. By Equation 27, we obtain that

$$f^{-1}(\theta_0) - f^{-1}(\theta) = \int_{\theta_0}^{\theta_1} \beta(\theta) d\theta.$$

It follows that f^{-1} is in $C^{1,h}(\mathbb{T})$ with $df^{-1} = \beta$. Since β never vanishes, f is actually is a diffeomorphism, and G belongs to $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$. Q.e.d.

9.3.2 First step: Conjugation in $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$

We recall that $H(\mathbb{T})$ acts on $T^*\mathbb{T}$. We denote by $\alpha \to \dot{\alpha}$ the projection from $H(\mathbb{T})$ to $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$. We first prove

Proposition 9.12 *Let ρ_0 and ρ_1 be two ∞-Hitchin representations with the same associated cross ratio. Then there exists a unique element $H = H^{\rho_0,\rho_1}$ in $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ which intertwines $\dot{\rho}_0$ and $\dot{\rho}_1$, that is*

$$\forall \gamma \in \pi_1(\Sigma), \forall y \in T^*\mathbb{T}, \quad H(\dot{\rho}_0(\gamma) \cdot y) = \dot{\rho}_1(\gamma) \cdot H(y).$$

Moreover, for any representation ρ, for any neighbourhood V of the identity in $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$, there exists a neighbourhood U of ρ, such that if two representations ρ_1 and ρ_0 in U have the same cross ratio, then H^{ρ_0,ρ_1} belongs to V.

PROOF: Applying Proposition 9.3 twice, we obtain a unique Hölder homeomorphism H of $T^*\mathbb{T}$ and a homeomorphism f of $\mathbb{T}$ such that H is above f and intertwines $\dot{\rho}_0$ and $\dot{\rho}_1$.

If ρ_0 and ρ_1 have the same associated cross ratio, then H preserves the area. By Lemma 9.11, H belongs to $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$. The first part of the proposition now follows. The second part follows from Lemma 8.10. Q.E.D.

9.3.3 Second step: from $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ to $H(\mathbb{T})$

We prove the following Lemma

Lemma 9.13 *Let $\phi : \gamma \to \phi_\gamma$ be a representation from $\pi_1(\Sigma)$ to $\mathrm{Diff}^h(\mathbb{T})$ with nonzero Euler class. Let α be a continuous one-form. Let $f : \gamma \to f_\gamma$ be a map from $\pi_1(\Sigma)$ to the space $C^1(\mathbb{T})$ of C^1-functions on $\mathbb{T}$. Assume that*

$$\begin{aligned} \phi_\gamma^*(f_\eta) + f_\gamma &= f_{\eta\gamma}, \\ \phi_\gamma^*(\alpha) - \alpha &= df_\gamma. \end{aligned}$$

Then α is exact.

PROOF: Let $\kappa := \int_{\mathbb{T}} \alpha$. We aim to prove that $\kappa = 0$. We write $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We choose a lift all our data to $\mathbb{R}$, and use $\tilde{u}$ to describe a lift of u. In particular, there exists an element c in $H^2(\pi_1(\Sigma), \mathbb{Z})$ such that

$$\forall x \in \mathbb{R}, \quad c(\gamma, \eta) = \tilde{\phi}_{\gamma\eta}(x) - \tilde{\phi}_\gamma \circ \tilde{\phi}_\eta(x) \in \mathbb{Z}.$$

By definition, c is a representative of the Euler class of the representation ϕ. Let $h \in C^1(\mathbb{R})$ such that

$$\tilde{\alpha} = dh.$$

By definition

$$\forall m \in \mathbb{Z}, \quad h(x + m) = h(x) + m\kappa.$$

Observe now that for all γ

$$d(\tilde{\phi}_\gamma^*(h) - h - \tilde{f}_\gamma) = \tilde{\phi}_\gamma^*\tilde{\alpha} - \tilde{\alpha} - d\tilde{f}_\gamma = 0.$$

Therefore, there exists

$$\eta : \pi_1(\Sigma) \to \mathbb{R},$$

such that

$$\tilde{\phi}_\gamma^*(h) - h = \tilde{f}_\gamma + \eta(\gamma).$$

Finally

$$\begin{aligned}
&c(\gamma,\eta)\kappa \\
&\quad = h \circ \tilde{\phi}_{\gamma\eta} - h \circ \tilde{\phi}_\gamma \circ \phi_\eta \\
&\quad = \tilde{\phi}_{\gamma\eta}^*(h) - \tilde{\phi}_\eta^*\tilde{\phi}_\gamma^*(h) \\
&\quad = (\tilde{\phi}_{\gamma\eta}^*(h) - h) - (\tilde{\phi}_\eta^*\tilde{\phi}_\gamma^*(h) - \tilde{\phi}_\eta^*(h)) - (\tilde{\phi}_\eta^*(h) - h) \\
&\quad = \tilde{f}_{\gamma\eta} - \tilde{\phi}_\eta^*\tilde{f}_\gamma - \tilde{f}_\eta + \eta(\gamma\eta) - \eta(\gamma) - \eta(\eta) \\
&\quad = \eta(\gamma\eta) - \eta(\gamma) - \eta(\eta).
\end{aligned}$$

Hence κc is the coboundary of η. Since the Euler class is nonzero in cohomology by hypothesis, we obtain that $\kappa = 0$, hence α is exact. Q.E.D.

9.3.4 Proof of Theorem 9.10

We first prove

Proposition 9.14 *If two representations have the same cross ratio then, after a Ghys deformation, they are conjugated by an element of $H(\mathbb{T})$.*

PROOF: Let ρ_0 and ρ_1 be two Hitchin representations with the same cross ratio. Write

$$\rho_i(\gamma) = (f_\gamma^i, \phi^i(\gamma)),$$

with $f_\gamma^i \in C^{1,h}(\mathbb{T})$ and $\phi^i(\gamma) \in \mathrm{Diff}^h(\mathbb{T})$. By Proposition 9.12, the representations $\dot{\rho}_0$ and $\dot{\rho}_1$ – with values in $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ – are conjugated in $\Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ by some element $H = (\alpha, F)$, where $F \in \mathrm{Diff}^h(\mathbb{T})$.

In particular, after conjugating by $(0, F) \in C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T}) = H(\mathbb{T})$, we may assume that F is the identity and thus

$$\phi_\gamma^0 = \phi_\gamma^1 := \phi_\gamma.$$

Since ρ_i is a representation, we have for $i = 0, 1$

$$\phi_\eta^*(f_\eta^i) + f_\gamma^i = f_{\eta\gamma}^i.$$

Since $H = (\alpha, \mathrm{id})$ intertwines $\dot{\rho}_0$ and $\dot{\rho}_1$, we also have

$$\phi_\gamma^*\alpha - \alpha = df_\gamma^1 - df_\gamma^0.$$

Lemma 9.13 applied to $f_\gamma = f^1_\gamma - f^0_\gamma$ yields that $\alpha = dg$ for some function g in $C^{1,h}(\mathbb{T})$.

Conjugating by (g, id), we may now as well assume that $\alpha = 0$. It follows that there exists a homomorphism

$$\omega : \pi_1(\Sigma) \to \mathbb{R},$$

such that

$$f^1_\gamma - f^0_\gamma = \omega(\gamma).$$

This exactly means that $\rho_0 = \rho_1^\omega$. Q.E.D.

This proves that two representations with the same cross ratio are conjugated after a Ghys deformation. Consequently two representations with the same cross ratio and spectrum are conjugated. Indeed if ρ and ρ^ω have the same spectrum and cross ratio, then by Formula (26), for all $\gamma \in \pi_1(\Sigma)$, $\omega(\gamma) = 0$. Hence $\rho = \rho^\omega$.

Finally, the proof of properness goes as follows. Suppose that we have a sequence of representations $\{\rho_n\}_{n\in\mathbb{N}}$ converging to ρ_0, a sequence of elements $\{\psi_n\}_{n\in\mathbb{N}}$ in $H(\mathbb{T})$ such that $\{\psi_n^{-1} \circ \rho_n \circ \psi_n\}_{n\in\mathbb{N}}$ converges to ρ_1. Let d be the homomorphism $H(\mathbb{T}) \to \Omega^h(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$. We aim to prove that $\{d\psi_n\}_{n\in\mathbb{N}}$ converges.

Since ρ_n and $\psi_n^{-1} \circ \rho_n \circ \psi_n$ have the same spectrum and cross ratio, it follows that ρ_1 and ρ_0 have the same spectrum and cross ratio. It follows from the first part of the proof that there exists F such that

$$\rho_0 = F^{-1} \circ \rho_1 \circ F.$$

We may therefore assume that $\rho_0 = \rho_1$.

By the second part of Proposition 9.12, $\{d(\psi_n)\}_{n\in\mathbb{N}}$ converges to the identity. The properness is now proved.

10 A characterisation of $C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$.

We aim to characterise in a geometric way the action of group $H(\mathbb{T}) = C^{1,h}(\mathbb{T}) \rtimes \mathrm{Diff}^h(\mathbb{T})$ generalising Proposition 6.3. Roughly speaking, we will describe it as a subgroup of the central extension of "exact symplectic homeomorphisms" of the Annulus (i.e. $T^*\mathbb{T}$). The notion of exact symplectic homeomorphism does not make sense in an obvious way, but some homeomorphisms may be coined as "exact symplectic" as we shall see.

We shall define in this section *π-exact symplectomorphism of $T^*\mathbb{T}$* which are characterised by several equivalent properties (see Proposition 10.9).

Using this notion, we can state our main result

Lemma 10.1 *A Hölder homeomorphism of J belongs to $H(\mathbb{T})$ if and only if*

1. *it preserves the foliation $\mathcal{F}$,*
2. *it commutes with the canonical flow,*
3. *it is above a π-exact symplectic homeomorphism of $T^*\mathbb{T}$.*

The aim of the following sections is

- to recall basic facts about symplectic and Hamiltonian actions on the Annulus,
- to extend this construction to a situation with less regularity, and in particular to give a "symplectic" interpretation of the action of $H(\mathbb{T})$ on $T^*\mathbb{T}$.

10.1 C^∞-exact symplectomorphisms

Let L be an orientable real line bundle over a manifold M, equipped with a connection ∇ whose curvature form ω is symplectic.

- For any curve γ joining x and y, let

$$\mathrm{Hol}(\gamma) : L_x \to L_y,$$

  be the holonomy of ∇ along γ. If γ is a closed curve – that is x equals y – we identify $GL(L_x)$ with the multiplicative group $\mathbb{R} \setminus \{0\}$ and consider $\mathrm{Hol}(\gamma)$ as a real number.

- Let ν be a nonzero section of L. Let λ_ν be the primitive of ω defined by

$$\nabla_X \nu = \lambda_\nu(X)\nu.$$

  If γ is a smooth closed curve, then

$$\mathrm{Hol}(\gamma) = e^{\int_\gamma \lambda_\nu}.$$

Definition 10.2 [Exact symplectomorphisms] *A symplectic diffeomorphism of M is* ∇-exact[2] *if for any closed curve*

$$\mathrm{Hol}(\sigma) = \mathrm{Hol}(\phi(\sigma)). \tag{28}$$

When the action of ϕ on $H^1(M)$ is non trivial the notion actually depends on the choice of ∇. We shall however usually say *exact* instead of ∇-exact in order to simplify our exposition.

The following proposition clarifies this last condition

Proposition 10.3 *Let ϕ be a symplectic diffeomorphism. The following conditions are equivalent:*

- *For any curve σ,* $\mathrm{Hol}(\sigma) = \mathrm{Hol}(\phi(\sigma))$.

[2]This definition is *ad hoc*: exact symplectomorphisms exist in a greater generality.

- *For any non zero section ν of L, $\phi^*\lambda_\nu - \lambda_\nu$ is exact.*
- *The action of ϕ lifts to a connection preserving action on L.*

Let now $M = T^*\mathbb{T}$. Let $\sigma_0 : \mathbb{T} \to T^*\mathbb{T}$ be the zero section considered as a curve in M. Then the map

$$\phi \mapsto \mathrm{Hol}(\phi(\sigma_0)),$$

is a group homomorphism from the group of symplectomorphisms to the multiplicative real numbers whose kernel is the group of exact symplectomorphism.

PROOF: We prove for an exact symplectic diffeomorphism ϕ, the action lift to an action on the line bundle L. We choose – on purpose – a more complicated proof. This proof however has the advantage of using very little regularity of ϕ, so that we can reproduce it later in another context.

We define a lift $\hat{\phi}$ in the following way. We first choose a base point x_0 in $T^*\mathbb{T}$ and η a curve joining x_0 to $\phi(x_0)$. For any point x in $T^*\mathbb{T}$, we choose a curve γ joining x_0 to x, then we define

$$\hat{\Psi}_x = \mathrm{Hol}(\phi(\gamma))\mathrm{Hol}(\eta)\mathrm{Hol}(\gamma)^{-1} : L_x \to L_{\phi(x)}.$$

Since for every closed curve σ, by definition of exactness

$$\mathrm{Hol}(\sigma) = \mathrm{Hol}(\phi(\sigma)),$$

the linear map $\hat{\Psi}_x$ is independent of the choice of γ. Finally we define

$$\hat{\phi}(x, u) = (\phi(x), \hat{\Psi}_x(u)).$$

By construction, $\hat{\phi}$ preserve the parallel transport along any curve γ, that is

$$\hat{\phi}(\mathrm{Hol}(\gamma)u) = \mathrm{Hol}(\phi(\gamma)).\hat{\phi}(u).$$

This is what we wanted to prove. The other assertions of the proposition are obvious. Q.E.D.

10.2 π-Symplectic Homeomorphisms and π-curves

The group $\mathrm{Diff}(\mathbb{T})$ of C^1-diffeomorphisms of the circle acts by area preserving homeomorphisms on $T^*\mathbb{T}$. Let $\Omega^1(\mathbb{T})$ be the vector space of continuous one-forms on $\mathbb{T}$. This group also acts on $T^*\mathbb{T}$ by area preserving homeomorphisms in the following way

$$f d\theta.(\theta, t) = (\theta, t + f(\theta)).$$

We observe that $\mathrm{Diff}(\mathbb{T})$ normalises this action.

Definition 10.4 [π-SYMPLECTIC HOMEOMORPHISM] *The group of* π-symplectic homeomorphisms *is*

$$\mathrm{Symp}^\pi = \Omega^1(\mathbb{T}) \rtimes \mathrm{Diff}(\mathbb{T}).$$

Here is an immediate characterisation of π-symplectic homeomorphisms. The proof is identical to that of Lemma 9.11

Proposition 10.5 *An area preserving homeomorphism of $T^*\mathbb{T}$ is π-symplectic if and only if it is above a diffeomorphism of $\mathbb{T}$.*

In order to extend the notion of exact diffeomorphisms, using Proposition 10.3 as a definition, we need to have a class a less regular curves for which we can compute a holonomy.

Definition 10.6 [π-CURVES AND THEIR HOLONOMIES] *A* π-curve *is a continuous curve $c = (c_\theta, c_r)$ with values in $T^*\mathbb{T}$ such that c_θ is C^1. We define*

$$\int_c \lambda = \int c_r dc_\theta.$$

The holonomy *of a closed π-curve is*

$$\mathrm{Hol}(c) = e^{\int_c \lambda}.$$

We collect in the following proposition a few elementary facts.

Proposition 10.7 *1. The image of a π-curve by a π-symplectic homeomorphism is a π-curve,*

2. Let $\phi = (\alpha, \psi)$ be a π-symplectic homeomorphism. Let c be a closed π-curve which projects isomorphically on $\mathbb{T}$, then

$$\mathrm{Hol}(\phi(c)) = \mathrm{Hol}(c) e^{\int_{\mathbb{T}} \alpha}.$$

Definition 10.8 [π-EXACT SYMPLECTIC HOMEOMORPHISM] *The group of π-*exact symplectomorphisms *is*

$$\mathrm{Exact}^\pi = (C^1(\mathbb{T})/\mathbb{R}) \rtimes \mathrm{Diff}(\mathbb{T}),$$

where $C^1(\mathbb{T})/\mathbb{R}$ is identified with the space of exact continuous one-forms on $\mathbb{T}$.

By the above remarks, we deduce immediately, reproducing the proof of Proposition 10.3

Proposition 10.9 *1. The group of π-exact symplectomorphisms is the group of π-symplectic homomorphisms ϕ such that for any closed π-curve c,*

$$\mathrm{Hol}(c) = \mathrm{Hol}(\phi(c)).$$

2. Let $x_0 \in T^*\mathbb{T}$. The action of any π-exact symplectomorphism ϕ lifts to an action of a homeomorphism $\hat{\phi}$ on L such that for any π-curve c starting from x_0, for any $u \in L_{x_0}$

$$\hat{\phi}(\mathrm{Hol}(c)u) = \mathrm{Hol}(\phi(c)).\hat{\phi}(u). \tag{29}$$

3. *Furthermore, $\hat{\phi}$ is determined by Equation (29) up to a multiplicative constant. As a consequence, there exists an element (α, ψ) of $C^1(\mathbb{T}) \rtimes \mathrm{Diff}(\mathbb{T})$ such that the induced action of $\hat{\phi}$ on J, identified with the frame bundle of L is (α, ψ). We observe that α is defined uniquely up to an additive constant.*

4. *Finally, if ϕ is Hölder, so is $\hat{\phi}$.*

From this we obtain immediately Lemma 10.1.

10.3 Width

We generalise the notion of width for a π-exact symplectomorphism ϕ in the following obvious way.

Definition 10.10 [Action difference] *Let x and y two fixed points of ϕ. Let c be a π-curve joining x to y. If q is a curve, we denote by $\overline{q}$ the curve with the opposite orientation. The* action difference *of x and y is*

$$\Delta(\phi; x, y) = \mathrm{Hol}(c \cup \overline{\phi(c)}) = e^{\int_c \lambda - \int_{\phi(c)} \lambda}.$$

By Proposition 10.9, this quantity does not depend on c. Moreover, let $\hat{\phi}$ be a lift of the action of ϕ on L. Then, for any fixed point z of ϕ, $\hat{\phi}(z)$ is an element of $GL(L_z)$ – identified with $\mathbb{R} \setminus \{0\}$ – and for any two fixed points x and y of ϕ

$$\Delta(\phi; x, y) = \hat{\phi}(x)/\hat{\phi}(y).$$

Definition 10.11 [Action difference] *The* width *of ϕ is*

$$w(\phi) = \sup_{x,y \text{ fixed points}} \Delta(\phi; x, y).$$

This quantity is invariant by conjugation under π-symplectic homeomorphisms. We now relate it to the period of some cross ratio.

Proposition 10.12 *Let ρ be an ∞-Hitchin homomorphism from $\pi_1(\Sigma)$ to $H(\mathbb{T})$. Let P be the projection*

$$P : H(\mathbb{T}) \to \mathrm{Exact}^\pi.$$

Let $\dot{\rho} = P \circ \rho$. Let ℓ_ρ be the spectrum of ρ (cf Paragraph 9.2.1) and $\mathbb{B}_\rho$ be its associated cross ratio (cf Paragraph 9.2.2) with periods $\ell_{\mathbb{B}_\rho}$. Then

$$\log(w(\dot{\rho}(\gamma)) = \ell_\rho(\gamma) + \ell_\rho(\gamma^{-1}) = 2\ell_{\mathbb{B}_\rho}(\gamma).$$

Proof: Let γ be a non trivial element of $\pi_1(\Sigma)$. Let w be a point in J such that

$$\varphi_{l_\rho(\gamma)}(w) = \rho(\gamma)(w).$$

The projection $v = \delta(w)$ is a fixed point of $\dot{\rho}(\gamma)$.

Since L is a bundle associated to J, it follows that the associated action of $\rho(\gamma)$ on L is a lift $\widehat{\dot{\rho}(\gamma)}$ of $\dot{\rho}(\gamma)$. Then

$$\widehat{\dot{\rho}(\gamma)}(v) = e^{\ell_\rho(\gamma)}.$$

Since $\dot{\rho}(\gamma)$ has precisely two fixed points v and u, we obtain that

$$\log(w(\dot{\rho}(\gamma)) = \ell_\rho(\gamma) + \ell_\rho(\gamma^{-1}).$$

We still have to identify the width with the period of the associated cross ratio. This requires some construction. Let q be the quadruple $(\gamma_+, y, \gamma_-, \gamma(y))$ of points of $\partial_\infty \pi_1(\Sigma)$, where γ_+ and γ_- are respectively the attractive and repulsive fixed points of γ and y is any point different to γ^+ and γ^-.

According to Proposition 9.3, there exists a $\pi_1(\Sigma)$-equivariant map Θ from $T^*\mathbb{T}$ to $\partial_\infty \pi_1(\Sigma)^{2*}$ which identifies the fibres of $T^*\mathbb{T} \to \mathbb{T}$ to the first factor. Then, there exists a π-curve C such that $\Theta(C) = \widehat{q}$ and by definition of the associated cross ratio $\mathbb{B}_\rho$, we have

$$2\ell_{\mathbb{B}_\rho}(\gamma) = 2\log(|\mathbb{B}_\rho(\gamma_+, y, \gamma_-, \gamma(y))|) \quad = \quad \int_C r d\theta. \tag{30}$$

Let us describe C. Let t and s be the points in $T^*\mathbb{T}$ such that $\Theta_\rho(t) = (\gamma^+, y)$ and $\Theta(s) = (\gamma^-, y)$ respectively. We denote by $[a, b]$ the arc joining a and b along that fibre, whenever a and b belong to the same fibre of $T^*\mathbb{T}$. Using this notation, we can write

$$C = c \cup [t, \dot{\rho}(\gamma)(t)] \cup \overline{\dot{\rho}(\gamma)(c)} \cup [\rho(\gamma)(s), s]),$$

where c is a π-curve. Finally let

$$\tilde{c} = [v, t] \cup c \cup [s, u],$$

where $v = \Theta(\gamma^+, \gamma^-)$ and $u = \Theta(\gamma^-, \gamma^+)$ are the two fixed points of $\dot{\rho}(\gamma)$ – See Figure (3).

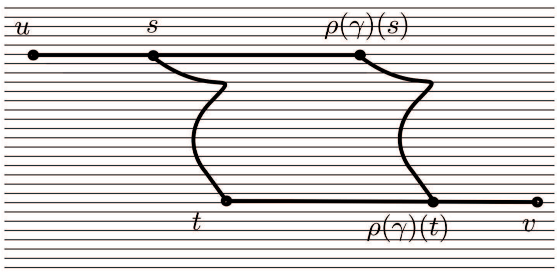

Figure 3: The π-curves C and $\tilde{c}$

We have

$$\begin{aligned}
\log(w(\dot\rho(\gamma))) &= \log(\mathrm{Hol}(\tilde c \cup \overline{\dot\rho(\gamma)(\tilde c)})\\
&= \log(\mathrm{Hol}(c \cup [t, \dot\rho(\gamma)(t)] \cup \overline{\dot\rho(\gamma)(c)} \cup [\rho(\gamma)(s), s]))\\
&= \int_C r d\theta\\
&= 2\ell_{\mathbb{B}_\rho}(\gamma).
\end{aligned}$$

The Assertion follows. Q.E.D.

11 A Conjugation Theorem

We use in this section the following setting and notations.

1. Let κ be a Hölder homeomorphism of $\mathbb{T}$. Let
$$D_\kappa = \{(s,t) \in \mathbb{T}\times\mathbb{T} \mid \kappa(s) \neq t\}.$$
2. Let $p = (p_1, p_2) : M \to D_\kappa$ be a principal $\mathbb{R}$-bundle over D_κ equipped with a connection ∇. Let ω be the curvature of ∇. Let f be such that $\omega = f(s,t)ds \wedge dt$. We suppose that f is positive and Hölder.
3. For all $s \in \mathbb{T}$, we denote by $c_s : t \mapsto (s,t)$ the curve in D_κ whose first factor is constant.
4. Let $\mathcal{L}$ be the one-dimensional foliation of M by the ∇-horizontal sections of M along the curves c_s.

Definition 11.1 [Anosov Homomorphism] *A homomorphism ρ_0 from $\pi_1(\Sigma)$ to the group $\mathrm{Diff}^{1,h}(M)$ of C^1-diffeomorphisms of M with Hölder derivatives is* Anosov *if*

1. *the action of $\rho_0(\pi_1(\Sigma))$ preserves ∇,*
2. *the quotient $\rho_0(\pi_1(\Sigma))\backslash M$ is compact,*
3. *the $\mathbb{R}$-action on $\rho_0(\pi_1(\Sigma))\backslash M$ contracts uniformly the leaves of $\mathcal{L}$ (*cf *Definition 7.3),*
4. *there exist two H-Fuchsian homomorphisms ρ_1 and ρ_2 from $\pi_1(\Sigma)$ to $C^{1,h}(\mathbb{T})$, such that*
   - *$\rho_2 = \kappa^{-1}\rho_1\kappa$,*
   - *$\forall \gamma \in \pi_1(\Sigma), u \in M, \quad p(\rho_0(\gamma)u) = (\rho_1(\gamma)(p_1(u)), \rho_2(\gamma)(p_2(u)))$.*

Our main result is that every Anosov representation on M is conjugated to an Anosov representation on J. We define more precisely what we mean by conjugation

Definition 11.2 [ANOSOV CONJUGATED] *An Anosov homomorphism ρ_0 from $\pi_1(\Sigma)$ to $\mathrm{Diff}^{1,h}(M)$ is* Anosov conjugated *to a homomorphism ρ from $\pi_1(\Sigma)$ to $H(\mathbb{T})$ by an homeomorphism $\hat{\psi}$ from M to J, if:*

- *The homeomorphism $\hat{\psi}$ intertwines the homomorphisms ρ and ρ_0, the foliations $\mathcal{L}$ and $\mathcal{F}$ as well as the $\mathbb{R}$-actions on M and J.*
- *There exists an area preserving homeomorphism ψ from D_κ to $T^*\mathbb{T}$ such that $\delta \circ \hat{\psi} = \psi \circ p$.*

REMARKS:

1. The representation ρ with values in $H(\mathbb{T})$ is Anosov in the sense of Definition 7.4.
2. As a consequence of the next result, $\hat{\psi}$ is unique up to conjugation by elements of $H(\mathbb{T})$.
3. If ρ_0 is Anosov conjugated to ρ_a and ρ_b, then ρ_a and ρ_b are conjugated by a π-symplectic homeomorphism.

Theorem 11.3 [CONJUGATION THEOREM] *Every Anosov homomorphism ρ_0 from $\pi_1(\Sigma)$ to $\mathrm{Diff}^{1,h}(M)$ is Anosov conjugated to an Anosov homomorphism ρ from $\pi_1(\Sigma)$ to $H(\mathbb{T})$ by a homeomorphism $\hat{\psi}$.*

Moreover, $\hat{\psi}$ and ρ are unique up to conjugation by an element of $H(\mathbb{T})$.

Finally, if κ, ∇ depends continuously on a parameter, then we can choose $\hat{\psi}$ to depend continuously on this parameter.

To simplify the notations, we fix a trivialisation of $M = \mathbb{R} \times D_\kappa$ so that the sections $U_{(s,u)} : t \mapsto (u, s, t)$ are horizontal along c_s. Let

$$
\begin{array}{rcl}
\pi & : & \left\{ \begin{array}{rcl} M & \to & \mathbb{T}, \\ (u,s,t) & \mapsto & t, \end{array} \right. \\
p & : & \left\{ \begin{array}{rcl} M & \to & D_\kappa, \\ (u,s,t) & \mapsto & (s,t). \end{array} \right.
\end{array}
$$

We also observe that the foliation $\mathcal{L}$ is the foliation by the fibres of the projection $(u,s,t) \to (u,s)$.

11.1 Contracting the leaves (bis)

For later use, using the above notations, we prove the following easy result which allows us to restate Condition (3) of Theorem 11.3.

Proposition 11.4 *Let ρ_0 be a faithful homomorphism of $\pi_1(\Sigma)$ to $\mathrm{Diff}^{1,h}(M)$ such that $\rho_0(\pi_1(\Sigma))\backslash M$ is compact. Then the following conditions (2) and (2') are equivalent*

(2) The action of $\mathbb{R}$ on $\rho_0(\pi_1(\Sigma))\backslash M$ contracts the leaves of $\mathcal{L}$.

(2') Let $\{t\}_{m\in\mathbb{N}}$ be a sequence of real numbers going to $+\infty$. Let $\{\gamma\}_{m\in\mathbb{N}}$ be a sequence of elements of $\pi_1(\Sigma)$. Let (u,s,t) be an element of M such that

$$\{\rho_0(\gamma_m)(u+t_m,s,t)\}_{m\in\mathbb{N}},$$

converges to (u_0,s_0,t_0) in M. Then for all w in $\mathbb{T}$, with $w\neq\kappa(s)$, the sequence $\{\rho_0(\gamma_m)(s,w)\}_{m\in\mathbb{N}}$ converges to (s_0,t_0).

Proof: To simplify the proof, we omit ρ_0. Let Π be the projection of M on $\pi_1(\Sigma)\backslash M$. To say that the $\mathbb{R}$ action on M contracts the leaves of $\mathcal{L}$ is to say that

$$\lim_{h\to\infty} d(\Pi(u+h,s,t),\Pi(u+h,s,w))=0.$$

Assume Condition (2'). Let $\{t\}_{m\in\mathbb{N}}$ be a sequence of real numbers going to $+\infty$. Let $\{\gamma\}_{m\in\mathbb{N}}$ be a sequence of elements of $\pi_1(\Sigma)$ such that after extracting a subsequence – since $\pi_1(\Sigma)\backslash M$ is compact – the sequence

$$\{\gamma_m(u+t_m,s,t)\}_{m\in\mathbb{N}},$$

converges to (u_0,s_0,t_0) in M. Then by hypothesis,

$$\{\gamma_m(s,w)\}_{m\in\mathbb{N}},$$

converges to (s_0,t_0). It follows that

$$\{\gamma_m(u+t_m,s,w)\}_{m\in\mathbb{N}},$$

converges to (u_0,s_0,w) in M. Hence

$$\lim_{h\to\infty} d(\Pi(u+h,s,t),\Pi(u+h,s,w))=0.$$

The converse relation is a similar yoga. Q.E.D.

11.2 A Conjugation Lemma

We prove a preliminary result. Let $\kappa:\mathbb{T}\to\mathbb{T}$ and D_κ be as before. Let ds (respectively dt) be the Lebesgue probability measure on the first (respectively second) factor of $\mathbb{T}\times\mathbb{T}$.

Definition 11.5 [κ-INFINITE] *A positive continuous function f defined on D_κ is* κ-infinite *if for all s,t such that $\kappa(s)\neq t$*

$$\int_t^{\kappa(s)} f(s,u)du=\int_{\kappa(s)}^t f(s,u)du=\infty. \tag{31}$$

Lemma 11.6 *Assume that f is κ-infinite and Hölder. Then, there exists a Hölder homeomorphism ψ from D_κ to $T^*\mathbb{T}$ such that*

- *$\psi_*\omega = d\theta\wedge dr$, where $\omega = f(s,t)ds\wedge dt$ and where $d\theta\wedge dr$ is the canonical symplectic form of $T^*\mathbb{T}$.*
- *$\pi\circ\psi = \pi_D$, where π_D be the projection from D_κ to the first $\mathbb{T}$ factor,*
- *if c is a C^1-curve in D_κ, then $\psi(c)$ is a π-curve. Furthermore if c is a closed curve then*
$$e^{\int_c \lambda} = \mathrm{Hol}(\psi(c)).$$
- *For any two elements γ_1, γ_2 of $\mathrm{Diff}^{1,h}(\mathbb{T})$ such that $\gamma = (\gamma_1,\gamma_2)$ preserves the 2-form ω then*
$$\psi\circ\gamma\circ\psi^{-1},$$
is Hölder and is π-symplectic above γ_1.

The homeomorphism ψ is unique up to right composition with a π-exact symplectomorphism. Finally ψ depends continuously on κ and f.

Proof: The uniqueness part of this statement follows from the characterisation of π-exact symplectomorphisms given in Proposition 10.5. The proof of the existence is completely explicit. Let g be a C^1-diffeomorphism of the circle $\mathbb{T}$ such that $\forall s, g(s)\neq\kappa(s)$. We consider

$$\psi : \left\{ \begin{array}{ccc} D_\kappa & \rightarrow & T^*\mathbb{T} = \mathbb{T}\times\mathbb{R}, \\ (s,t) & \mapsto & (s, \int_{g(s)}^t f(s,u)du). \end{array}\right.$$

It is immediate to check that

- $\psi_*\omega = d\theta\wedge dr$,
- $\pi\circ\psi = \pi_D$.

We also observe that ψ is a Hölder map and a homeomorphism. We now prove that ψ^{-1} is Hölder. We have

$$\psi^{-1}(s,u) = (s,\alpha(s,u)),$$

where

$$\int_{g(s)}^{\alpha(s,u)} f(s,w)dw = u.$$

It is enough to prove that α is Hölder. We work locally so that f is bounded from below by a positive constant k. Firstly,

$$|\alpha(s,v)-\alpha(s,u)| \leq \frac{1}{k}|\int_{\alpha(s,v)}^{\alpha(s,u)} f(s,w)dw| = \frac{1}{k}|u-v|.$$

This proves that α is Hölder with respect to the second variable. Let us take care of the first variable. We have by definition of α, for all s and t

$$\int_{g(s)}^{\alpha(s,u)} f(s,w)dw = u = \int_{g(t)}^{\alpha(t,u)} f(t,w)dw.$$

Hence,

$$\int_{\alpha(s,u)}^{\alpha(t,u)} f(t,w)dw = \int_{g(t)}^{g(s)} f(t,w)dw + \int_{g(t)}^{\alpha(t,u)} (f(t,w) - f(s,w))dw. \quad (32)$$

Since we work locally, we may assume that $k \leq |f(t,w)| \leq K$. Since f is a ζ-Hölder function for some ζ, we also have $|f(t,w) - f(s,w)| \leq C|t-s|^\zeta$. Therefore, Equality (32) yields

$$k|\alpha(s,u) - \alpha(t,u)| \leq K|t-s| + C|t-s|^\zeta.$$

This finishes the proof that α is a Hölder function.

By the construction of ψ, if c is a C^1-curve in D_κ, then $\psi(c)$ is a π-curve. Note that

$$\lambda(s,t) = \left(\int_{g(s)}^{t} f(s,u)du\right)ds,$$

is a primitive of ω. It follows that if $c : s \mapsto (c_1(s), c_2(s))$ is a C^1 curve in D_κ. Then

$$\int_c \lambda = \int_{\mathbb{T}}\int_{g(s)}^{c_2(s)} f(s,u)\dot{c}_1(s)duds = \log(\mathrm{Hol}(\psi(c))).$$

By construction, $\psi \circ \gamma \circ \psi^{-1}$ is π-symplectic and Hölder. The continuity also of ψ on κ and f follows from the construction. Q.E.D.

11.3 Proof of Theorem 11.3

11.3.1 A preliminary lemma

We prove

Lemma 11.7 *Let ρ_1 and ρ_2 be two H-Fuchsian representations from $\pi_1(\Sigma)$ in* $\mathrm{Diff}^h(\mathbb{T})$*. Let κ be a Hölder homeomorphism of $\mathbb{T}$ such that $\kappa \circ \rho_1 = \rho_2 \circ \kappa$. Let $f(s,t)$ be a positive continuous function on D_κ.*

Assume that, for all γ in $\pi_1(\Sigma)$, $\omega = f(s,t)ds \wedge dt$ is invariant under the action of $(\rho_1(\gamma), \rho_2(\gamma))$. Then f is κ-infinite.

Proof: For the sake of simplicity, we write $\rho_i(\gamma) = \gamma^i$. We first observe that the invariance of ω yields that for all γ in $\pi_1(\Sigma)$

$$f(s,t) = \frac{d\gamma^1}{ds}(s)\frac{d\gamma^2}{dt}(t)f(\gamma^1(s), \gamma^2(t)). \quad (33)$$

For any (s,t) in D_κ, we may find a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ as well as (s_0, t_0) in D_κ such that

$$\lim_{n\to\infty} \gamma_n^1(s) = s_0, \quad (34)$$

$$\lim_{n\to\infty} \gamma_n^2(t) = t_0, \quad (35)$$

$$\lim_{n\to\infty} \frac{d\gamma_n^1}{ds}(s) = +\infty. \quad (36)$$

Relation (33) yields

$$\begin{aligned}
\int_{\kappa(s)}^{t} f(s,u)du &= \int_{\kappa(s)}^{t} \frac{d\gamma_n^1}{ds}(s)\frac{d\gamma_n^2}{du}(u)f(\gamma_n^1(s),\gamma_n^2(u))du \\
&= \frac{d\gamma_n^1}{ds}(s)\int_{\kappa(s))}^{t} \frac{d\gamma_n^2}{du}(u)f(\gamma_n^1(s),\gamma_n^2(u))du \\
&= \frac{d\gamma_n^1}{ds}(s)\int_{\gamma_n^2(\kappa(s))}^{\gamma_n^2(t)} f(\gamma_n^1(s),u)du \\
&= \frac{d\gamma_n^1}{ds}(s)\int_{\kappa(\gamma_n^1(s))}^{\gamma_n^2(t)} f(\gamma_n^1(s),u)du.
\end{aligned}$$

From Assertions (34) and (35), we deduce that

$$\lim_{n\to\infty}\int_{\kappa(\gamma_n^1(s))}^{\gamma_n^2(t)} f(\gamma_n^1(s),u)du = \int_{\kappa(s_0)}^{t_0} f(s_0,u)du > 0.$$

Hence Assertion (36) shows that

$$\int_{\kappa(s)}^{t} f(s,u)du = \infty.$$

A similar argument yields

$$\int_{t}^{\kappa(s)} f(s,u)du = \infty.$$

Q.E.D.

11.3.2 Proof of Theorem 11.3

Proof: Recall our hypothesis and notations. Let ρ_1 and ρ_2 be two H-Fuchsian representations. Let κ be a Hölder homeomorphism of $\mathbb{T}$ such that $\kappa\circ\rho_1 = \rho_2\circ\kappa$. Let $p : M = \mathbb{R}\times D_\kappa \to D_\kappa$ be a principal $\mathbb{R}$-bundle over D_κ equipped with a connection ∇. Assume the curvature ω of ∇ is such that $\omega = f(s,t)ds\wedge dt$ with f positive and Hölder. Let

$$\begin{aligned}
\pi &: \left\{\begin{array}{rcl} M & \to & \mathbb{T}, \\ (u,s,t) & \mapsto & t, \end{array}\right. \\
p &: \left\{\begin{array}{rcl} M & \to & D_\kappa, \\ (u,s,t) & \mapsto & (s,t). \end{array}\right.
\end{aligned}$$

Assume that $\pi_1(\Sigma)$ acts on M by C^1-diffeomorphisms with Hölder derivatives. Assume that this action preserves ∇, and that

- $\pi_1(\Sigma)\backslash M$ is compact,
- the action of $\mathbb{R}$ on $\pi_1(\Sigma)\backslash M$ contracts uniformly the fibres of π_1,

- we have $p(\gamma u) = (\rho_1(\gamma)(p(u)), \rho_2(\gamma)(p(u)))$.

We want to prove there exists an $\mathbb{R}$-commuting Hölder homeomorphism $\hat{\psi}$ from M to J over a homeomorphism ψ from D_κ to $T^*\mathbb{T}$, a representation ρ from $\pi_1(\Sigma)$ to $H(\mathbb{T})$ element of Hom_H, such that

$$\hat{\psi} \circ \gamma = \rho(\gamma) \circ \hat{\psi}.$$

By Lemma 11.7, f is κ-infinite. Hence, by Lemma 11.6 there exists a Hölder homeomorphism ψ from D_κ to $T^*\mathbb{T}$ unique up to right composition with a π-exact symplectic homeomorphism, such that

1. $\psi_*\omega = d\theta \cdot dr$,
2. $\pi \circ \psi = \pi_D$,
3. if γ_1, γ_2 are two C^1 diffeomorphisms with Hölder derivatives of the circle such that $\gamma = (\gamma_1, \gamma_2)$ preserves the 2-form ω then
$$\psi \circ \gamma \circ \psi^{-1},$$
is Hölder and π-symplectic above γ_1,
4. if c is a C^1-curve in D, then $\psi(c)$ is a π-curve and $\mathrm{Hol}_\omega(c) = \mathrm{Hol}(\psi(c))$, where $\mathrm{Hol}_\omega(c) = \int_c \lambda$.

Let γ be an element of $\pi_1(\Sigma)$. Since γ acts on M preserving the connection ∇, it follows that $\mu(\gamma) = (\rho_1(\gamma), \rho_2(\gamma))$ acts on D_κ in such a way that for all C^1-curves $\mathrm{Hol}_\nabla(c) = \mathrm{Hol}_\nabla(\mu(\gamma)c)$.

We now show that

$$\mathrm{Hol}_\omega(c) = \mathrm{Hol}_\omega(\mu(\gamma)c). \tag{37}$$

We choose a trivialisation of L. In this trivialisation $\nabla = D + \lambda + \alpha$, where D is the trivial connection and α is a closed form. Then

$$\mathrm{Hol}_\nabla(c) = \mathrm{Hol}_\omega(c).e^{\int_c \alpha}.$$

To prove Equality (37), it suffices to show

$$\int_{f(\gamma)(c)} \alpha = \int_c \alpha. \tag{38}$$

But $\rho_1(\gamma)$ preserves the orientation of $\mathbb{T}$, hence is homotopic to the identity by a family of mapping f_t. It follows that $\mu(\gamma)$ is also homotopic to the identity through the family $(f_t, \kappa f_t \kappa^-1)$. This implies that $\mu(\gamma)$ acts trivially on the homology. Hence Equality (38).

It follows from (3) and (4) that $g(\gamma) = \psi \circ \mu(\gamma) \circ \psi^{-1}$ is a π-exact symplectomorphism.

Finally, we describe the construction of a map $\hat{\psi}$ from M to J above ψ, commuting with $\mathbb{R}$ action, "preserving the holonomy " well defined up right composition by an element of $\mathbb{R}$. Let fix an element y_0 of the fibre of p above x_0 in D_κ and an element z_0 of the fibre of δ above $\psi(x_0)$. Let y be an element of the fibre of p above some point x of D_κ. Let c be path joining x_0 to x. We observe that there exists ζ in $\mathbb{R}$ such that

$$y = \zeta + \mathrm{Hol}_\nabla(c) y_0.$$

We define

$$\hat{\psi}(y) = \zeta + \mathrm{Hol}(\psi(c)) z_0.$$

Since for any closed curve $\mathrm{Hol}_\nabla(c) = \mathrm{Hol}(\psi(c))$, it follows that $\hat{\psi}(y)$ is independent of the choice of c.

By construction, $\hat{\psi}$ satisfies the required conditions. The uniqueness statement follows from the Lemma 10.1 The continuity statement follows by the corresponding continuity statement of Lemma 11.6 and the construction. Q.E.D.

12 Negatively curved metrics

We first use our conjugation Theorem 11.3 to prove the space Rep_H contains an interesting space.

Theorem 12.1 *Let $\mathcal{M}$ be the space of negatively curved metrics on the surface Σ identified up to diffeomorphisms isotopic to the identity. Then, there exists a continuous injective map*

$$\psi : \mathcal{M} \to \mathrm{Rep}_H.$$

Moreover, for every g, $\psi(g)$ is coherent. Furthermore, for any γ in $\pi_1(\Sigma)$

$$\ell_g(\gamma) = \ell_{\psi(g)}(\gamma).$$

Here $\ell_g(\gamma)$ is the length of the closed geodesic for g freely homotopic to γ, and $\ell_{\psi(g)}$ is the $\psi(g)$ -length of γ.

We first recall some facts about the geodesic flow and the boundary at infinity of negatively curved manifolds.

12.1 The boundary at infinity and the geodesic flow

Let Σ be a compact surface equipped with a negatively curved metric. Let $\tilde{\Sigma}$ be its universal cover. Let $U\tilde{\Sigma}$ (respectively $U\Sigma$) be the unitary tangent bundle of $\tilde{\Sigma}$ (respectively Σ). Let φ_t be the geodesic flow on these bundles. Let $\partial_\infty\tilde{\Sigma}$ be the boundary at infinity of $\tilde{\Sigma}$.

We collect in the following proposition classical facts :

Proposition 12.2 1. *The boundaries at infinity of $\pi_1(\Sigma)$ and $\tilde{\Sigma}$ coincide $\partial_\infty\tilde{\Sigma} = \partial_\infty\pi_1(\Sigma)$.*

2. *$\partial_\infty\tilde{\Sigma}$ has a C^1-structure depending on the choice of the metric such that the action of $\pi_1(\Sigma)$ on it is by $C^{1,h}$-diffeomorphisms. The action of $\pi_1(\Sigma)$ is Hölder conjugate to a Fuchsian one.*

3. *Let $\mathcal{G}$ be the space of geodesics of $\tilde{\Sigma}$. Then $\mathcal{G}$ is $C^{1,h}$-diffeomorphic to $(\partial_\infty\pi_1(\Sigma))^{2*}$.*

4. *$U\tilde{\Sigma} \to \mathcal{G}$ is a principal $\mathbb{R}$-bundle (with the action of the geodesic flow). Furthermore the Liouville form is a connection form for this bundle which is invariant under $\pi_1(\Sigma)$, and its curvature is symplectic.*

We recall in the next Proposition the identification of the unitary tangent bundle with a suitable subset of the cube of the boundary at infinity. Let

$$\partial_\infty\pi_1(\Sigma)^{3+} = \{ \text{ oriented } (x,y,z) \in (\partial_\infty\pi_1(\Sigma))^3 \mid x \neq y \neq z \neq x\}.$$

For any (x,y) let

$$\mathcal{L}_{(x,y)} = \{(w,x,y) \in (\partial_\infty\pi_1(\Sigma))^{3+}\}.$$

Proposition 12.3 *There exist a homeomorphism f of $\partial_\infty\pi_1(\Sigma)^{3^+}$ with $U\tilde{\Sigma}$ such that if $\psi_t = f^{-1}\circ\varphi_t\circ f$ then $\psi_t(\mathcal{L}_{(x,y)}) = \mathcal{L}_{(x,y)}$.*

Moreover, for any sequence of real number $\{t\}_{m\in\mathbb{N}}$ going to infinity, let $(x,y,z) \in \partial_\infty\pi_1(\Sigma)^{3+}$, let $\{\gamma\}_{m\in\mathbb{N}}$ be a sequence of elements of $\pi_1(\Sigma)$, such that

$$\{\gamma_m \circ \psi_{t_m}(z,x,y)\}_{n\in\mathbb{N}} \text{ converges to } (z_0,x_0,y_0),$$

Then, for any w, v such that $(w,x,v) \in \partial_\infty\pi_1(\Sigma)^{3+}$, there exists v_0 such that

$$\{\gamma_m \circ \psi_{t_m}(w,x,v)\}_{n\in\mathbb{N}} \text{ converges to } (v_0,x_0,y_0).$$

Proof: We first explain the construction of the map f. Let $(z,x,y) \in \partial_\infty\pi_1(\Sigma)^{3+}$. Let c the geodesic in $\tilde{\Sigma}$ going from x to y. Let $c(t_0)$ be the image of the projection of z on c, that is the unique minimum on c of the horospherical function associated to z. We set $f(z,x,y) = \dot{c}(t_0)$. Then, the first property of f is obvious, and the second one is a classical consequence of negative curvature, namely that two geodesics with the same endpoints at infinity go exponentially closer and closer. Q.e.d.

12.2 Proof of Theorem 12.1

Using Proposition 11.4, 12.2 and 12.3 yields that the hypotheses of Theorem 11.3 are satisfied

Therefore, we obtain a continuous map ψ from $\mathcal{M}$ to $\mathrm{Hom}^*/H(\mathbb{T})$. For a hyperbolic metric, we obtain precisely the associated ∞-Fuchsian representation. Since the space of negatively curved metrics on a compact surface is connected, all the representations are actually in Rep_H.

Finally if $\psi(g_0) = \psi(g_1)$, then the two metrics have the same length spectrum and therefore are isometric by Otal's Theorem [27]. This proves injectivity. The continuity follows from the continuity statement of Theorem 11.3.

13 Hitchin component

We now prove that Rep_H contains all these Hitchin components.

Theorem 13.1 *There exists a continuous injective map*

$$\psi : \mathrm{Rep}_H(\pi_1(\Sigma), \mathrm{PSL}(n,\mathbb{R})) \to \mathrm{Rep}_H,$$

such that, if $\rho \in \mathrm{Rep}_H(\pi_1(\Sigma), \mathrm{PSL}(n,\mathbb{R}))$ then, for any γ in $\pi_1(\Sigma)$, we have

$$\ell_{\psi(\rho)}(\gamma) = w_\rho(\gamma), \tag{39}$$

where $\ell_{\psi(\rho)}(\gamma)$ is the $\psi(\rho)$-length of γ, and $w_\rho(\gamma)$ is the width of γ with respect to ρ.

Moreover, the cross ratio associated to ρ and $\psi(\rho)$ coincide, and the representation $\psi(\rho)$ is coherent (cf Definition 9.8).

Remarks: It follows in particular that if ρ belongs to $\rho \in \mathrm{Rep}_H(\pi_1(\Sigma), \mathrm{PSL}(n,\mathbb{R}))$, and if ρ^* is the contragredient representation, then $\psi(\rho)$ and $\psi(\rho^*)$ have the same spectrum although they are different representations.

13.1 Hyperconvex curves

We recall a proposition obtained in Section 2.2.1

Proposition 13.2 *Let ρ be a hyperconvex representation. Let*

$$\xi = (\xi^1, \dots, \xi^{n-1}),$$

be the limit curve of ρ. Then, there exist

- *two C^1 embeddings with Hölder derivatives, η_1 and η_2, of $\mathbb{T}$ in respectively $\mathbb{P}(\mathbb{R}^n)$ and $\mathbb{P}(\mathbb{R}^{*n})$,*
- *two representations ρ_1 and ρ_2 from $\pi_1(\Sigma)$ in $\mathrm{Diff}^h(\mathbb{T})$, the group of C^1-diffeomorphisms of $\mathbb{T}$ with Hölder derivatives,*
- *a Hölder homeomorphism κ of $\mathbb{T}$,*

such that

1. *$\eta_1(\mathbb{T}) = \xi^1(\partial_\infty\pi_1(\Sigma))$ and $\eta_2(\mathbb{T}) = \xi^{n-1}(\partial_\infty\pi_1(\Sigma))$,*
2. *the map η_i is ρ_i equivariant,*
3. *if $\kappa(s) \neq t$, then the sum $\eta_1(s) + \eta_2(t)$ is direct,*
4. *κ intertwines ρ_1 and ρ_2.*

We shall use we following the continuous map

$$\dot\eta = (\eta_1^{-1} \circ \xi^1, \eta_2^{-1} \circ \xi^{n-1}), \tag{40}$$

from $\partial_\infty\pi_1(\Sigma)^{2*}$ to D_κ.

13.2 Proof of Theorem 13.1

We use in this section the independent results proved in the Section 4.4.

Let ρ be a hyperconvex representation. Let η_1, η_2 and κ as in Proposition 2.7. Let $\eta = (\eta_1, \eta_2)$. Let D_κ be defined as usual. We observe that $\pi_1(\Sigma)$ acts by $\dot\rho = (\rho_1, \rho_2)$ on D_κ. By Proposition 2.7, η is a ρ-equivariant C^1 map from D_κ to

$$\mathbb{P}(n)^{2*} = \mathbb{P}(\mathbb{R}^n) \times \mathbb{P}(\mathbb{R}^{*n}) \setminus \{(D, P) | D \subset P^\perp\}.$$

In Section 4.7, we show there exists a principal $\mathbb{R}$-bundle L on $\mathbb{P}(n)^{2*}$ equipped with an action of $\mathrm{PSL}(n, \mathbb{R})$ and an invariant connection whose curvature is a symplectic form Ω.

We pull back this structure on D_κ:

- Let M be the induced bundle by η. Note that M is equipped with an action of $\pi_1(\Sigma)$ induced from the $\mathrm{PSL}(n, \mathbb{R})$ action on L.
- Let φ_t be the flow of the induced $\mathbb{R}$-action on M.
- Let $\mathcal{L}$ be the foliation of M by horizontal sections along the curves $c_s : t \mapsto (s, t)$ in D_κ.

We observe the following

Proposition 13.3 *Let γ be an element of $\pi_1(\Sigma)$. Let $x = \dot\eta(\gamma^+, \gamma^-)$ be a fixed point of γ in D_κ. Let $\lambda_{\max}$ and $\lambda_{\min}$ be the largest and smallest eigenvalues (in absolute values) of $\rho(\gamma)$. Then the action of γ on the fibre M_x of M above x is given by the translation by*

$$\log \left| \frac{\lambda_{\max}}{\lambda_{\min}} \right|.$$

PROOF: This follows from the last point of Proposition 4.7. Q.E.D.

13.2.1 Proof

By Propositions 5.4 and 5.5, the 2-form $\omega = \eta^*\Omega$ is non degenerate and Hölder To complete the proof of Theorem 13.1, we prove

- the quotient $\pi_1(\Sigma)\backslash M$ is compact (in Proposition 13.4),
- the action of $\mathbb{R}$ on M contracts the leaves of $\mathcal{L}$ (in Proposition 13.6).

We now explain how these properties imply the theorem: by Theorem 11.3, there exists a homeomorphism $\hat\psi$ from M to J, an area preserving homeomorphism ψ from D to $T^*\mathbb{T}$ such that

- $\hat\psi$ commutes with the $\mathbb{R}$-action,
- $\delta \circ \hat\psi = \psi \circ p$,
- $\hat\psi$ sends $\mathcal{L}$ to $\mathcal{F}$,

and a representation ρ from $\pi_1(\Sigma)$ to $H(\mathbb{T})$ such that

$$\hat{\psi} \circ \gamma = \rho(\gamma) \circ \hat{\psi}.$$

In particular, ρ belongs to Hom^*.

Let us prove that ρ actually belongs to Hom_H. By the definition of the Hitchin component and the fact that we can choose $\hat{\psi}$ to depend continuously on our parameters, it suffices to verify this assertion whenever ρ is an n-Fuchsian representation. In this case, the action of $\pi_1(\Sigma)$ extends to a transitive action of $\mathrm{PSL}(2, \mathbb{R})$ and is by definition in Hom_H.

The statement (39) about the spectrum follows from Proposition 13.3 which implies the spectrum is symmetric. Therefore, by definition, $\psi(\rho)$ is symmetric or, equivalently, coherent. Finally, by construction and Proposition 4.7, the two cross ratios coincide.

By Theorem 9.13, an ∞-Hitchin representation which is symmetric is determined by its cross ratio. It follows that ψ is injective.

Finally the continuity statement follows from the continuity statement of Theorem 11.3

13.2.2 Compact quotient

Let M be the $\mathbb{R}$-bundle over D_κ defined by $M = \eta^* L$. Inducing the connection form L, M is equipped with a connection whose curvature form is ω. Furthermore $\pi_1(\Sigma)$ acts on M, by the pull back of the action of $\mathrm{PSL}(n, \mathbb{R})$ on L. We now prove

Proposition 13.4 *The quotient $\pi_1(\Sigma)\backslash M$ is compact.*

Recall that $\pi_1(\Sigma)$ acts with a compact quotient on

$$\partial_\infty \pi_1(\Sigma)^{3+} = \{\text{oriented triples } (x, y, z) \in (\partial_\infty \pi_1(\Sigma))^3, x \neq y, y \neq z, x \neq z\},$$

We first prove:

Proposition 13.5 *There exists a continuous onto $\pi_1(\Sigma)$-equivariant proper map l from $\partial_\infty \pi_1(\Sigma)^{3+}$ to M. Moreover, the map l is above the map $\dot{\eta} = (\dot{\eta}_1, \dot{\eta}_2)$ from $\partial_\infty \pi_1(\Sigma)^{2*}$ to D_κ.*

PROOF: Let (z, x, y) be an element of $\mathbb{T}^{3+} = \partial_\infty \pi_1(\Sigma)^{3+}$. The two transverse flags $\xi(x)$ and $\xi(y)$ define a decomposition

$$\mathbb{R}^n = L_1(x, y) \oplus L_2(x, y) \oplus \ldots \oplus L_n(x, y),$$

such that $\xi^1(x) = L_1(x, y)$ and

$$\xi^{n-1}(y) = L_2(x, y) \oplus \ldots \oplus L_n(x, y). \tag{41}$$

Let u be a nonzero element of $\xi^1(z)$. Let u_i be the projection of u on $L_i(x, y)$. By hyperconvexity, $u_i \neq 0$. We choose u, up to sign, so that

$$|u_1 \wedge \ldots \wedge u_n| = 1.$$

Finally we choose f in $\xi^{n-1}(y)^{\perp}$ so that $\langle f, u_1\rangle = 1$. The pair (u_1, f) is well defined up to sign, and hence defines a unique element $l(z,x,y)$ of the fibre of M above $\dot\eta(x,y)$.

Our first assertion is that l :

$$\left\{\begin{array}{rcl} \mathbb{T}^{3+} & \to & M, \\ (z,x,y) & \mapsto & l(z,x,y), \end{array}\right.$$

is proper.

Let $\{(z_m, x_m, y_m)\}_{m\in\mathbb{N}}$ be a sequence of elements of $\mathbb{T}^{3+}$ such that

$$\{l(z_m, x_m, y_m) = (u_m, f_m)\}_{m\in\mathbb{N}},$$

converges to (u_0, f_0) with $\langle f_0, u_0\rangle = 1$. In particular, $\{(x_m, y_m)\}_{m\in\mathbb{N}}$ converges to (x_0, y_0), with $x_0 \neq y_0$. We may assume after extracting a subsequence that $\{z_m\}_{m\in\mathbb{N}}$ converges to z_0. To prove l is proper, it suffices to show that $x_0 \neq z_0$ and $y_0 \neq z_0$.

Suppose that this is not the case and let us first assume that $z_0 = x_0$. Let π_m be the projection of $\xi^1(z_m)$ on $\xi^1(x_m)$ along $\xi^{n-1}(y_m)$. We observe that π_m converges to the identity from $\xi^1(z_0) = \xi^1(x_0)$ to $\xi^1(x_0)$. Let $v_m \in \xi^1(z_m)$ such that $\pi_m(v_m) = u_m$. Since $\{u_m\}_{m\in\mathbb{N}}$ converges to a nonzero element u_0, $\{v_m\}_{m\in\mathbb{N}}$ converges to u_0. As a consequence, all the projections $\{v_m^i\}_{m\in\mathbb{N}}$ of v_m on $L_i(x_m, y_m)$ converge to zero for $i > 1$. Hence,

$$1 = |v_m^1 \wedge \ldots \wedge v_m^n| \to 0,$$

and the contradiction.

Suppose now that $z_0 = y_0$. Using the volume form of $\mathbb{R}^n$, we identify $\xi^{n-1}(y)^{\perp}$ with

$$L_2(x,y) \wedge \ldots \wedge L_n(x,y).$$

We use the same notations as in the previous paragraph. Then $v_m^2 \wedge \ldots \wedge v_m^n$ is identified with f_m. It follows that

$$v_m^2 \wedge \ldots \wedge v_m^n \to f_0. \tag{42}$$

By hyperconvexity

$$\xi^1(z_m) \oplus \xi^p(y_m) \to \xi^{p+1}(y_0).$$

We also have,

$$\xi^p(y) = L_{n-p+1}(x,y) \oplus \ldots \oplus L_n(x,y).$$

Since $\xi^1(z_m) \to \xi^1(y_0)$, it follows that for all k

$$\frac{\|v_m^1\|}{\|v_m^k\|} \to 0.$$

In particular

$$\frac{\|v_m^1\|^{n-1}}{\|v_m^2 \wedge \ldots \wedge v_m^n\|} \to 0.$$

Thanks to Assertion (42), we finally obtain that $\|v_m^1\| \to 0$. Hence

$$\begin{aligned} 1 &= \|v_m^1 \wedge \ldots \wedge v_m^n\| \\ &\leq \|v_m^1\| \|v_m^2 \wedge \ldots v_m^n\| \to 0, \end{aligned}$$

and the contradiction. Therefore $(z_0, x_0, y_0) \in \mathbb{T}^{3+}$ and l is proper.

It remains to prove that l is onto. Note first that for every (x, y),

$$L_{(x,y)} = l(\mathbb{T}^{3+}) \cap M_{\dot{\eta}(x,y)},$$

is an interval, being the image of an interval. If (γ_+, γ_-) is a fixed point of γ, then $L_{(\gamma_+,\gamma_-)}$ is invariant by $\rho(\gamma)$ that acts as a translation on $M_{(\gamma_+,\gamma_-)}$. It follows that

$$L_{(\gamma_+,\gamma_-)} = M_{(\gamma_+,\gamma_-)}.$$

Since the set of fixed points of elements of $\pi_1(\Sigma)$ is dense in $\partial_\infty \pi_1(\Sigma)^2$, and $l(\mathbb{T}^{3+})$ is closed by properness, we conclude that $l(\mathbb{T}^{3+}) = M$ and that l is onto. Q.E.D.

As a corollary, we prove Proposition 13.5

Proof of Proposition 13.5:

Since $\pi_1(\Sigma)$ acts properly on $\mathbb{T}^{3+}$ and l is proper and onto, the action of $\pi_1(\Sigma)$ on M is proper. Indeed for any compact K in M

$$\{\gamma \in \pi_1(\Sigma) \mid \gamma(K) \cap K \neq \emptyset\} \subset \{\gamma \in \pi_1(\Sigma) \mid \gamma(l^{-1}(K)) \cap l^{-1}(K) \neq \emptyset\},$$

Hence,

$$\sharp\{\gamma \in \pi_1(\Sigma) \mid \gamma(K) \cap K \neq \emptyset\} \leq \sharp\{\gamma \in \pi_1(\Sigma) \mid \gamma(l^{-1}(K)) \cap l^{-1}(K) \neq \emptyset\} < \infty.$$

Since $\pi_1(\Sigma)$ has no torsion elements and acts properly, it follows that the action of $\pi_1(\Sigma)$ on M is free. The space $\pi_1(\Sigma)\backslash M$ is therefore a topological manifold. Finally, the quotient map of l from $\pi_1(\Sigma)\backslash\mathbb{T}^{3+}$ (which is compact) being onto, it follows $\pi_1(\Sigma)\backslash M$ is compact.

13.2.3 Contracting the leaves

We notice that M is topologically a trivial bundle. Let $\mathcal{L}$ be the foliation of M by horizontal sections along the curves $c_s : t \mapsto (s, t)$ in D_κ. We choose a $\pi_1(\Sigma)$-invariant metric on M. We prove now:

Proposition 13.6 *The flow φ_t contracts the leaves of $\mathcal{L}$ on M.*

Proof: In the proof of Proposition 13.5, we exhibited a proper onto continuous $\pi_1(\Sigma)$-equivariant map l from $\partial_\infty \pi_1(\Sigma)^{3+}$ to M. This map is such that

$$l(z, x, y) \in M_{\dot{\eta}(x,y)}. \tag{43}$$

By Proposition 12.3, the choice of a hyperbolic metric on Σ gives rises to a flow ψ_t by proper homeomorphisms on $\partial_\infty \pi_1(\Sigma)^{3+}$ such that

1. for all $t \in \mathbb{R}$, for all $(z, x, y) \in \partial_\infty \pi_1(\Sigma)^{3+}$, there exists w such that
$$\psi_t(z, x, y) = (w, x, y),$$
2. For any sequence $\{t\}_{m\in\mathbb{N}}$ of real numbers going to infinity, let $(z, x, y) \in \partial_\infty \pi_1(\Sigma)^{3+}$ and let $\{\gamma\}_{m\in\mathbb{N}}$ be a sequence of elements of $\pi_1(\Sigma)$, such that
$$\{\gamma_m \circ \psi_{t_m}(z, x, y)\}_{n\in\mathbb{N}} \text{ converges to } (z_0, x_0, y_0).$$
Then for any w, v such that $(w, x, v) \in \partial_\infty \pi_1(\Sigma)^{3+}$, there exists v_0 such that
$$\{\gamma_m \circ \psi_{t_m}(w, x, v)\}_{n\in\mathbb{N}} \text{ converges to } (w_0, x_0, y_0).$$

Since Σ is compact and since l and the flows are proper, there exist positive constants a and $\mathbb{B}$ such that
$$\forall u, \forall T, \exists t \in]T/a - b, Ta + b[\text{ such that } l(\psi_t(u)) = \varphi_T(l(u)).$$

Therefore, the following assertion holds:

Let $\{t\}_{m\in\mathbb{N}}$ be a sequence of real numbers going to infinity, let $u \in M$, let $\{\gamma\}_{m\in\mathbb{N}}$ be a sequence of elements of $\pi_1(\Sigma)$, such that
$$\{\gamma_m \circ \varphi_{t_m}(u)\}_{n\in\mathbb{N}} \text{ converges to } u_0 \in M_{\dot{\eta}(x_0,y_0)},$$
then for any w, s such that $w \in M_{\dot{\eta}(x,s)}$, there exists v_0 such that
$$\{\gamma_m \circ \varphi_{t_m}(w)\}_{n\in\mathbb{N}} \text{ converges to } v_0 \in M_{\dot{\eta}(x_0,y_0)}.$$
By Proposition 11.4, the action contracts of ϕ_t contracts the leaves of $\mathcal{L}$. Q.E.D.

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